A Non-Newtonian rheology model application to complex flows (PyroClast)

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Outline

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  - Lubrication model
  - Numerical results
- Bingham fluid (Known model)
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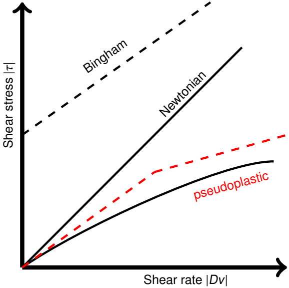
  - Lubrication model
  - Numerical results
- Two viscosities model (New model)
  - Two-viscosities model for pseudo-viscoplastic fluid
  - Lubrication model
  - Numerical results

Motivation

Pyroclast flow (Source: Internet)



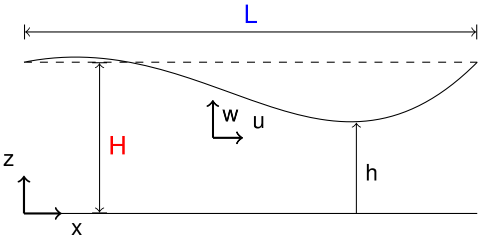
Rheology - types

Idea: To approximate the pseudo-plastic rheology by a piecewise affine function.

Main features

- $H/L \ll 1$ Lubrication approximation
- Rheology - type (Stress-Strain relation).
  - Newtonian fluid : Water, oil,....
  - Bingham fluid
  - Two - viscosities model: "Pseudo-viscoplastic fluid"

Navier-Stokes model



Newtonian fluid

Navier-Stokes model

$$u_x + w_z = 0$$

$$\rho(u_t + uu_x + wu_z) = -p_x + 2\mu u_{xx} + \mu u_{zz} + \mu w_{xz}$$
$$\rho(w_t + uw_x + ww_z) = -p_z + \mu u_{xz} + \mu w_{zz} + 2\mu w_{zz} - \rho g$$

Boundary condition

$$u = w = 0 \qquad z = 0$$

$$(1 - h_x^2)p + 2\mu(1 + h_x)^2 u_x = 0 \qquad z = h(x,t)$$
$$(1 - h_x^2)(u_z + w_x) - 4h_x u_x = 0 \qquad z = h(x,t)$$

Kinematic condition

$$h_t + uh_x = w \qquad z = h(x,t)$$

Lubrication Method

Scale independent

$$x = L\tilde{x}, \quad z = H\tilde{z}, \quad u = U\tilde{u}$$

Scale dependent

$$h = H\tilde{h}, \quad t = \frac{L}{U}\tilde{t}, \quad w = \frac{H}{LU}\tilde{w}, \quad p = \rho g H\tilde{p}$$

$$\varepsilon = \frac{H}{L}, \quad Fr^2 = \frac{U^2}{gH}, \quad Re = \frac{\rho HU}{\mu}$$

Dimentionless equation

$$\varepsilon Re(u_t + uu_x + wu_z) = -\frac{\varepsilon Re}{Fr^2}p_x + 2\varepsilon^2\mu u_{xx} + u_{zz} + \varepsilon^2 w_{xz}$$

$$\varepsilon^3 Re(w_t + uw_x + ww_z) = -\frac{\varepsilon Re}{Fr^2}p_z + \varepsilon^4\mu w_{xx} + \varepsilon^2 u_{xz} + 2\varepsilon^2 w_{zz} - 1$$

Lubrication model

Limit problem

$$u_x + w_z = 0, \quad p_x = u_{zz}, \quad p_z = -1$$

Boundary condition

$$\begin{aligned} u = w &= 0 \qquad z = 0 \\ p = u_z &= 0 \qquad z = h(x,t) \end{aligned}$$

Kinematic condition

$$h_t + u h_x = w \qquad z = h(x,t)$$

Lubrication model

$$h_t - \partial_x \left(\frac{h^3}{3} h_x \right) = 0$$

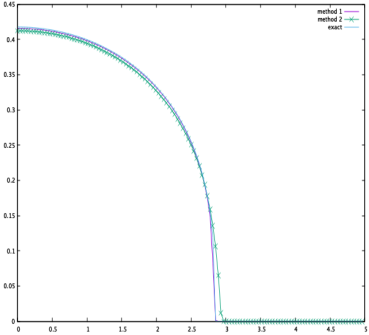
Numerical results

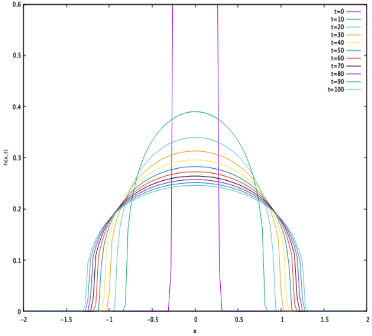
- Method 1:

$$F(H_l, H_r) = -\left(\frac{H_L + H_R}{2}\right)^3 \frac{H_R - H_L}{3\Delta x}$$

- Method 2:

$$F(H_l, H_r) = \frac{(H_L - H_R)(H_L^3 + H_R^3)}{6\Delta x} - \frac{|H_L - H_R|\max(H_L^2, H_R^2)(H_R - H_L)}{2\Delta x}$$





Bingham fluid

- $v = (u, w)^T$
- $Dv = \frac{\nabla v + \nabla v^T}{2}, \quad \tau = \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{xz} & \tau_{zz} \end{pmatrix}$

Laws of behavior

Newtonian fluid	**Bingham fluid**
$\tau = 2\mu Dv$	$\begin{cases} \lvert\tau\rvert < Bi & Dv = 0 \quad \textbf{solid} \\ \tau = Dv + Bi\frac{Dv}{\lvert Dv\rvert} & Dv \neq 0 \quad \textbf{fluid} \end{cases}$

Limit problem

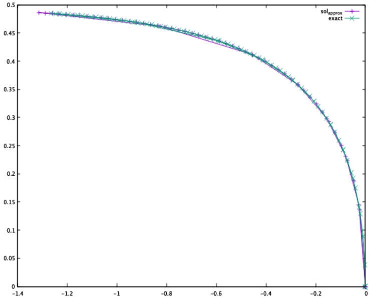
Newtonian fluid	**Bingham fluid**
$p_x = u_{zz}$	$p_x = 2\partial_z \tau_{xz} = \partial_z \left(u_z \left(1 + \frac{Bi\sqrt{2}}{\lvert u_z\rvert} \right) \right)$

$$h_t - \partial_x \left(\frac{h_x Y^2}{6} [3h - Y] \right) = 0$$

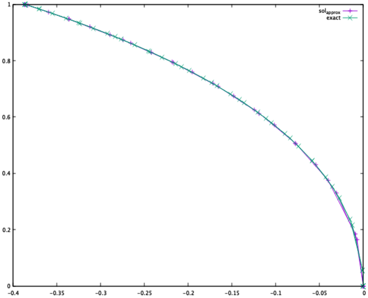
$$Y = \max \left(h - \frac{Bi\sqrt{2}}{|h_x|}, 0 \right)$$

Remark: $Bi = 0$ Bingham $\rightarrow$ Newtonian

$Bi = 0.5$



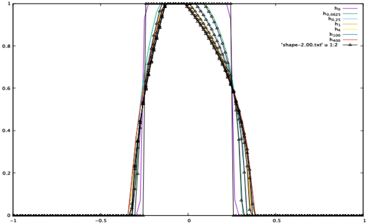
$Bi = 2$



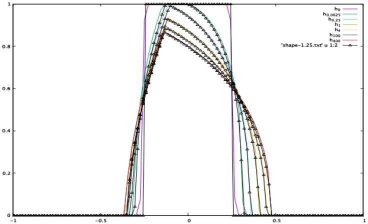
$nx = 128$

Numerical results - Slump test

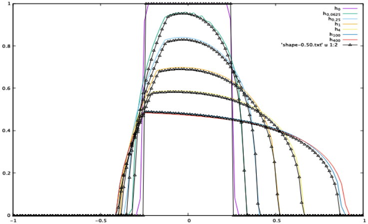
$Bi = 2$



$Bi = 1.25$



$Bi = 0.5$



Two-viscosities model

$$\tau = \begin{cases} 2\mu_1 Dv & |Dv| \leq \frac{\tau^*}{2\mu_1} \quad \textbf{high viscosity} \\ 2\mu_2 Dv + \left(1 - \frac{\mu_2}{\mu_1}\right)\tau^* \frac{Dv}{|Dv|} & |Dv| > \frac{\tau^*}{2\mu_1} \quad \textbf{low viscosity} \end{cases}$$

$\mu_2 = \mu_1$

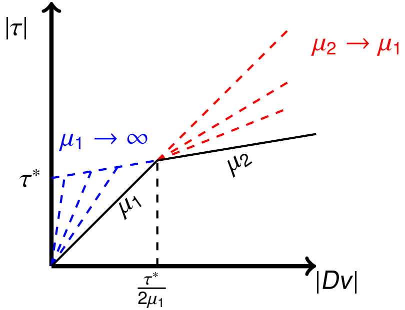
$$\tau = 2\mu_1 Dv$$

Newtonian fluid

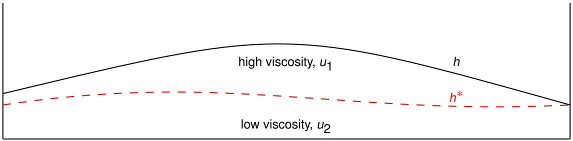
$\mu_1 \to +\infty$

$$\tau = \begin{cases} |\tau| < \tau^* & |Dv| \leq 0 \\ 2\mu_2 Dv + \tau^* \frac{Dv}{|Dv|} & |Dv| > 0 \end{cases}$$

Bingham fluid



Two-viscosities model



Difficulty

1. Induce unknown interface h^*
2. Identify BC at interface h^*

Resolving

1. Choice of scaling + constraint on $|Du|$
2. Preserve continuity of u and τ
3. Scale the viscosity: $\mu_i = \alpha_i \mu$

Lubrication model

Two-viscosities model

$$\partial_t h - \partial_x\left(h_x\left(\frac{1}{\mu_2}\left[\frac{h^{*3}}{3} - \frac{Yh^{*2}}{2} + Yhh^* - \frac{hh^{*2}}{2}\right] - \frac{1}{\mu_1}\left[\frac{h^{*3}-h^3}{3} + h^2h^* - hh^{*2}\right]\right)\right) = 0$$

$$Y = \max\left(h - \left(1 - \frac{\mu_2}{\mu_1}\right)\frac{\sqrt{2}B}{|h_x|}, 0\right), \qquad h^* = \max\left(h - \frac{\sqrt{2}B}{|h_x|}, 0\right)$$

$\mu_2 = \mu_1$

$$\partial_t h - \partial_x\left(\frac{h^3}{3}h_x\right) = 0$$

Newtonian fluid

$\mu_1 \to +\infty$

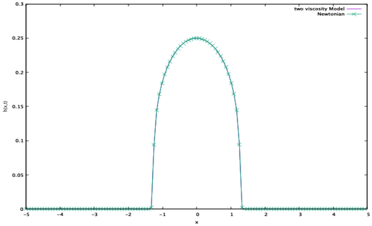
$$h_t - \partial_x\left(\frac{h_x Y^2}{6}[3h - Y]\right) = 0$$

$$Y = \max\left(h - \frac{Bi\sqrt{2}}{|h_x|}, 0\right)$$

Bingham fluid

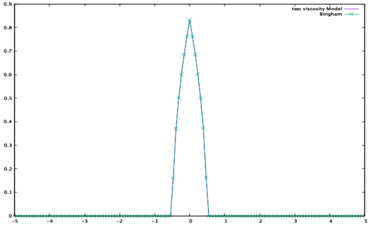
Numerical results

$\mu_2 = \mu_1$



Newtonian fluid

$\mu_1 = 1000 \gg \mu_2$



Bingham fluid

Slump test for two-viscosities fluid

Thanks CEMRACS, 2019