Compressible viscoplastic models for granular flows

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Mathematical model

- stress tensor σ : matrix symmetric $n \times n$
- strain tensor $Du := \frac{\nabla u + \nabla u^T}{2}$

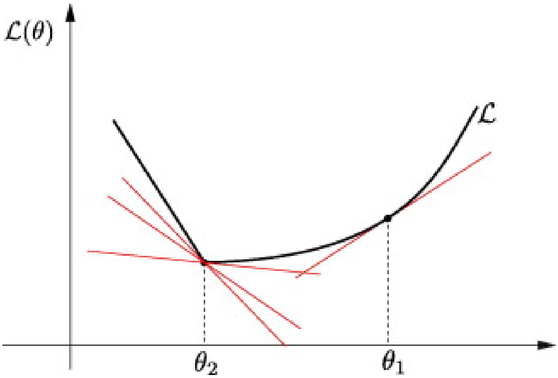
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0$$

$$\frac{\partial(\rho u)}{\partial t} + \operatorname{div}(\rho u \otimes u) - \operatorname{div} \sigma + \nabla p = f$$

$$\sigma \in \partial F(Du)$$

+ Initial condition, Boundary condition

Subgradient - Subdifferential



Subgradient

Newtonian and Non-Newtonian fluids

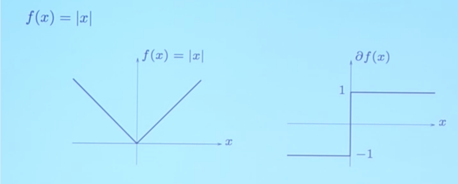
Newtonian fluids

$$F(Du) = \frac{\eta}{2}|Du|^2 \quad \Rightarrow \quad \sigma = \eta Du$$

Stress tensor is linearly dependent on Strain rate

Example of Non-Newtonian fluid

$$F(Du) = |Du| \quad \Rightarrow \quad \sigma = \begin{cases} \frac{Du}{|Du|} & Du \neq 0 \quad \textbf{fluid} \\ |\sigma| \leq 1 & Du = 0 \quad \textbf{solid} \end{cases}$$



Splitting scheme

- Using Finite Volume Method with Suliciu's solver for:

$$\partial_t \begin{pmatrix} \rho \\ \rho u \end{pmatrix} + \operatorname{div} \begin{pmatrix} \rho u \\ \rho u \otimes u + pI_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Using Finite Element Method for:

$$\partial_t \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} 0 \\ \operatorname{div} \sigma + f \end{pmatrix}$$

Algorithm for viscoplastic models

$$\alpha u - \operatorname{div} \sigma = f$$
$$\sigma \in \partial F(Du)$$

Regularization method

$$\alpha u_\varepsilon - \operatorname{div} \sigma_\varepsilon = 0$$
$$\sigma_\varepsilon = F'_\varepsilon(Du_\varepsilon)$$

In inviscid Bingham case:

$$\sigma = \begin{cases} \sigma_0 \frac{Du}{|Du|} & Du \neq 0 \\ |\sigma| \leq \sigma_0 & \text{otherwise} \end{cases} \Rightarrow \sigma_\varepsilon = \sigma_0 \frac{Du_\varepsilon}{\sqrt{|Du_\varepsilon|^2 + \varepsilon^2}}$$

- Necessarity of finding the optimal ε
- Advantages: Regularization method is natural, fast.
- Disadvantages (for inviscid Bingham): Cannot solve exactly plug zones $Du = 0$

Alternative approach

- Augmented Lagrange Method
- Bermudez-Moreno Method [Bresch and al 2014]
- Bi-projection method [Laurent Chupin, Thierry Dubois 2015]
- Duality method [Chambolle, A. and Pock, T. (2011)]
- ...

Goal

- Solving for the general viscoplastic model $\sigma \in \partial F(Du)$
- Proving the convergence in space for the scheme.
- Comparing with other methods.

Main results

Proposition (Projection formulation)

For any $r > 0$*:*

$$\sigma \in \partial F(Du) \Leftrightarrow \mathbb{P}_r(\sigma + rDu) = \sigma$$

where

$$\mathbb{P}_r(A) = (Id + r\partial F^*)^{-1}(A)$$

In inviscid Bingham case:

$$\sigma = \begin{cases} \frac{Du}{|Du|} & Du \neq 0 \\ |\sigma| \leq 1 & \text{otherwise} \end{cases} \Leftrightarrow \sigma = \begin{cases} \frac{\sigma + rDu}{|\sigma + rDu|} & |\sigma + rDu| \geq 1 \\ \sigma + rDu & |\sigma + rDu| < 1 \end{cases}$$

Algorithm I

$$\begin{cases} \alpha u - \operatorname{div} \sigma = f \\ \hat{\sigma} = \mathbb{P}_r(\hat{\sigma} + rD\hat{u}) \end{cases} \tag{1}$$

Convergence [F.Bouchut, D.N.]

Suppose: $|Du| < L|u|$.
Condition: $L^2 \tau r < 1$.

$$\begin{cases} \alpha u_{k+1} - \operatorname{div} \sigma_{k+1} + \frac{u_{k+1} - u_k}{\tau} = f \\ \\ \sigma_{k+1} = \mathbb{P}_r(\sigma_k + r(2Du_k - Du_{k-1})) \end{cases}$$

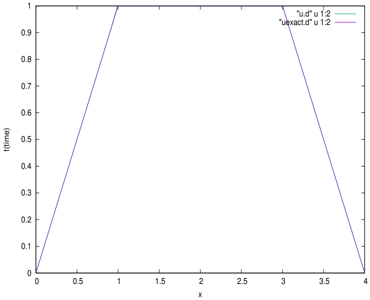
Algorithm II

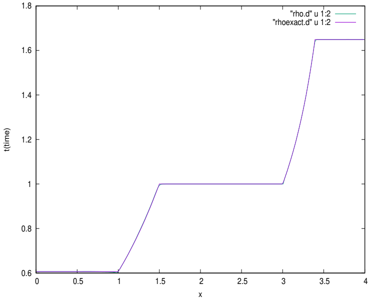
Convergence [F.Bouchut, D.N.]

Suppose: $|Du| < L|u|$.
Condition: $L^2\tau_0 r_0 < 1$.

$$\begin{cases} \theta_k = \frac{1}{\sqrt{1+2\alpha\tau_{k-1}}} \\ \tau_k = \theta_k \tau_{k-1} \\ r_k = \frac{r_{k-1}}{\theta_{k-1}} \\ \sigma_{k+1} = \mathbb{P}_{r_k}(\sigma_k + r_k(Du_k + \theta_k(Du_k - Du_{k-1})) \\ \alpha u_{k+1} - \operatorname{div}\sigma_{k+1} + \frac{u_{k+1}-u_k}{\tau_k} = f \end{cases}$$

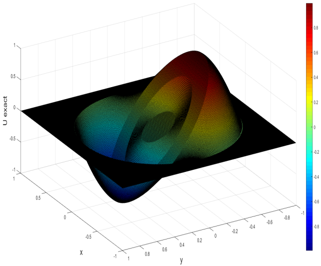
Numerical results 1D

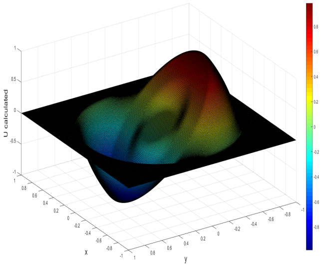




$$nx = 300, \quad \varepsilon \approx C\frac{dx^2}{dt}$$

Numerical results 2D - Viscoplastic model





- The proposed numerical scheme works for unstructured mesh.
- The second algorithm is not faster than the first one.
- Both scheme are faster than Lagrange Augmented and Bermudez-Moreno Method, but slower than Regularization method.

THANK YOU FOR YOUR ATTENTION !