A first order staggered scheme for the shallow water equations

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The shallow water flow

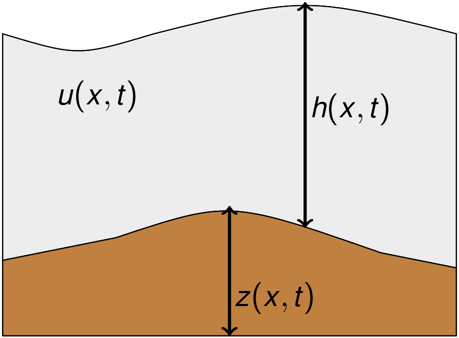


Figure: Horizontal plane.

Equations:

Ω being an open bounded domain of $\mathbb{R}^d$ with $d = 1, 2$ and $T > 0$.

$$
\begin{aligned}
&\partial_t h + \mathrm{div}(h\boldsymbol{u}) = 0 && \text{in } \Omega \times (0, T),\\
&\partial_t (hu_i) + \mathrm{div}(h\boldsymbol{u}\, u_i) + \partial_x p + gh\partial_x z = 0,\ i = 1, 2 && \text{in } \Omega \times (0, T),\\
&p = \frac{1}{2} g h^2 && \text{in } \Omega \times (0, T),\\
&\boldsymbol{u} \cdot \boldsymbol{n} = 0 && \text{on } \partial\Omega \times (0, T),\\
&h(\boldsymbol{x}, 0) = h_0,\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 && \text{in } \Omega.
\end{aligned}
$$

Some generic properties

Stability:

Positivity of the water height $\rightsquigarrow h \geq 0$

Lake at rest steady state:

$\boldsymbol{u} = \boldsymbol{u}_0 = 0$ and $h + z = h_0 + z = Cst.$

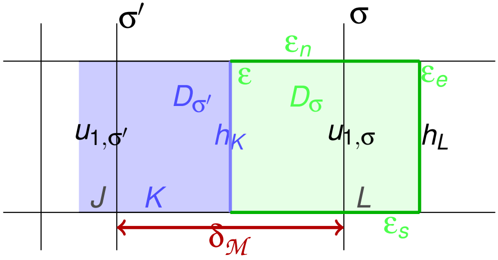
Entropy balance equation:

$$\partial_t \eta + \mathrm{div}\Phi = 0,$$

$$\eta = \frac{1}{2} h |\boldsymbol{u}|^2 + \frac{1}{2} g h^2 + g h z \text{ and } \Phi = (\eta + \frac{1}{2} g h^2) \boldsymbol{u}.$$

Outline

Decoupled forward Euler scheme (1/3)



Primal mesh: $\mathcal{M} = \{J, K, L, \dots\}$, $\mathcal{E}^{(1)} = \{\sigma', \sigma\}$,

Dual mesh: $\widetilde{\mathcal{M}} = \{D_{\sigma'}, D_\sigma, \dots\}$, $\widetilde{\mathcal{E}}^{(1)} = \{\varepsilon, \varepsilon_e, \varepsilon_n, \varepsilon_s\}$.

$$(h, p, z)(\boldsymbol{x}) = \sum_{K \in \mathcal{M}} (h, p, z)_K \mathbf{1}_K(\boldsymbol{x})$$

$$u_i(\boldsymbol{x}) = \sum_{D_\sigma \in \widetilde{\mathcal{M}}} u_{i,\sigma} \mathbf{1}_{D_\sigma}(\boldsymbol{x}), \; i = 1, 2$$

Decoupled Explicit scheme (2/3)

$$h_K^{n+1} = h_K^n - \delta t \sum_{i=1}^{2} \mathrm{div}_K(h^n u_i^n), \quad p_K^{n+1} = \frac{1}{2} g (h_K^{n+1})^2,$$

$$h_{D_\sigma}^{n+1} u_{1,\sigma}^{n+1} = h_{D_\sigma}^n u_{1,\sigma}^n - \delta t \left(\mathrm{div}_{\mathcal{E}^{(1)}}(h^n \boldsymbol{u}^n u_1^n) + \eth_1(p^{n+1})_\sigma + g h_{\sigma,c}^{n+1} \eth_1(z)_\sigma \right),$$

With:

$$\mathrm{div}_K(h u_1) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}^{(1)}(K)} F_{K,\sigma}, \quad F_{K,\sigma} = |\sigma| \left(h_K (u_{1,\sigma})^+ + h_L (u_{1,\sigma})^- \right),$$

$$\mathrm{div}_{\mathcal{E}^{(1)}}(h \boldsymbol{u} u_1^n) = \frac{1}{|D_\sigma|} \sum_{\varepsilon \in \widetilde{\mathcal{E}}^{(1)}(D_\sigma)} \left(u_{1,\sigma'} (F_{\sigma,\varepsilon})^+ + u_{1,\sigma} (F_{\sigma,\varepsilon})^- \right),$$

$$\eth_1(p)_\sigma = \frac{|\sigma|}{|D_\sigma|} (p_L - p_K), \quad h_{\sigma,c} = \frac{h_K + h_L}{2}, \ \sigma = K|L.$$

Decoupled Explicit scheme (3/3)

$$2|D_\sigma|\, h_{D_\sigma} = |K|\, h_K + |L|\, h_L, \text{ with } 2|D_\sigma| = |K| + |L|, \sigma = K|L,$$

For $\varepsilon = D_{\sigma'}|D_\sigma \in \widetilde{\mathcal{E}}^{(1)}$, we have:

$$F_{\sigma,\varepsilon} = \begin{cases} \frac{1}{2}|\varepsilon|\, (h_\sigma\, u_{1,\sigma} + h_{\sigma'}\, u_{1,\sigma'}), & \text{if } \varepsilon \perp \boldsymbol{e}^{(1)} (\subset K) \\ \\ \frac{1}{2}\, (|\tau|\, h_\tau\, u_{2,\tau} + |\tau'|\, h_{\tau'}\, u_{2,\tau'}), & \text{if } \varepsilon = (\frac{1}{2}\tau \cup \frac{1}{2}\tau') \perp \boldsymbol{e}^{(2)} \end{cases}$$

Both definitions satisfy the relation:

$$|D_\sigma|\, (h^{n+1}_{D_\sigma} - h^n_{D_\sigma}) + \delta t \sum_{\varepsilon \in \widetilde{\mathcal{E}}(D_\sigma)} F^n_{\sigma,\varepsilon} = 0.$$

Discrete properties (1/6)

Let $n \in \mathbb{N}$, let $(h_K^n,\ \boldsymbol{u}_\sigma^n)_{K \in \mathcal{M},\ \sigma \in \mathcal{E}}$ be the approximate solution.

Positivity preservation

If $h_K^n > 0$, for all $K \in \mathcal{M}$, then $h_K^{n+1} > 0$ for all $K \in \mathcal{M}$, under the time step restriction:

$$\Big(\sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma|\, |u_{i,\sigma}^n| \Big)\, \delta t \le |K|.$$

'Lake at rest' preservation

If $u_{i,\sigma}^n = 0$ for all $\sigma \in \mathcal{E}^{(i)}$ and $h_K^n + z_K = C$ for all $K \in \mathcal{M}$, then $u_{i,\sigma}^{n+1} = 0$ for all $\sigma \in \mathcal{E}^{(i)}$ and $h_K^{n+1} + z_K = C$ for all $K \in \mathcal{M}$.

Discrete properties (2/6)

Discrete kinetic energy balance:

$$\frac{1}{2\,\delta t}\big(h_{D_\sigma}^{n+1}(u_{i,\sigma}^{n+1})^2 - h_{D_\sigma}^{n}(u_{i,\sigma}^{n})^2\big) + \frac{1}{2\,|D_\sigma|}\sum_{\varepsilon\in\widetilde{\mathcal{E}}^{(i)}(D_\sigma)} F_{\sigma,\varepsilon}^{n}(u_{i,\varepsilon}^{n})^2 + u_{i,\sigma}^{n+1}(\eth_i p^{n+1})_\sigma + g\,h_{\sigma,c}^{n+1}\,u_{i,\sigma}^{n+1}(\eth_i z)_\sigma = -R_{i,\sigma}^{n+1},$$

where $u_{i,\varepsilon}^{n}(= u_{i,\sigma'}^{n}$ or $u_{i,\sigma}^{n}$ for $\varepsilon = D_{\sigma'}|D_\sigma)$ is upwinded respect to $F_{\sigma,\varepsilon}^{n}$ and with $R_{i,\sigma}^{n+1} \geq 0$ under the CFL like condition:

$$\forall\sigma\in\mathcal{E}^{(i)}, \qquad \delta t \leq \frac{|D_\sigma|\,h_{D_\sigma}^{n+1}}{\displaystyle\sum_{\varepsilon\in\widetilde{\mathcal{E}}(D_\sigma)} (F_{\sigma,\varepsilon}^{n})^-}.$$

Discrete properties (3/6)

Discrete potential energy balance:

$$\frac{1}{2} g\, \eth_t (h_K^{n+1})^2 + \frac{1}{2} g\, \mathrm{div}_K((h^n)^2 \boldsymbol{u}^n) + p_K^n \mathrm{div}_K(\boldsymbol{u}^n) = -R_K^{n+1},$$

with

$$R_K^{n+1} \geq \frac{1}{|K|} g \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma|\; u_{i,\sigma}^n h_\sigma^n (h_K^{n+1} - h_K^n)\, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}.$$

Discrete properties (4/6)

- Discrete total potential energy associated to K:

$$(\varepsilon_p)_K^n = g\, h_K^n (\frac{1}{2} h_K^n + z_K)$$

- Discrete kinetic energy associated to K:

$$(\varepsilon_k)_K^n = \frac{1}{4\,|K|} \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}(K) \cap \mathcal{E}^{(i)}} |D_\sigma|\; h_{D_\sigma}^n (u_{i,\sigma}^n)^2$$

- Discrete kinetic energy flux associated to K, for $\sigma \in \mathcal{E}^{(1)}(K)$:

$$G_{K,\sigma}^n = \frac{1}{4} \sum_{\varepsilon \in \widetilde{\mathcal{E}}^{(i)}(D_\sigma)} \left((u_{i,\varepsilon'}^n)^2 (F_{\sigma,\varepsilon'}^n)^+ + (u_{i,\varepsilon}^n)^2 (F_{\sigma,\varepsilon}^n)^+ \right).$$

Discrete properties (5/6)

Discrete entropy balance:

$$\frac{|K|}{\delta t}\left[(\varepsilon_k)_K^{n+1}+(\varepsilon_p)_K^{n+1}-(\varepsilon_k)_K^{n}-(\varepsilon_p)_K^{n}\right]$$
$$+\sum_{i=1}^{2}\sum_{\sigma\in\mathcal{E}^{(i)}(K)}\left[G_{K,\sigma}^n+\frac{1}{2}g\,h_\sigma^n\,F_{K,\sigma}^n+\frac{1}{2}\,F_{K,\sigma}^n\,(z_K+z_L)\right]$$
$$+\sum_{i=1}^{2}\sum_{\sigma\in\mathcal{E}^{(i)}(K),\ \sigma=K|L}|\sigma|\,\frac{1}{2}\,(p_K^{n+1}+p_L^{n+1})\,u_{i,\sigma}^{n+1}\,\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{e}^{(i)}$$
$$=-(R_e)_K^{n+1},$$

Discrete properties 6/6

with

$$(R_e)_K^{n+1} \geq g \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma| \; u_{i,\sigma}^n \, h_\sigma^n (h_K^{n+1} - h_K^n) \, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}$$

$$+ g \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} \Big[\frac{1}{2} F_{K,\sigma}^n - \frac{1}{4} |\sigma| \; (h_K^{n+1} + h_L^{n+1}) \; u_{i,\sigma}^{n+1}\Big] \; (z_L - z_K) \, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}$$

$$+ \sum_{i=1}^{2} \sum_{\sigma \in \mathcal{E}^{(i)}(K)} |\sigma| \; \frac{1}{2} \, (p_K^{n+1} \, u_{i,\sigma}^{n+1} - p_K^n \, u_{i,\sigma}^n) \, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)},$$

Consistency analysis (1/3)

A weak solution to the shallow water equations satisfies, for any $\varphi \in C_c^\infty(\Omega \times [0,T])$ ($\varphi \in C_c^\infty(\Omega \times [0,T])^2$):

$$-\int_0^T \int_\Omega \Big[h \partial_t \varphi + h u \cdot \nabla \varphi \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega h_0(\boldsymbol{x}) \, \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} = 0,$$

$$-\int_0^T \int_\Omega \Big[h \boldsymbol{u} \cdot \partial_t \varphi + (h \boldsymbol{u} \otimes \boldsymbol{u}) : \varphi + \frac{1}{2} g h^2 \mathrm{div}(\varphi) + g h \nabla(z) \varphi \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega h_0(\boldsymbol{x}) \, \boldsymbol{u}_0(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} = 0.$$

A weak solution is entropic if for any nonnegative test functions $\varphi \in C_c^\infty(\Omega \times [0,T), \mathbb{R}_+)$:

$$-\int_0^T \int_\Omega \Big[\eta \, \partial_t \varphi + \Phi \cdot \nabla \varphi \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega \eta_0(\boldsymbol{x}) \, \varphi(\boldsymbol{x}, 0) \, \mathrm{d}\boldsymbol{x} \le 0.$$

Consistency analysis (2/3)

Let $(\mathcal{M}^{(m)}, \mathcal{E}^{(m)})_{m\in\mathbb{N}}$ be a sequence of meshes and let $(h^{(m)}\ \boldsymbol{u}^{(m)})_{m\in\mathbb{N}}$ be the associated sequence of solutions of the scheme.

Assumed estimates

$$1/C < (h^{(m)})_K^n \leq C, \quad 1/C' < 1/(h^{(m)})_K^n \leq C' \quad \forall K \in \mathcal{M}^{(m)}$$

$$|(\boldsymbol{u}^{(m)})_\sigma^n| \leq \bar{C}, \quad \forall \sigma \in \mathcal{E}^{(m)}$$

$$\eta_{\mathcal{M}^{(m)}} = \max\left\{\frac{|\sigma|}{|\tau|},\ \sigma \in (\mathcal{E}^{(m)})^{(1)},\ \tau \in (\mathcal{E}^{(m)})^{(2)}\right\} \leq \eta.$$

Consistency of the forward Euler scheme

If $(h^{(m)}, \boldsymbol{u}^{(m)})$ converges in $L^1(\Omega\times(0,T))\times L^1(\Omega\times(0,T))^2$ to $(\bar{h}, \bar{\boldsymbol{u}})$ when $m\to+\infty$ with $\delta t^{(m)}$ and $\delta_{\mathcal{M}^{(m)}} \to 0$, then $(\bar{h}, \bar{\boldsymbol{u}})$ satisfies the weak formulation of the shallow water equations.

Consistency analysis (3/3)

Additional assumptions

$$\frac{\delta t^{(m)}}{\delta_{\mathcal{M}^{(m)}}} \to 0 \text{ as } m \to +\infty.$$

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} |K| \, |(h^{(m)})_K^{n+1} - (h^{(m)})_K^n| \leq C.$$

Consistency of the entropy inequality

If $(h^{(m)}, \boldsymbol{u}^{(m)})$ converges in $L^1(\Omega \times (0,T)) \times L^1(\Omega \times (0,T))^2$ to $(\bar{h}, \bar{\boldsymbol{u}})$, then $(\bar{h}, \bar{\boldsymbol{u}})$ satisfies the weak entropy inequality.

Numerical results

The computational domain is $(0, L) \times (0, L)$ and the elevation of the support is:

$$z = -h_0 \left(1 - (x - \frac{L}{2})^2 - (y - \frac{L}{2})^2\right),$$

with $L = 4$ and $h_0 = 0.1$. The fluid height is given by

$$h = h_0 \; \max\Big(0, \; (x - \frac{L}{2}) \cos(\omega t) + (y - \frac{L}{2}) \sin(\omega t) - z - 0.5\Big),$$

and the velocity is

$$\boldsymbol{u} = \frac{1}{2} \omega \begin{bmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{bmatrix}.$$

Numerical results

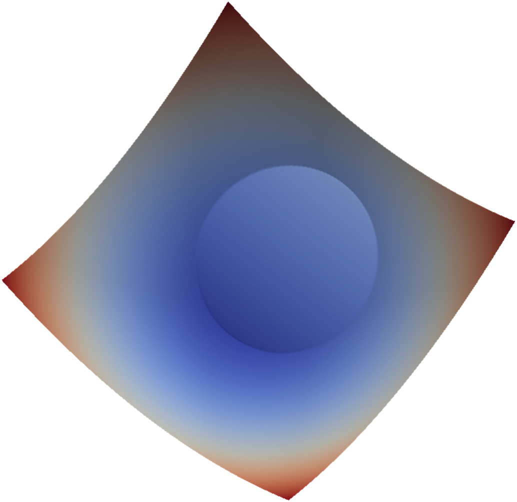


Figure: Rotation in a paraboloid: $t = 2\pi/w$ and $dt = dx/8$.

Numerical results

grid	error (discrete L^1 norm of h)
100×100	$3.02\,10^{-3}$
200×200	$1.54\,10^{-3}$
400×400	$0.896\,10^{-3}$
800×800	$0.511\,10^{-3}$

This yields an order of convergence between 0.8 and 1, which is consistent with a first-order approximation of the fluxes and the time derivative.

Thanks for your attention !