Numerical boundary conditions for transport equations

Nguyen Thi Hoai Thuong, Abraham Sylla, Sébastien Tran Tien
Supervisors: Benjamin Boutin and Jean-François Coulombel

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Transport IBVP:

$$\begin{cases} \partial_t u + a\partial_x u = 0 & t \geqslant 0 \quad x \in (0, L) \\ u(0, x) = u_0(x) & \quad\quad\quad\; x \in (0, L) \\ u(t, 0) = h(t) & t \geqslant 0 \end{cases}$$

Explicit solution:

$$u(t, x) = \begin{cases} u_0(x - at), & \text{if } x \geqslant at, \\ h\left(t - \dfrac{x}{a}\right), & \text{if } x \leqslant at, \end{cases}$$

which is in $H^{k+1}((0, L) \times (0, +\infty))$ provided that $u_0 \in H^{k+1}((0, L))$, $h \in H^{k+1}((0, +\infty))$ satisfy:

$$\forall m = 0, \dots, k, \qquad u_0^{(m)}(0) = (-a)^{-m} h^{(m)}(0).$$

Regular grid:

$$\Delta x := L/J, \qquad x_j := j\Delta x \qquad (j \in \mathbb{Z}), \qquad \Delta t := \lambda \Delta x$$

Two-time step finite difference scheme in the interior domain:

$$u_j^{n+1} = \sum_{\ell=-r}^{p} a_\ell u_{j+\ell}^n, \qquad n \in \mathbb{N}, \qquad j = 1, \dots, J.$$

Assumption

For some integer $k \geqslant 1$, there holds:

$$\forall m = 0 \dots k, \qquad \sum_{\ell=-r}^{p} \ell^m a_\ell = (-\lambda a)^m, \qquad (\textit{consistency of order } k)$$

$$\sup_{\theta \in [0,2\pi]} \left| \sum_{l=-r}^{p} a_\ell e^{i\ell\theta} \right| \leqslant 1, \qquad (\ell^2 - \textit{stability}).$$

Examples

- **Lax-Wendroff scheme** $(r = 1,\ p = 1)$.

$$u_j^{n+1} = \frac{\nu(\nu+1)}{2} u_{j-1}^n + \left(1-\nu^2\right) u_j^n + \frac{\nu(\nu-1)}{2} u_{j+1}^n;$$

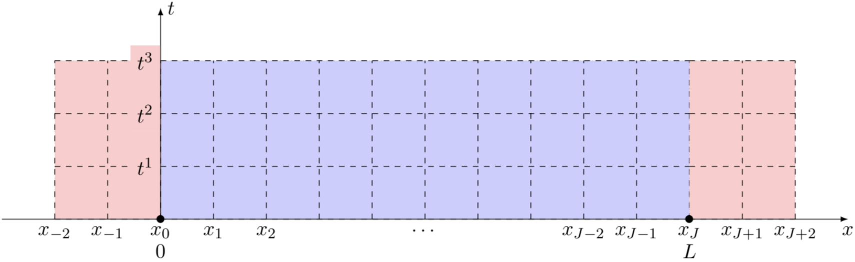
- **O3 scheme** $(r = 2,\ p = 1)$.

$$u_j^{n+1} = -\frac{\nu(\nu+1)(1-\nu)}{6} u_{j-2}^n + \frac{\nu(\nu+1)(2-\nu)}{2} u_{j-1}^n + \frac{(\nu+1)(1-\nu)(2-\nu)}{2} u_j^n - \frac{\nu(1-\nu)(2-\nu)}{6} u_{j+1}^n,$$

where

$$\nu = a\lambda.$$

What values should we prescribe for u_j^n, for $1 - r \leqslant j \leqslant 0$ and $J + 1 \leqslant j \leqslant J + p$?



- Inflow boundary: Taylor expansion
- Outflow boundary: Neumann condition of order $k_b \geqslant 0$

Theorem

The solution $(u_j^n)_{1\leqslant j\leqslant J, 0\leqslant n\leqslant N}$ *to the scheme along with suitable numerical boundary conditions and initial datum satisfies:*

$$\sup_{0\leqslant n\leqslant N}\left(\sum_{j=1}^{J}\left|u_j^n-\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j}u(t^n,y)dy\right|^2\Delta x\right)^{1/2}$$
$$\leqslant C\Delta x^{\min(k,k_b)}\left(\|u_0\|_{H^{k+1}((0,L))}+\|h\|_{H^{k+1}((0,+\infty))}\right)$$

and

$$\sup_{0\leqslant n\leqslant N}\sup_{1\leqslant j\leqslant J}\left|u_j^n-\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j}u(t^n,y)dy\right|$$
$$\leqslant C\Delta x^{\min(k,k_b)-1/2}\left(\|u_0\|_{H^{k+1}((0,L))}+\|h\|_{H^{k+1}((0,+\infty))}\right)$$

- Numerical condition at the outflow boundary ($1 \leqslant \ell \leqslant p$):

$$(D_-^{k_b} u^n)_{J+\ell} = 0, \qquad \text{where} \qquad (D_- v)_j := v_j - v_{j-1}$$

- Using an energy method we obtain the following stability estimate:

Proposition

The solution $(u_j^n)_{j \leqslant J, 0 \leqslant n \leqslant N}$ *to:*

$$\begin{cases} u_j^0 = f_j & j \leqslant J & \\ (D_-^{k_b} u^n)_{J+\ell} = g_{J+\ell}^n & 1 \leqslant \ell \leqslant p & 0 \leqslant n \leqslant N-1 \\ u_j^{n+1} = \sum_{l=-r}^{p} a_l u_{j+l}^n + \Delta t F_j^{n+1} & j \leqslant J & 0 \leqslant n \leqslant N-1 \end{cases}$$

satisfies:

$$\sup_{0 \leqslant n \leqslant N} \sum_{j \leqslant J} \Delta x (u_j^n)^2 \leqslant C \Big[\sum_{j \leqslant J} \Delta x (f_j)^2$$

$$+ (N\Delta t)^2 \sup_{1 \leqslant n \leqslant N} \sum_{j \leqslant J} \Delta x (F_j^n)^2 + \sum_{n=0}^{N-1} \Delta t \sum_{\ell=1}^{p} (g_{J+\ell}^n)^2 \Big]$$

Theorem

The solution $(u_j^n)_{j\leqslant J,\,0\leqslant n\leqslant N}$ *to*

$$\begin{cases} u_j^0 = \dfrac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u_0(y)dy & j\leqslant J \\ (D_-^{k_b}u^n)_{J+\ell} = 0 & 1\leqslant \ell\leqslant p \quad 0\leqslant n\leqslant N-1 \\ u_j^{n+1} = \sum_{l=-r}^{p} a_l u_{j+l}^n & j\leqslant J \quad 0\leqslant n\leqslant N-1 \end{cases}$$

satisfies, with $k_0 := min\ (k, k_b)$

$$\sup_{0\leqslant n\leqslant N}\left(\sum_{j\leqslant J}\left|u_j^n - \frac{1}{\Delta x}\int_{x_j-1}^{x_j} u(t^n,y)dy\right|^2\Delta x\right)^{1/2} \leqslant C\Delta x^{k_0}\|u_0\|_{H^{k_0+1}((-\infty,L))}$$

- From a Taylor expansion:

$$\int_{x_{\ell-1}}^{x_\ell} h\left(t^n - \frac{y}{a}\right) dy = \int_{x_{\ell-1}}^{x_\ell} \sum_{s=0}^{k-1} \left(-\frac{y}{a}\right)^s \frac{h^{(s)}(t^n)}{s!} dy + O(\Delta x^{k+1})$$

$$= \sum_{s=0}^{k-1} \frac{(\ell\Delta x)^{s+1} - ((\ell-1)\Delta x)^{s+1}}{(-a)^s (s+1)!} h^{(s)}(t^n) + O(\Delta x^{k+1})$$

- We derive the numerical condition at the inflow boundary:

$$u_\ell^n = \sum_{s=0}^{k-1} \frac{\ell^{s+1} - (\ell-1)^{s+1}}{(-a)^s (s+1)!} \Delta x^s h^{(s)}(t^n), \qquad 1 - r \leqslant \ell \leqslant 0$$

for which we get the following convergence estimate:

Theorem

The solution $(u_j^n)_{j\geqslant 1,\, 0\leqslant n\leqslant N}$ *to the scheme*

$$\begin{cases} u_j^0 = \dfrac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u_0(y)dy & j\geqslant 1 \\ u_\ell^n = \displaystyle\sum_{s=0}^{k-1} \frac{\ell^{s+1}-(\ell-1)^{s+1}}{(-a)^s(s+1)!}\Delta x^s h^{(s)}(t^n) & 1-r\leqslant \ell\leqslant 0 \quad 0\leqslant n\leqslant N-1 \\ u_j^{n+1} = \displaystyle\sum_{l=-r}^{p} a_l u_{j+l}^n & j\geqslant 1 \quad 0\leqslant n\leqslant N-1 \end{cases}$$

satisfies

$$\sup_{0\leqslant n\leqslant N}\left(\sum_{j\geqslant 1}\left|u_j^n-\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(t^n,y)dy\right|^2\Delta x\right)^{1/2} \leqslant C\Delta x^k\left(\|u_0\|_{H^{k+1}((0,+\infty))}+\|h\|_{H^{k+1}((0,+\infty))}\right)$$

Superposition argument along with the previously obtained estimates to yield:

Theorem

The solution $(u_j^n)_{1\leqslant j\leqslant J,\,0\leqslant n\leqslant N}$ *to the scheme*

$$\begin{cases} u_j^0 = \dfrac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u_0(y)dy & 1\leqslant j\leqslant J & \\ (D_-^{k_b}u^n)_{J+\ell} = 0 & 1\leqslant \ell\leqslant p & 0\leqslant n\leqslant N-1 \\ u_\ell^n = \displaystyle\sum_{s=0}^{k-1}\frac{\ell^{s+1}-(\ell-1)^{s+1}}{(-a)^s(s+1)!}\Delta x^s h^{(s)}(t^n) & 1-r\leqslant \ell\leqslant 0 & 0\leqslant n\leqslant N-1 \\ u_j^{n+1} = \displaystyle\sum_{l=-r}^{p} a_l u_{j+l}^n & 1\leqslant j\leqslant J & 0\leqslant n\leqslant N-1 \end{cases}$$

satisfies, with $k_0 = min(k, k_b)$*:*

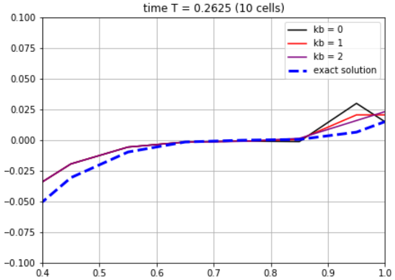
$$\sup_{0\leqslant n\leqslant N}\left(\sum_{j=1}^{J}\left|u_j^n - \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u(t^n,y)dy\right|^2\Delta x\right)^{1/2}$$

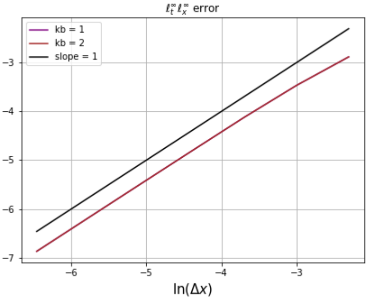
$$\leqslant C\Delta x^{k_0}\left(\|u_0\|_{H^{k+1}((0,L))} + \|h\|_{H^{k+1}((0,+\infty))}\right)$$

Naive

Naive treatment of the left boundary in the Lax-Wendroff scheme:

$$u_0^n = h(t^n), \quad n \in \mathbf{N}.$$

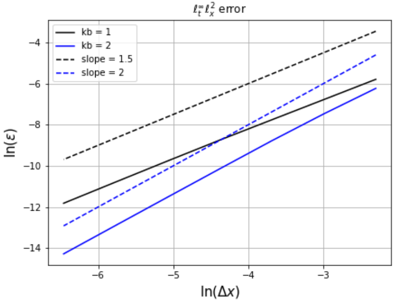


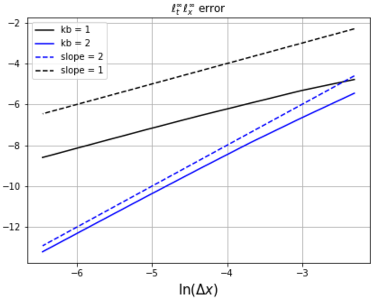


Smarter choice

Taylor expansion in the Lax-Wendroff scheme:

$$u_0^n = h(t^n) + \frac{\Delta x}{2a} h'(t^n), \quad n \in \mathbf{N}.$$

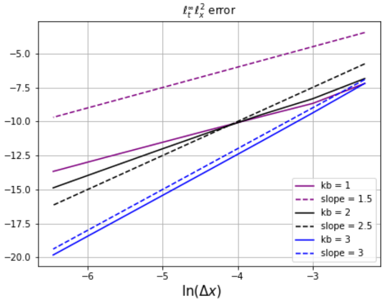


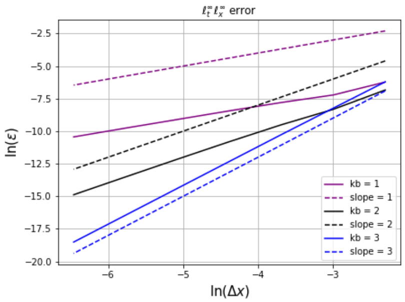


With the O3 scheme

Taylor expansion in the O3 scheme:

$$\begin{cases} u_0^n = h(t^n) + \frac{1}{2}\left(\frac{\Delta x}{a}\right) h'(t^n) + \frac{1}{6}\left(\frac{\Delta x}{a}\right)^2 h^{(2)}(t^n), \\ u_{-1}^n = h(t^n) + \frac{3}{2}\left(\frac{\Delta x}{a}\right) h'(t^n) + \frac{7}{6}\left(\frac{\Delta x}{a}\right)^2 h^{(2)}(t^n) \end{cases}$$





- How to explain the order $\min(k, k_b + 1/2)$ in $\ell_t^\infty \ell_x^2$ norm?
- Denote $(\rho_j^{(1)})_{j \leqslant J+p}, \dots, (\rho_j^{(p)})_{j \leqslant J+p}$ the p steady states of the scheme belonging to ℓ^2 and let:

$$w_j^n := u_j^n - u_{\text{exact},j}^n + \sum_{i=1}^{p} z_i^n \cdot \rho_j^{(i)}$$

where $(z_1^n)_n, \dots, (z_p^n)_n$ are to be found so that:

$$\forall 0 \leqslant n \leqslant N, \, \forall 1 \leqslant \ell \leqslant p, \qquad (D_-^{k_b} w^n)_{J+\ell} = 0$$

- The boundary layer term $\sum_{i=1}^{p} z_i^n \cdot \rho_j^{(i)}$ is of size $\Delta x^{k_b + 1/2}$.
- Writing $\|u_j^n - u_{\text{exact},j}^n\| \leqslant \|w_j^n\| + \|\sum_{i=1}^{p} z_i^n \cdot \rho_j^{(i)}\|$ we obtain the estimate:

Proposition

The solution $(u_j^n)_{j\leqslant J, 0\leqslant n\leqslant N}$ *to:*

$$\begin{cases} u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} u_0(y)dy & j\leqslant J \\ (D_-^{k_b}u^n)_{J+\ell} = 0 & 1\leqslant \ell\leqslant p \quad 0\leqslant n\leqslant N-1 \\ u_j^{n+1} = \sum_{l=-r}^{p} a_l u_{j+l}^n & j\leqslant J \quad 0\leqslant n\leqslant N-1 \end{cases}$$

satisfies:

$$\sup_{0\leqslant n\leqslant N}\left(\sum_{j\leqslant J}\left|u_j^n - \frac{1}{\Delta x}\int_{x_j-1}^{x_j} u(t^n,y)dy\right|^2 \Delta x\right)^{1/2} \leqslant C\left[\Delta x^k \|u_0\|_{H^{k+1}} + \Delta x^{k_b+1/2}\|u_0\|_{H^{k_b+2}}\right]$$

Conclusion: Development of a systematic construction and stability analysis for high order discretization of linear transport equations with boundary conditions.
Possible perspectives:

- In the case $k = k_b$, how to prove that the convergence order for the $\ell^\infty_t \ell^\infty_x$ norm is k?
- Tackle the nonlinear case

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & t \geqslant 0 & x \in (0, L) \\ u(0, x) = u_0(x) & & x \in (0, L) \\ u(t, 0) = h(t) & t \geqslant 0 & \end{cases}$$

- Tackle the multi-dimensional case

$$\begin{cases} \partial_t u + \vec{a} \cdot \nabla u = 0 & t \geqslant 0 & x \in \Omega \subset \mathbf{R}^n \\ u(0, x) = u_0(x) & & x \in \Omega \\ u(t, \sigma) = h(t, \sigma) & t \geqslant 0 & \sigma \in \partial\Omega. \end{cases}$$

References

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- F. Filbet, C. Yang. An inverse Lax-Wendroff method for boundary conditions applied to Boltzmann type models (2013)
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