Positive and entropic schemes by relaxation techniques

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Introduction

System of conservation laws

$$\partial_t U + \partial_x F(U) = 0.$$

Goal :

- Design robust and accurate scheme

Difficulties :

- Preserves positivy of densities, temperatures, ...
- Compute admissible shocks
- Compromise between accuracy and stability

Godunov scheme

Godunov scheme = Finite volume scheme

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{\Delta x}\left(F(U_i, U_{i+1}) - F(U_{i-1}, U_i)\right) = 0.$$

Numerical flux $F(U_i, U_{i+1})$
given via the solution of the Riemann pb with initial data U_i,U_{i+1}.

$F(U, U) = F(U)$ consistant to first order and conservative.

Godunov scheme

Entropy inequality

$$\partial_t \eta(U) + \partial_x G(U) \leq 0.$$

Discrete entropy inequality

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{\Delta x}\left(G(U_i, U_{i+1}) - G(U_{i-1}, U_i)\right) \leq 0.$$

Interests :

- Convergence proof for scalar and 2×2 systems
- A priori bounds
- Comupted shocks are physically relevant

Godunov scheme

Drawbacks of the exact solution :

- complicated to solve
- contains details which will be averaged

Self similar function $R(\xi, U_l, U_r)$ is an Approximate Riemann solver.

$$F(U_l) - \int_{-\infty}^{0} \left(R(\xi, U_l, U_r) - U_l\right) d\xi = F(U_r) + \int_{0}^{\infty} \left(R(\xi, U_l, U_r) - U_r\right) d\xi$$

Example : HLL solver

$$R_{\mathrm{HLL}}(\xi, U_l, U_r) = \begin{cases} U_l, & \xi < \sigma_1 \\ \dfrac{\sigma_2 U_r - \sigma_1 U_l - F(U_r) + F(U_l)}{\sigma_2 - \sigma_1}, & \sigma_1 < \xi < \sigma_2 \\ U_r, & \sigma_2 < \xi \end{cases}$$

Stability conditions

Signal speeds Choice of σ_1 and σ_2 is crucial for stability.

Subcharacteristic conditions

Remark : if the signal speeds are too large, the scheme will be too diffusive.

Drawback of HLL scheme

- 2-wave solver, too diffusive on contact discontinuities.

Goal : Design 3-wave or more solver with the following properties.

- preserves positivy,
- satisfies entropy inequality,
- is sharp across contact discontinuities.

Suliciu solver

Compressible Euler system
Pressure term $p(\rho, \varepsilon)$.

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = 0,$$
$$\partial_t E + \partial_x \left(u\left(E + p\right)\right) = 0.$$

Equation on ρp

$$\partial_t \rho p + \partial_x(\rho p\, u) + \rho^2 p'(\rho)\, \partial_x u = 0,$$

Complex Riemann problem

$$u - \sqrt{p'(\rho)} \quad ; \quad u + \sqrt{p'(\rho)}$$

Suliciu relaxation system
New variable π

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t \rho u + \partial_x \left(\rho u^2 + \pi\right) = 0,$$
$$\partial_t E + \partial_x \left(u\left(E + \pi\right)\right) = 0.$$

Definition of π

$$\partial_t \rho\pi + \partial_x(\rho p\, u) + c^2\, \partial_x u = 0,$$

Easy to solve Riemann problem

$$u - \sqrt{\rho^{-2}\, a^2};\, u; \quad u + \sqrt{\rho^{-2}\, a^2}$$

Suliciu solver

Relaxation system

$$RHS = \frac{p - \pi}{\varepsilon}$$

Transport-projection between t_n and t_{n+1} :

- Initialize $\pi_l = p(\rho_l, \varepsilon_l)$, $\pi_r = p(\rho_r, \varepsilon_r)$.
- Solve homogenenous relaxation system $V = (\rho, \rho u, E, \pi)$.
- Keep variables U=$(\rho, \rho u, E)$.

Properties of the scheme [Bouchut, 2004]

- Solver is exact on contact discontinuities,
- Positive and entropic under the subcharacteristic condition

$$\rho^2 p'(\rho, \varepsilon) \leq a^2$$

Argument of stability proof

How to prove characteristic conditions are sufficient stability conditions

- For smooth solutions, by Chapman-Enskog expansion
- For discontinuous solutions, by Entropy extension

Relaxation framework

$$\partial_t f + \partial_x \mathcal{A}(f) = \frac{Q(f)}{\varepsilon},$$

with equilibrium $f = M(U)$.
Relaxation system admits an entropy extension if there exists $(\mathcal{H}, G)$ satisfying

- Consistency

$$\mathcal{H}(M(U)) = \eta(U)$$

$$\mathcal{G}(M(U)) = G(U)$$

- Minimization principle [Chen, Levermore, Liu, 1994]

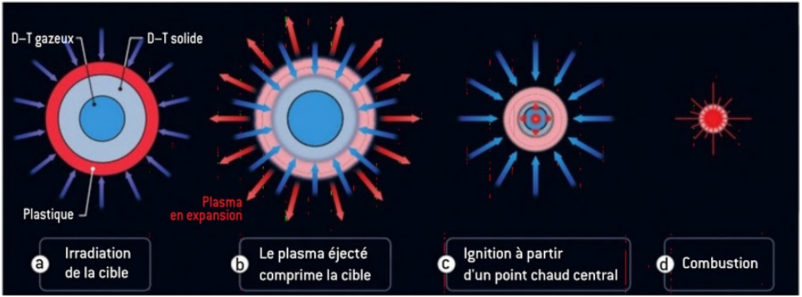
$$\mathcal{H}(M(U)) \leq \mathcal{H}(f), \quad \forall U = Lf$$

Bibliography

- 1994, [Chen, Levermore, Liu]
- 1999, 2004, [Bouchut]
- 2011, [Bouchut, Klingenberg, Waagan]
- 2012, 2015, [Berthon, Dubroca, Sangam]
- 2013, [Bouchut, Boyaval]
- 2016 [Bouchut, L]

Application

Application : Inertial confinement fusion.



Bitemperature Euler system with transverse magnetic field

Nonconservative hyperbolic system

$$\begin{cases} \partial_t \rho & +\partial_x(\rho u_1) = 0, \\ \partial_t(\rho u_1) & +\partial_x(\rho u_1^2 + p_e + p_i + B_3^2/2) = 0, \\ \partial_t(\rho u_2) & +\partial_x(\rho u_1 u_2) = 0, \\ \partial_t B_3 & +\partial_x(u_1 B_3) = 0, \\ \partial_t \overline{\mathcal{E}_e} & +\partial_x(u_1(\overline{\mathcal{E}_e} + p_e + c_e B_3^2/2)) - u_1(c_i \partial_x p_e - c_e \partial_x p_i) = S_{ei}, \\ \partial_t \overline{\mathcal{E}_i} & +\partial_x(u_1(\overline{\mathcal{E}_i} + p_i + c_i B_3^2/2)) \ \ + u_1(c_i \partial_x p_e - c_e \partial_x p_i) = -S_{ei}, \end{cases}$$

Two pression laws and two temperatures :

$$p_\alpha = (\gamma_\alpha - 1)\rho_\alpha \varepsilon_\alpha = n_\alpha k_B T_\alpha, \quad \alpha = e, i.$$

Result : This model has been obtained as the hydrodynamic limit of an underlying conservative kinetic model.

Suliciu relaxation for the nonconservative bitemperature MHD system

Bitemperature MHD system

$$\partial_t \rho + u_1 \partial_x \rho + \rho \partial_x u_1 = 0,$$
$$\partial_t u_1 + u_1 \partial_x u_1 + \rho^{-1} \partial_x (p_e + p_i + B_3^2/2) = 0,$$
$$\partial_t u_2 + u_1 \partial_x u_2 = 0,$$
$$\partial_t B_3 + B_3 \partial_x u_1 + u_1 \partial_x B_3 = 0,$$
$$\partial_t \varepsilon_e + u_1 \partial_x \varepsilon_e + \rho_e^{-1} p_e \partial_x u_1 = 0,$$
$$\partial_t \varepsilon_i + u_1 \partial_x \varepsilon_i + \rho_i^{-1} p_i \partial_x u_1 = 0.$$

Equations on p_e and p_i

$$\partial_t p_e + u_1 \partial_x p_e + \gamma_e p_e \partial_x u_1 = 0,$$
$$\partial_t p_i + u_1 \partial_x p_i + \gamma_i p_i \partial_x u_1 = 0,$$

Complicated to solve Riemann problem

$$u;\, u \pm \sqrt{\rho^{-2}(\gamma_e \rho p_e + \gamma_i \rho p_i + \rho B_3^2)}$$

Relaxation system

New nariables π_e and π_i

$$\partial_t \rho + u_1 \partial_x \rho + \rho \partial_x u_1 = 0,$$
$$\partial_t u_1 + u_1 \partial_x u_1 + \rho^{-1} \partial_x (\pi_e + \pi_i + B_3^2/2) = 0,$$
$$\partial_t u_2 + u_1 \partial_x u_2 = 0,$$
$$\partial_t B_3 + B_3 \partial_x u_1 + u_1 \partial_x B_3 = 0,$$
$$\partial_t \varepsilon_e + u_1 \partial_x \varepsilon_e + \rho_e^{-1} \pi_e \partial_x u_1 = 0,$$
$$\partial_t \varepsilon_i + u_1 \partial_x \varepsilon_i + \rho_i^{-1} \pi_i \partial_x u_1 = 0,$$

Definition of π_e and π_i

$$\partial_t \pi_e + u_1 \partial_x \pi_e + \frac{c_e}{\rho}(a^2 - \rho B_3^2) \partial_x u_1 = 0,$$
$$\partial_t \pi_i + u_1 \partial_x \pi_i + \frac{c_i}{\rho}(a^2 - \rho B_3^2) \partial_x u_1 = 0.$$

Easy to solve system

$$u;\, u \pm \sqrt{\rho^{-2}\, a^2}$$

Resolution of Riemann pb

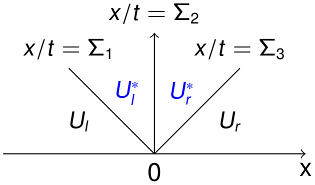
$$V = (\rho, u_1, \varepsilon_e, \varepsilon_i, B_3, \pi_e, \pi_i) \in \mathbb{R}^7$$

Eigenvalues	Mult.	Riemann Invariants
$\lambda_1 = u_l - \dfrac{a_l}{\rho_l}$	1	$B_3/\rho, \quad w_{1,e}, \quad w_{1,i}, \quad w_{2,e}, \quad w_{2,i}$
$\lambda_2 = u_l^* = u_r^r$	5	$\pi_e + \pi_i + B_3^2/2$
$\lambda_3 = u_r + \dfrac{a_r}{\rho_r}$	1	$B_3/\rho, \quad w_{1,e}, \quad w_{1,i}, \quad w_{2,e}, \quad w_{2,i}$

with

$$w_{1,\alpha} = \pi_\alpha + c_\alpha B_3^2/2 + \frac{a^2 c_\alpha}{\rho}, \quad w_{2,\alpha} = \varepsilon_\alpha + \frac{B_3^2}{2\rho} - \frac{\left(\pi_\alpha + c_\alpha B_3^2/2\right)^2}{2(c_\alpha\, a)^2}.$$

Explicit resolution of intermediate states.



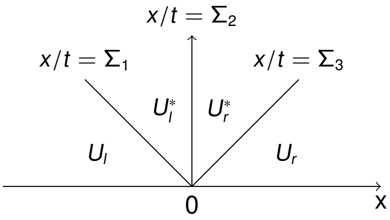
Nonconservative product

We notice that

$$c_i w_1 - c_e w_2 = c_i \pi_e - c_e \pi_i$$

Thus $c_i \pi_e - c_e \pi_i$ is a Riemann invariant for extremes discontinuities.

Thus the nonconservative product $u\,\partial_x \left(c_i \pi_e - c_e \pi_i\right)$ is well defined



Accuracy and stability

The Suliciu solver exactly resolved contact discontinuities.

Under the following subcharacteristic condition on relaxation speed a

$$a^2 \geq \rho B_3^2 + \rho \max(a_e^2, a_i^2), \quad a_\alpha = \sqrt{\frac{\gamma_\alpha p_\alpha}{\rho_\alpha}},$$

- It preserves positivity of densities and temperatures,
- It satisfies a discrete entropy inequality.

Implicit proof : monoT Euler case

We use Riemann invariants

$$\varphi(\tau,\varepsilon,\pi) = \pi + a^2\tau$$
$$\phi(\tau,\varepsilon,\pi) = \varepsilon - \frac{\pi^2}{2a^2}$$

Under subcharcateristic condition, we can define the following change of variable

$$\Theta(\tau,\varepsilon) = \begin{pmatrix} \varphi(\tau,\varepsilon,p(\tau,\varepsilon)) \\ \phi(\tau,\varepsilon,p(\tau,\varepsilon)) \end{pmatrix}, \quad \Theta^{-1}(X,Y) = \begin{pmatrix} \bar{\tau}(X,Y) \\ \bar{\varepsilon}(X,Y) \end{pmatrix}$$

We define the l'extended entropy $\mathcal{S}$ from the initial entropy s :

$$\mathcal{S}(\tau,\varepsilon,\pi) = s\left(\ \bar{\tau}\left(\varphi(\tau,\varepsilon,\pi),\phi(\tau,\varepsilon,\pi)\right),\ \bar{\varepsilon}\left(\varphi(\tau,\varepsilon,\pi),\phi(\tau,\varepsilon,\pi)\ \right)\right.$$

At equilibrium,

$$\begin{aligned}\mathcal{S}(\tau,\varepsilon,p(\tau,\varepsilon)) &= s\left(\bar{\tau}\left(\Theta(\tau,\varepsilon)\right),\bar{\varepsilon}\left(\Theta(\tau,\varepsilon)\right)\right) \\ &= s\left(\Theta^{-1}\circ\Theta(\tau,\varepsilon)\right) \\ &= s(\tau,\varepsilon)\end{aligned}$$

Moreover, we can prove that $\pi = p$ is the unique maximum de $\mathcal{S}$.

Implicit proof : bitemperature MHD sytem

Cas monotempérature

$$\varphi(\tau, \varepsilon, \pi) = \pi + a^2\tau$$

$$\phi(\tau, \varepsilon, \pi) = \varepsilon - \frac{\pi^2}{2a^2}$$

Sous condition sous caractéristique, on peut définir le changement de variable

$$\Theta(\tau, \varepsilon) = \begin{pmatrix} \varphi(\tau, \varepsilon, p) \\ \phi(\tau, \varepsilon, p) \end{pmatrix}$$

$$\Theta^{-1}(X, Y) = \begin{pmatrix} \bar{\tau}(X, Y) \\ \bar{\varepsilon}(X, Y) \end{pmatrix}$$

Résultat connu : la fonction $\mathcal{S} : \Sigma \mapsto \mathcal{S}(\Sigma)$ définie par

$$\mathcal{S}(\Sigma) \quad = \quad s\left(\bar{\tau}\left(\varphi(\Sigma), \phi(\Sigma)\right), \bar{\varepsilon}\left(\varphi(\Sigma), \phi(\Sigma)\right)\right)$$

avec $\Sigma = (\tau, \varepsilon, \pi)$ est une entropie étendue.

Cas bitempérature TM

$$\varphi_e(\tau, \varepsilon_e, B_3, \pi) = \pi + c_e B_3^2/2 + a^2 c_e \tau$$

$$\phi_e(\tau, \varepsilon_e, B_3, \pi) = \varepsilon_e + \tau B_3^2/2 - \frac{(\pi + c_e B_3^2/2)^2}{2(c_e a)^2}$$

$$\psi_e(\tau, \varepsilon_e, B_3, \pi) = \tau B_3$$

Sous condition sous caractéristique, on peut définir le changement de variable

$$\Theta(\tau, \varepsilon, B_3) = \begin{pmatrix} \varphi_e(\tau, \varepsilon, B_3, p) \\ \phi_e(\tau, \varepsilon, B_3, p) \\ \psi_e(\tau, \varepsilon, B_3, p) \end{pmatrix}$$

$$\Theta^{-1}(X, Y, Z) = \begin{pmatrix} \bar{\tau}(X, Y, Z) \\ \bar{\varepsilon}(X, Y, Z) \\ \bar{B}_3(X, Y, Z) \end{pmatrix}$$

Nouveau résultat : la fonction $\mathcal{S} : \Sigma \mapsto \mathcal{S}(\Sigma)$ définie par

$$\mathcal{S}(\Sigma) = s_e\left(\bar{\tau}\left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)\right), \bar{\varepsilon}_\alpha\left(\phi(\Sigma), \varphi(\Sigma), \psi(\Sigma)\right)\right),$$

avec $\Sigma = (\tau, \varepsilon, B_3, \pi)$ est une entropie étendue.

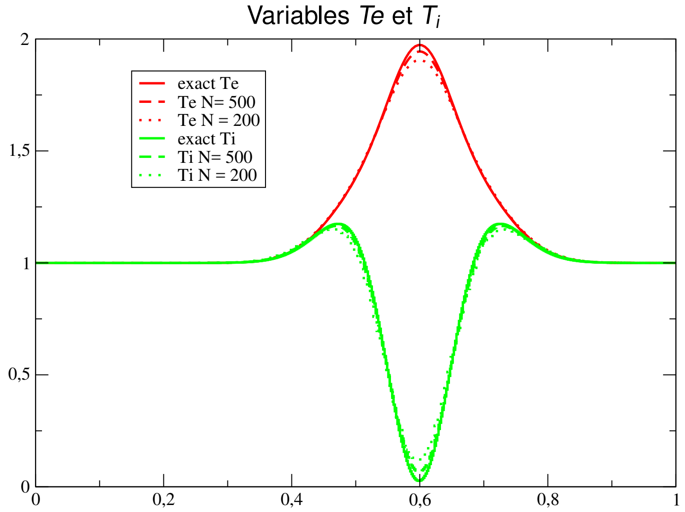
Numerical results

Test with smooth analytical solution

Initial conditions

$$\begin{aligned}
\rho(x,0) &= 1,\\
u_1(x,0) &= 10,\\
T_e(x,0) &= 1 + \exp(-200(x-1/2)^2),\\
T_i(x,0) &= 2 - T_e(x,0),\\
B_3(x,0) &= \exp(-50(x-1/2)^2).
\end{aligned}$$

Test with smooth analytical solution



Riemann problems

Five tests about accuracy and robustness

Through rarefaction wave, contact discontinuity and shocks

Test/Variables	ρ	u	B_3	T_e	T_i
Test 1 left	1	0.75	0.8164966	0.3336667	0.3336667
Test 1 right	0.125	0	0.2581989	0.2669333	0.2669333
Test 2 left	1	-2	0.5163978	0.1334667	0.1334667
Test 2 right	1	2	0.5163978	0.1334667	0.1334667
Test 3 left	1	0	14.142136	100.10000	100.1
Test 3 right	1	0	0.2581989	0.0333667	0.0333667
Test 4 left	5.9999924	19.5975	17.528909	25.630859	25.630860
Test 4 right	5.9999242	-6.19633	5.5434646	2.5634264	2.5634266
Test 5 left	1	-19.5975	8.1649658	33.366665	33.366667
Test 5 right	1	-19.5975	0.2581989	0.0333667	0.0333667

Riemann problems

Two tests about accuracy on isolated contact discontinuities

Test 6 left	1.4	0	0.8164966	0.2383333	0.2383333
Test 6 right	1	0	0.8164966	0.3336667	0.3336667
Test 7 left	1.4	0.1	0.8164966	0.2383333	0.2383333
Test 7 right	1	0.1	0.8164966	0.3336667	0.3336667

Référence : E. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics : A Practical Introduction*, 1997.

Comparison with an nonconservative HLL scheme

NC term : $\boxed{-u}\partial_x\phi$

$$F_l^{\overline{\mathcal{E}_e}}(U_l, U_r) = F_{\text{HLL}}^{\overline{\mathcal{E}_e}}(U_l, U_r)\boxed{-u_l}(\phi_r - \phi_l)$$

$$F_r^{\overline{\mathcal{E}_e}}(U_l, U_r) = F_{\text{HLL}}^{\overline{\mathcal{E}_e}}(U_l, U_r)\boxed{-u_r}(\phi_r - \phi_l)$$

Terme NC : $\boxed{+u}\partial_x\phi$

$$F_l^{\overline{\mathcal{E}_i}}(U_l, U_r) = F_{\text{HLL}}^{\overline{\mathcal{E}_i}}(U_l, U_r)\boxed{+u_l}(\phi_r - \phi_l)$$

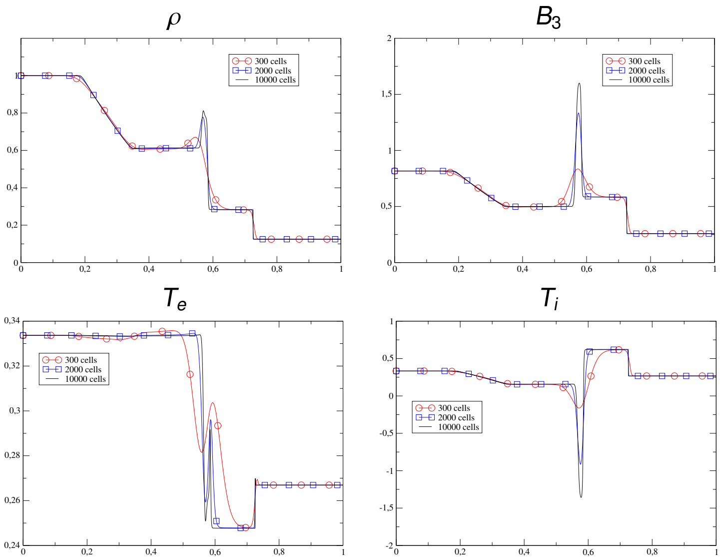
$$F_r^{\overline{\mathcal{E}_i}}(U_l, U_r) = F_{\text{HLL}}^{\overline{\mathcal{E}_i}}(U_l, U_r)\boxed{+u_r}(\phi_r - \phi_l)\,.$$

Numerical results

With ncHLL, test cases 2 and 5 show instabilities which ruin the simulation and fail to give a result.

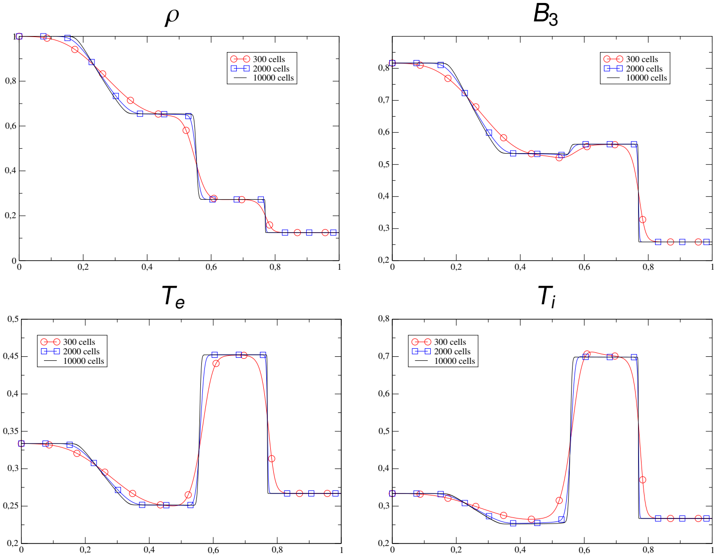
	ncHLL	Suliciu
test case 1	✓	✓
test case 2	✗	✓
test case 3	✓	✓
test case 4	✓	✓
test case 5	✗	✓
test case 6	✓	✓
test case 7	✓	✓

Test 1 - HLL nonconservatif



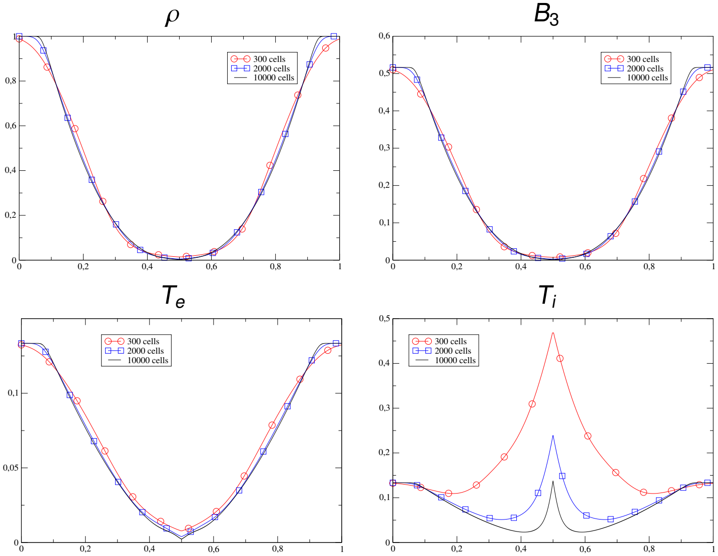
Parameter $x_0 = 0.3$ et final time $t = 0.2$.

Test 1 - Suliciu



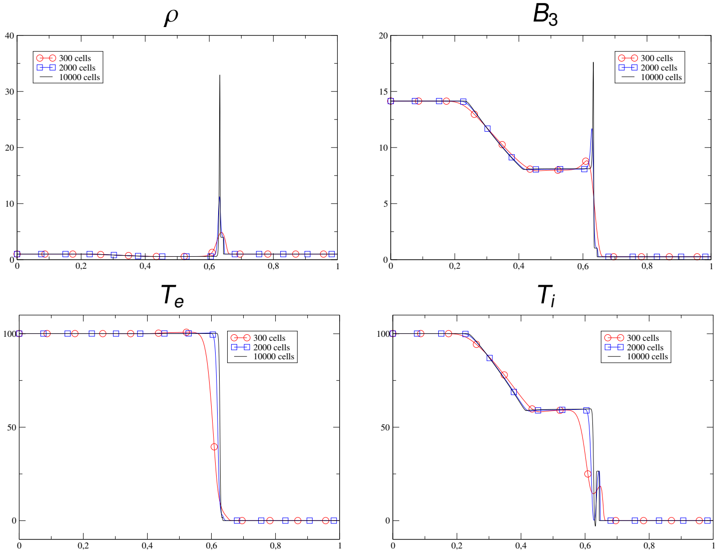
Parameter $x_0 = 0.3$ et final time $t = 0.2$.

Test 2 - Suliciu



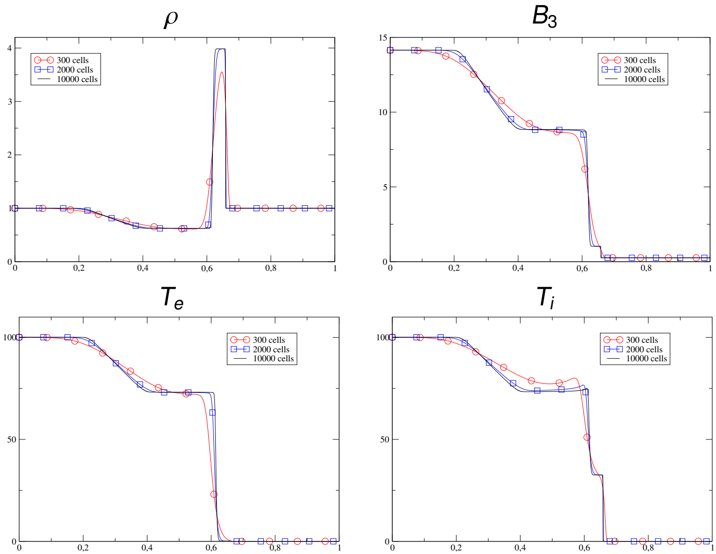
Parameter $x_0 = 0.5$ et final time $t = 0.15$.

Test 3 - HLL nonconservatif



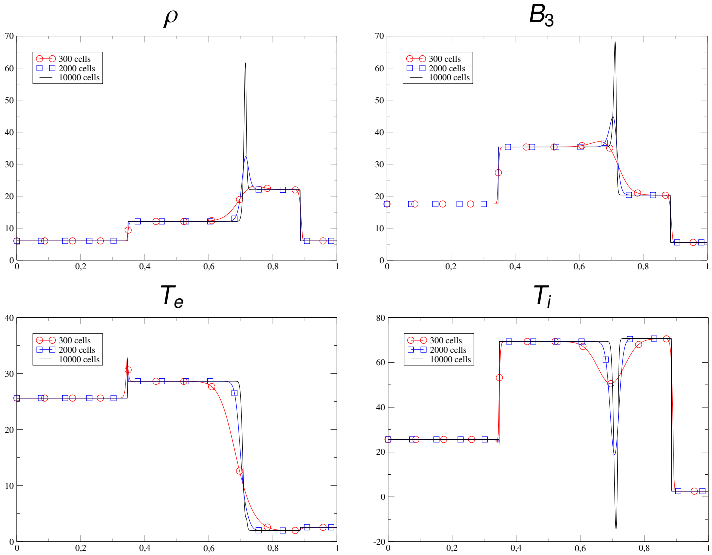
Parameter $x_0 = 0.5$ et final time $t = 0.012$.

Test 3 - Suliciu



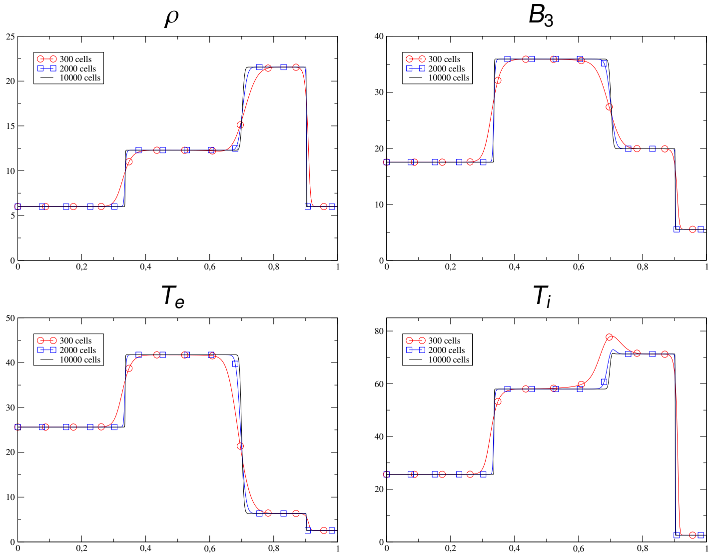
Parameter $x_0 = 0.5$ et final time $t = 0.012$.

Test 4 - HLL nonconservatif



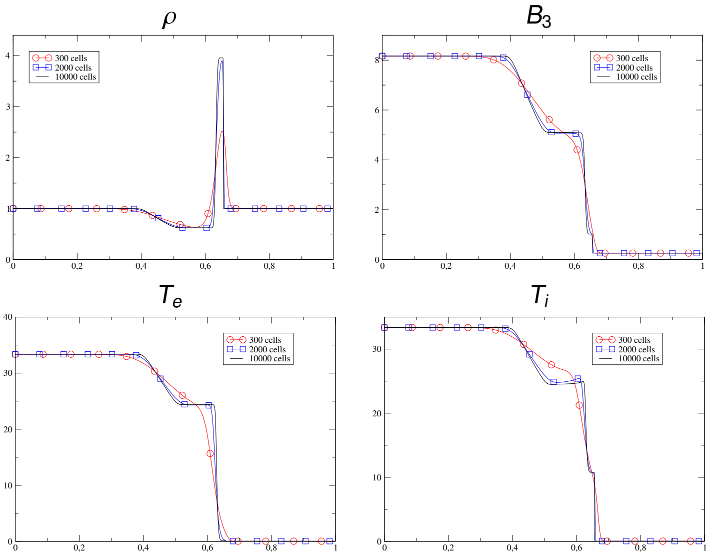
Parameter $x_0 = 0.4$ et final time $t = 0.035$.

Test 4 - Suliciu



Parameter $x_0 = 0.4$ et final time $t = 0.035$.

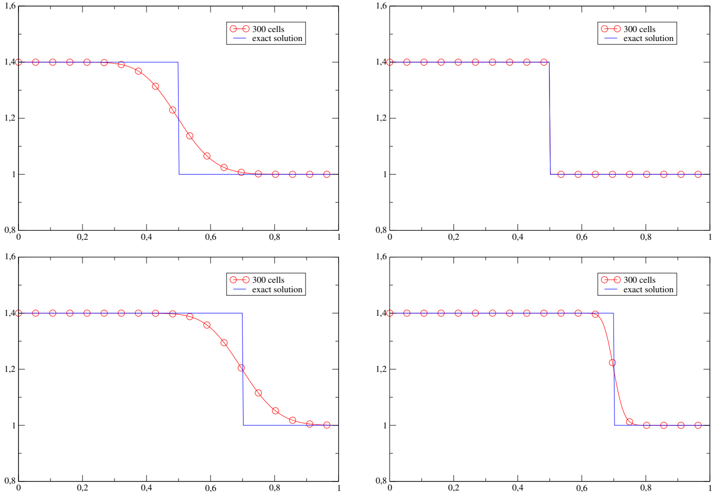
Test 5 - Suliciu



Parameter $x_0 = 0.8$ et final time $t = 0.012$.

Tests 6 et 7 : accuracy on contact discontinuities

densité HLL nonconservatif — densité Suliciu



Parameters pour le Test 6 : $x_0 = 0.5$ et $t = 2.0$.
Parameters pour le Test 7 : $x_0 = 0.3$ et $t = 2.0$.

THANK YOU FOR YOUR ATTENTION !