Chaotic behavior of an analog of the 2D Kuramoto-Sivashinsky equation

Jiao He

Université Lyon 1
Institut Camille-Jordan

August 9, 2019
CEMRACS

Outline

1D Kuramoto-Sivashinsky equation

In mathematics, the Kuramoto–Sivashinskys equation is a fourth-order nonlinear partial differential equation, named after Yoshiki Kuramoto and Gregory Sivashinsky, who derived the equation to model the diffusive instabilities in a laminar flame front in the late 1970s.
The Kuramoto-Sivashinsky equation is writen as :

$$u_t + uu_x = -u_{xx} - u_{xxxx} \tag{1}$$

The KS equations bridge the gap between infinite-dimensional behavior of PDEs and the finite-dimensional behavior in dynamical systems. The solutions to the KS equation reveal a complex interplay of simple spatial patterns and low fractal dimensional chaos.

Some PDEs

- In one variable, we consider the Heat equation :

$$u_t - u_{xx} = 0, \quad u(0,x) = f(x) \tag{2}$$

with general solution

$$u(x,t) = \int \Phi(x-y,t) f(y) dy$$

where $\Phi(x,t)$ is the fundamental solution.

Some PDEs

- In one variable, we consider the Heat equation :

$$u_t - u_{xx} = 0, \quad u(0,x) = f(x) \tag{2}$$

with general solution

$$u(x,t) = \int \Phi(x-y,t) f(y) dy$$

where $\Phi(x,t)$ is the fundamental solution.

- The Burgers' equation :

$$u_t + uu_x = u_{xx} \tag{3}$$

We can solve it by means of the Cole-Hopf transformation, which transform Burgers' equation to the linear diffusion equation by a nonlinear transformation.

Some PDEs

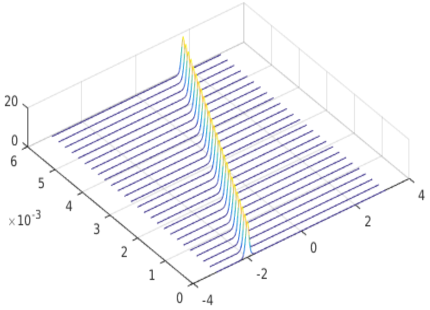
KdV equation :

$$u_t + uu_x + u_{xxx} = 0 \qquad (4)$$

with solution

$$-\frac{1}{2}\, c \operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - c\,t - a)\right]$$

where sech stands for the hyperbolic secant and a is an arbitrary constant. This describes a right-moving soliton.



1D Kuramoto-Sivashinsky equation

- Kuramoto-Sivashinsky equation :

$$\eta_t + \eta\eta_x + \eta_{xx} + \eta_{xxxx} = 0 \tag{5}$$

In this equation, the large scale dynamics are dominated by a destabilising 'diffusion' η_{xx}, whereas small scale dynamics are dominated by stabilising hyperdiffusion η_{xxxx}, and a nonlinear advective term $\eta\eta_x$ stabilises the system by transferring energy from the large unstable modes to the small stable modes

$$\eta_t + \eta\eta_x = -\eta_{xx} - \eta_{xxxx}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

non-linearity instability dissipation

1D Kuramoto-Sivashinsky equation

Applying the Fourier transformation to the linear part,

$$\partial_t \hat{\eta}(k) = (k^2 - k^4)\hat{\eta}(k),$$

it results in the stability of high frequencies ($|k| > 1$) and instability of low frequencies ($0 < |k| < 1$). Specifically, the

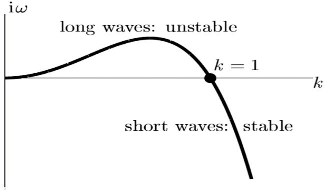


Fig. 1: Dispersion relation for the linear part of (2)

term η_{xx} leads to instability at large scales; the dissipative term η_{xxxx} is responsible for damping at small scales. When the nonlinear term $\eta\eta_x$ is added, stabilization occurs as this term transfers energy from the long wavelengths to the short wavelengths and balances the exponential growth due to the linear parts. This interaction between the unstable linear parts and a nonlinearity makes the solution to develop **chaotic dynamics**.

Dynamical systems

The focus of dynamical systems is to understand the qualitative behavior of the solutions. Typical questions include:

- What is the long-time asymptotic behavior of general solutions?
- Do solutions behave chaotically?
- What about attactors ?
- etc...

Dynamical systems

Abstract form of an evolution equation :

$$u_t = Au(t) + f(u(t)), u(0) = u_0 \tag{6}$$

where A is an infinitesimal generator of a semi-group $S(t) = e^{At}$.
Due to Duhamel's formula,

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(s, u(s))ds$$

Compact global attractor : A nonlinear semigroup $S(t)$ has a compact global attractor $\mathcal{A}$ if
(i) $\mathcal{A}$ is a compact subset of X;
(ii) $\mathcal{A}$ is an invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$;
(iii) $\mathcal{A}$ attract every bounded set $\mathcal{B}$ of X, i.e.

$$\text{dist}\,(S(t)\mathcal{B}, \mathcal{A}) \to 0 \;\text{ as }\; t \to \infty.$$

Compact global attractor : A nonlinear semigroup $S(t)$ has a compact global attractor $\mathcal{A}$ if
(i) $\mathcal{A}$ is a compact subset of X;
(ii) $\mathcal{A}$ is an invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$;
(iii) $\mathcal{A}$ attract every bounded set $\mathcal{B}$ of X, i.e.

$$\text{dist}\,(S(t)\mathcal{B}, \mathcal{A}) \to 0 \;\text{ as }\; t \to \infty.$$

A dynamic system with a chaotic attractor is locally unstable but globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the attractor.

Compact global attractor : A nonlinear semigroup $S(t)$ has a compact global attractor $\mathcal{A}$ if
(i) $\mathcal{A}$ is a compact subset of X;
(ii) $\mathcal{A}$ is an invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$;
(iii) $\mathcal{A}$ attract every bounded set $\mathcal{B}$ of X, i.e.

$$\text{dist}\,(S(t)\mathcal{B}, \mathcal{A}) \to 0 \;\text{ as }\; t \to \infty.$$

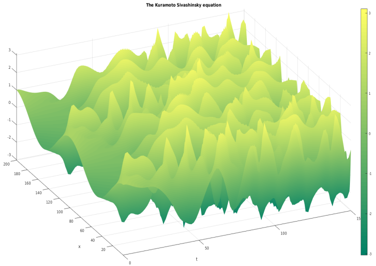
A dynamic system with a chaotic attractor is locally unstable but globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the attractor.

The K-S equation: a bridge between PDE's and dynamical systems : The existence of a compact global attractor with finite fractal dimension shows that the asymptotic behavior of solutions of equations is essentially finite-dimensional.

A small spoiler

A small spoiler

Simulation of 1D Kuramoto-Sivashinsky equation



An analog of the 2D KS equation

We consider the following canonical equation for overlying electrified films, which was derived by Tomlin, Papageorgiou & Pavliotis [1]:

$$\eta_t + \eta\eta_x + (\beta - 1)\eta_{xx} - \eta_{yy} - \gamma\Lambda^3\eta + \Delta^2\eta = 0 \qquad (7)$$

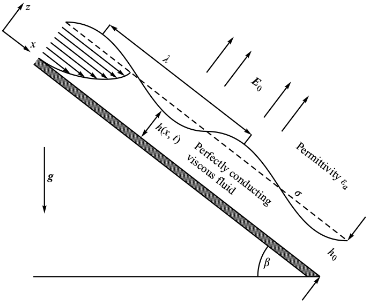
where $\beta > 0$ is the Reynolds number, $0 \leq \gamma \leq 2$ measures the electric field strength and Λ is a non-local operator corresponding to the electric field effect given on the Fourier variables as

$$\widehat{\Lambda\eta} = |\xi|\hat{\eta}(\xi) = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}\hat{\eta}(\xi).$$

We observe that the term corresponding to the electric field, $-\gamma\Lambda^3(\eta)$, always has a destabilizing effect,

R. Tomlin, D. Papageorgiou, and G. Pavliotis.
Three-dimensional wave evolution on electrified falling films.
Journal of Fluid Mechanics, 822:54–79, 2017.

Physical model



A Newtonian liquid of constant density ρ and viscosity μ, flows under gravity along an infinitely long flat plate which is inclined at an angle β to the horizontal. A coordinate system (x, z) is adopted with x measuring distance down and along the plate and z distance perpendicular to it.

The film thickness is $z = h(x, t)$ and its unperturbed value is h_0 .

An analog of the 2D KS equation

We will study the initial value problem for nonlocal 2D Kuramoto-Sivashinsky-type equation

$$\begin{cases} \eta_t + \eta\eta_x + (\beta - 1)\eta_{xx} - \eta_{yy} - \delta\Lambda^3(\eta) + \epsilon\Delta^2\eta = 0, \\ \eta(x, y, 0) = \eta_0(x, y), \ (x, y) \in \mathbb{T}^2. \end{cases} \tag{8}$$

with periodic boundary conditions and initial data with zero mean

$$\int_0^L \int_0^L \eta_0(x, y)dxdy = 0.$$

Step 1 : Global existence
Step 2 : Existence of an absorbing set
Step 3 : Analyticity
Step 4 : Existence of an attractor.
Step 5 : Oscillations

Global existence (R. Granero-Belinchón, J. He, 19):
If $\eta_0 \in H^2(\mathbb{T}^2)$, then for every $0 < T < \infty$ the initial value problem has a unique solution

$$\eta \in C([0,T]; H^2(\mathbb{T}^2)) \cap L^2(0,T; H^4(\mathbb{T}^2)).$$

By Picard's theorem and some *a priori* estimates

Global existence (R. Granero-Belinchón, J. He, 19):
If $\eta_0 \in H^2(\mathbb{T}^2)$, then for every $0 < T < \infty$ the initial value problem has a unique solution

$$\eta \in C([0,T]; H^2(\mathbb{T}^2)) \cap L^2(0,T; H^4(\mathbb{T}^2)).$$

By Picard's theorem and some *a priori* estimates

Absorbing set (R. Granero-Belinchón, J. He, 19)
Let $\eta_0 \in H^2(\mathbb{T}^2)$ be the zero mean initial data. Then the solution η of the initial-value problem (8) satisfies

$$\limsup_{t\to\infty} \|\eta(t)\|_{L^2(\mathbb{T}^2)} \leq R_{\epsilon,\delta}.$$

Based on the construction of a Lyapunov functional

Analyticity (R. Granero-Belinchón, J. He, 19)

Let η_0 be given in $H^2(\mathbb{T}^2)$. Then, there exists T_0 depending on $\eta_0, \epsilon, \beta, \delta$ such that the solution of (8) satisfies

$$\|e^{\sigma(t)\Lambda}\eta(t)\|_{L^2}^2 \leq 1 + 2C_{\epsilon,\delta,\eta_0}^2, \ \forall\ t \geq 0$$

where $\sigma(t) = \min\{\tanh(t), \tanh\left(\frac{T_0}{2}\right)\}$. In particular, it becomes analytic for $t > 0$.

Based on a priori estimates in a Gevrey class.

Analyticity (R. Granero-Belinchón, J. He, 19)

Let η_0 be given in $H^2(\mathbb{T}^2)$. Then, there exists T_0 depending on $\eta_0, \epsilon, \beta, \delta$ such that the solution of (8) satisfies

$$\|e^{\sigma(t)\Lambda}\eta(t)\|_{L^2}^2 \leq 1 + 2C_{\epsilon,\delta,\eta_0}^2, \ \forall \ t \geq 0$$

where $\sigma(t) = \min\{\tanh(t), \tanh\left(\frac{T_0}{2}\right)\}$. In particular, it becomes analytic for $t > 0$.

Based on a priori estimates in a Gevrey class.

Attractor (R. Granero-Belinchón, J. He, 19)

The system has a maximal, connected, compact attractor in the space $H^2(\mathbb{T}^2)$.

By showing that $S(\cdot)\eta_0 \in \mathcal{C}\left([0, T]; H^2\left(\mathbb{T}^2\right)\right)$ defines a compact semiflow in $H^2\left(\mathbb{T}^2\right)$.

Spatial oscillations (R. Granero-Belinchón, J. He, 19)

Let η be a solution of system for initial data $\eta_0 \in H^2(\mathbb{T}^2)$. Then, $\mathbb{T} = I \cup R$, where I is a union of at most $[\frac{4\pi}{\tanh(\frac{T_0}{2})}]$ open intervals in $\mathbb{T}$ and the following estimates hold for $t \geq \frac{T_0}{2}$ (T_0 is explicit),

$$|\partial_x \eta(x, y, t)| \leq 1, \text{ for all } x \in I, y \in \mathbb{T}$$

and

$$\text{card}\{x \in R : |\nabla \eta(x, y, t)| = 0\} \leq \frac{4\pi}{\log 2} \frac{\log C_{\epsilon,\delta,\eta_0}}{\tanh\left(\frac{T_0}{2}\right)}$$

where $C_{\epsilon,\delta,\eta_0}$ is a constant depending on ϵ, δ, η_0.

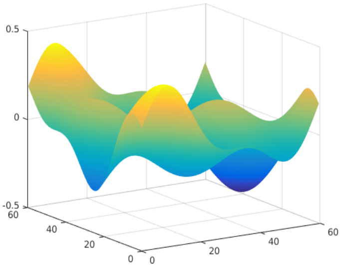
Spatial oscillations (R. Granero-Belinchón, J. He, 19)

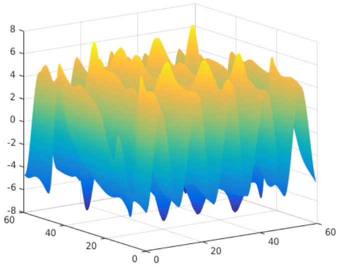
Let η be a solution corresponding to the initial data $\eta_0 \in H^2(\mathbb{T}^2)$, then for $t \geq \frac{T_0}{2}$, the number of peaks for η can be bounded as

$$\text{card}\,\{\text{peaks for } \eta\} \leq \frac{4\pi}{\log 2} \frac{\log C_{\epsilon,\delta,\eta_0}}{\tanh\left(\frac{T_0}{2}\right)}$$

where $C_{\epsilon,\delta,\eta_0}$ depends on ϵ, δ, η_0 and T_0 is defined as before.

Numerical simulations of usual 2D K-S equation





Thanks for your attentions!