The Gradient Discretisation Method applied to the elliptic part of a dispersive shallow water system

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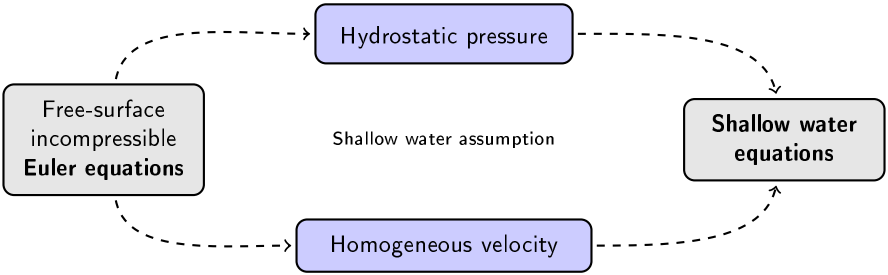
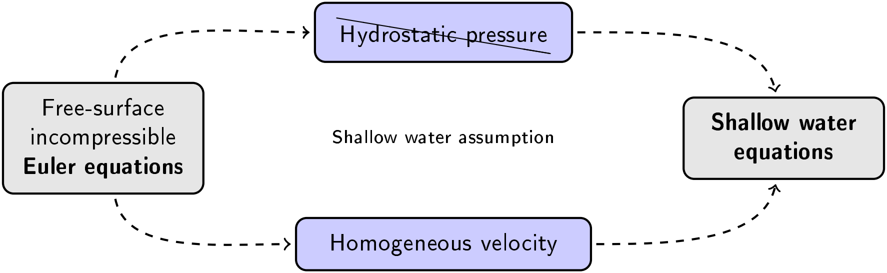
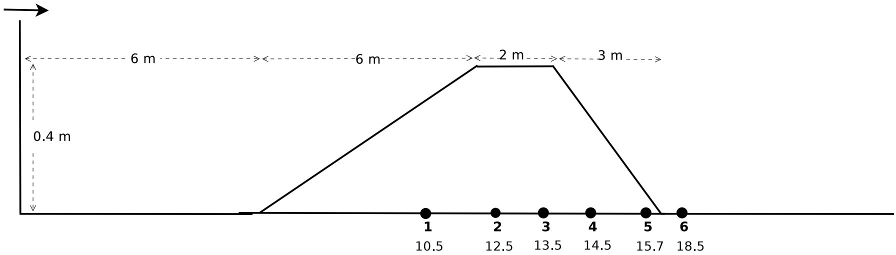
Free-surface flow modelling



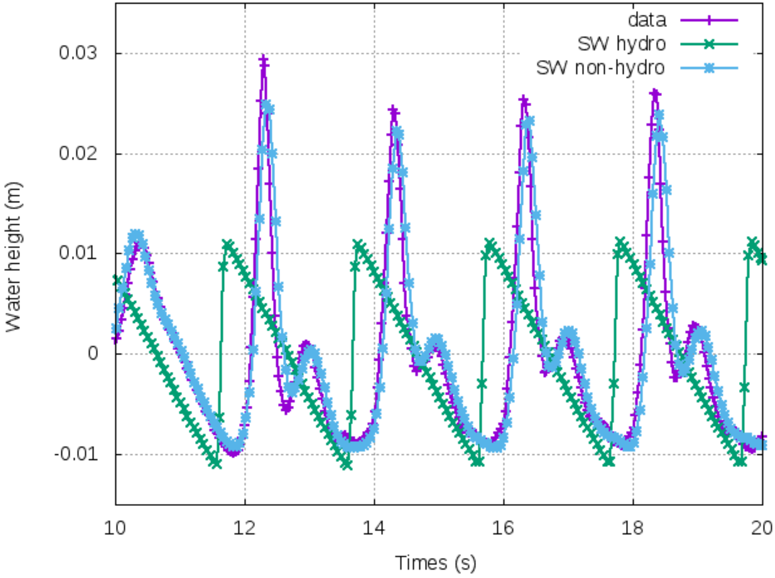

Hydrostatic pressure
Free-surface incompressible Euler equations
Shallow water assumption
Shallow water equations
Homogeneous velocity

Emphasis of non-hydrostatic effects



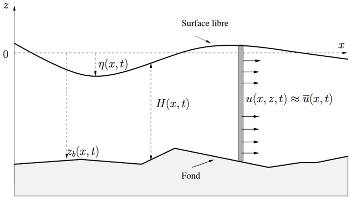
M.-W. Dingemans, *Wave propagation over uneven bottoms* (**Adv. Ser. Ocean Eng.**, 1997)

Emphasis of non-hydrostatic effects



A family of dispersive models

- water height H
- velocity $\overline{\mathbf{u}} = (\overline{\boldsymbol{v}}, \overline{w})$
- (non-hydrostatic) pressure p_{nh}

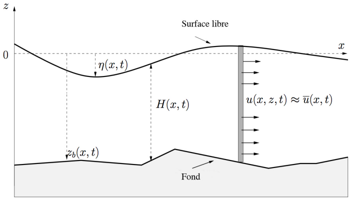


N. Aïssiouene, M.-O. Bristeau, E. Godlewski, J. Sainte-Marie, *A combined finite volume – finite element scheme for a dispersive shallow water system* (**Netw. Heterog. Media** 11(1), 2016)

N. Aïssiouene, M.-O. Bristeau, E. Godlewski, A. Mangeney, C. Parés, J. Sainte-Marie, *A two-dimensional method for a dispersive shallow water model* (submitted)

A family of dispersive models

- water height H
- velocity $\bar{\mathbf{u}} = (\bar{\mathbf{v}}, \bar{w})$
- (non-hydrostatic) pressure p_{nh}
- $\alpha = \sqrt{3}$: Serre-Green-Naghdi



$$\frac{\partial H}{\partial t} + \nabla_0 \cdot (H\bar{\mathbf{u}}) = 0$$

$$\frac{\partial (H\bar{\mathbf{u}})}{\partial t} + \nabla_0 \cdot (H\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) + \nabla_0 \left(g\frac{H^2}{2} \right) + \nabla^\alpha_{sw}\, p_{nh} + gH\nabla_0 z_b = 0$$

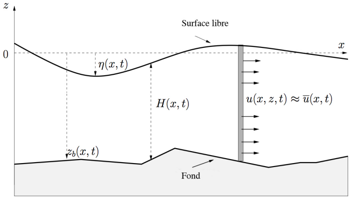
$$\mathrm{div}^\alpha_{sw}\, \bar{\mathbf{u}} = 0$$

Notations

$$\bar{\mathbf{u}} = \begin{pmatrix} \bar{\mathbf{v}} \\ \bar{w} \end{pmatrix} \qquad \nabla^\alpha_{sw}\, p = \begin{pmatrix} H\nabla_x p + p\nabla_x (H + \alpha^2 z_b) \\ -\alpha p \end{pmatrix}$$

$$\nabla_0 = \begin{pmatrix} \nabla_x \\ 0 \end{pmatrix} \qquad \mathrm{div}^\alpha_{sw}\, \bar{\mathbf{u}} = \nabla_x \cdot (H\bar{\mathbf{v}}) - \bar{\mathbf{v}} \cdot \nabla_x (H + 2z_b) + \alpha^2 \bar{w}$$

- water height H
- velocity $\overline{\mathbf{u}} = (\overline{\boldsymbol{v}}, \overline{w})$
- (non-hydrostatic) pressure p_{nh}
- $\alpha = \sqrt{3}$: Serre-Green-Naghdi



$$\frac{\partial H}{\partial t} + \nabla_0 \cdot (H\overline{\mathbf{u}}) = 0$$

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$$\mathrm{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} = 0$$

Time splitting: Hyperbolic solver / Dispersive solver

- water height H
- velocity $\overline{\mathbf{u}} = (\overline{\boldsymbol{v}}, \overline{w})$
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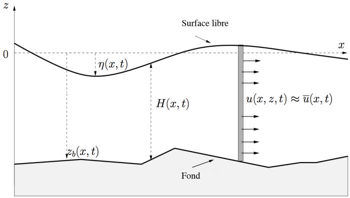

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$$\mathrm{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} = 0$$

Time splitting: Hyperbolic solver / Dispersive solver

Time Splitting: Correction step

Focus on the elliptic part - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

For $\delta > 0$, find $p_{nh} : \Omega \to \mathbb{R}$ and $\overline{\mathbf{u}} : \Omega \to \mathbb{R}^{d+1}$ s.t.

$$\begin{aligned} H\overline{\mathbf{u}} + \delta\, \nabla^{\alpha}_{sw}\, p_{nh} &= g \quad \text{on } \Omega \\ \mathrm{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} &= f \quad \text{on } \Omega \\ H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s &= \phi \quad \text{on } \Gamma_n \\ p_{nh} &= 0 \quad \text{on } \Gamma_d \end{aligned}$$

with

$$\alpha > 0,$$

$$\zeta := H + \frac{\alpha^2}{2} z_b, \qquad \nabla^{\alpha}_{sw}\, p_{nh} = (H\nabla p_{nh} + p_{nh}\nabla\zeta, -\alpha p_{nh}),$$

$$\partial\Omega = \Gamma = \Gamma_d \cup \Gamma_n, \qquad \mathrm{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} = \mathrm{div}(H\overline{\boldsymbol{v}}) - \overline{\boldsymbol{v}} \cdot \nabla\zeta + \alpha\overline{w},$$

$$\boldsymbol{n}_s = (\boldsymbol{n}_{\Gamma_n}, 0), \qquad \int_\Omega p_{nh}\, \mathrm{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} = -\int_\Omega \overline{\mathbf{u}}\, \nabla^{\alpha}_{sw}\, p_{nh} + \int_{\Gamma_n} H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s .$$

Conforming formulation - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

For $\delta > 0$, find $p_{nh} : \Omega \to \mathbb{R}$ s.t.

$$-\mathrm{div}_{sw}^{\alpha}\left(\frac{\delta}{H}\nabla_{sw}^{\alpha} p_{nh}\right) = f - \mathrm{div}_{sw}^{\alpha}\left(\frac{\mathbf{g}}{H}\right) \quad \text{on } \Omega$$

$$\delta\, \nabla_{sw}^{\alpha} p_{nh} \cdot \boldsymbol{n}_s = \boldsymbol{g} \cdot \boldsymbol{n}_s - \phi \quad \text{on } \Gamma_n$$

$$p_{nh} = 0 \quad \text{on } \Gamma_d$$

with

$$\alpha > 0,$$

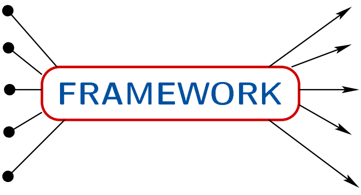
$$\zeta := H + \frac{\alpha^2}{2} z_b, \qquad \nabla_{sw}^{\alpha} p_{nh} = (H\nabla p_{nh} + p_{nh}\nabla\zeta, -\alpha p_{nh}),$$

$$\partial\Omega = \Gamma = \Gamma_d \cup \Gamma_n, \qquad \mathrm{div}_{sw}^{\alpha}\,\overline{\mathbf{u}} = \mathrm{div}(H\overline{\boldsymbol{v}}) - \overline{\boldsymbol{v}} \cdot \nabla\zeta + \alpha\overline{w},$$

$$\boldsymbol{n}_s = (\boldsymbol{n}_{\Gamma_n}, 0), \qquad \int_\Omega p_{nh}\,\mathrm{div}_{sw}^{\alpha}\,\overline{\mathbf{u}} = -\int_\Omega \overline{\mathbf{u}}\,\nabla_{sw}^{\alpha} p_{nh} + \int_{\Gamma_n} H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s.$$

METHODS

- FE
- mixed FE
- MPFA
- DDFV
- dG
- ...



PROBLEMS

- Heat equation
- Stefan problem
- Porous media flow
- Richards equation
- Incompressible Navier-Stokes
- ...

Objective

The framework identifies a few key properties that all methods satisfy, and that are sufficient for all convergence analyses

J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin, *The gradient discretisation method* Springer International Publishing AG, 82, 2018, Mathématiques et Applications

J. Droniou, R. Eymard, T. Gallouët, R. Herbin, *A unified analysis of elliptic problems with various boundary conditions and their approximation* (**hal-01823265**, 2019)

Conforming formulation - $\Omega \subset \mathbb{R}^d$ $(d = 1 \text{ or } d = 2)$

For $\delta > 0$, find $p_{nh} \; : \; \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\operatorname{div}_{sw}^{\alpha}\left(\frac{\delta}{H}\nabla_{sw}^{\alpha} p_{nh}\right) &= f - \operatorname{div}_{sw}^{\alpha}\left(\frac{\mathbf{g}}{H}\right) && \text{on } \Omega \\
\delta\,\nabla_{sw}^{\alpha} p_{nh}\cdot \boldsymbol{n}_s &= \boldsymbol{g}\cdot \boldsymbol{n}_s - \phi && \text{on } \Gamma_n \\
p_{nh} &= 0 && \text{on } \Gamma_d
\end{aligned}
$$

Weak formulation :

For $\delta > 0$, find $p \in H^1_{0,\Gamma_d}(\Omega)$ such that $\forall q \in H^1_{0,\Gamma_d}(\Omega)$,

$$
\int_\Omega \delta \frac{(H\nabla p + p\nabla\zeta)\cdot(H\nabla q + q\nabla\zeta) + \alpha^2 pq}{H}\, dx
$$

$$
= \int_\Omega fq + \frac{\mathbf{g}_1\cdot\nabla\zeta - \alpha g_2}{H} q \, dx + \int_\Omega \mathbf{g}_1 \cdot \nabla q \, dx - \int_{\Gamma_n} \phi\gamma(q)\, ds
$$

Elliptic part: GDM

Conforming formulation - $\Omega \subset \mathbb{R}^d$ $(d = 1 \text{ or } d = 2)$

For $\delta > 0$, find $p_{nh} \; : \; \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\operatorname{div}_{sw}^{\alpha}\left(\frac{\delta}{H}\nabla_{sw}^{\alpha}\, p_{nh}\right) &= f - \operatorname{div}_{sw}^{\alpha}\left(\frac{\mathbf{g}}{H}\right) && \text{on } \Omega \\
\delta\, \nabla_{sw}^{\alpha}\, p_{nh} \cdot \boldsymbol{n}_s &= \boldsymbol{g} \cdot \boldsymbol{n}_s - \phi && \text{on } \Gamma_n \\
p_{nh} &= 0 && \text{on } \Gamma_d
\end{aligned}
$$

Gradient scheme :

For $\delta > 0$, find $p_{\mathcal{D}} \in X_{\mathcal{D}}$ such that $\forall q_{\mathcal{D}} \in X_{\mathcal{D}}$,

$$
\int_{\Omega} \delta \frac{(H\nabla_{\mathcal{D}} p_{\mathcal{D}} + \Pi_{\mathcal{D}} p_{\mathcal{D}} \nabla\zeta) \cdot (H\nabla_{\mathcal{D}} q_{\mathcal{D}} + \Pi_{\mathcal{D}} q_{\mathcal{D}} \nabla\zeta) + \alpha^2 \Pi_{\mathcal{D}} p_{\mathcal{D}} \Pi_{\mathcal{D}} q_{\mathcal{D}}}{H} \, dx
$$

$$
= \int_{\Omega} f \Pi_{\mathcal{D}} q_{\mathcal{D}} + \frac{\mathbf{g}_1 \cdot \nabla\zeta - \alpha g_2}{H} \Pi_{\mathcal{D}} q_{\mathcal{D}} \, dx + \int_{\Omega} \mathbf{g}_1 \cdot \nabla_{\mathcal{D}} q_{\mathcal{D}} \, dx - \int_{\Gamma_n} \phi \mathbb{T}_{\mathcal{D},\Gamma_n} q_{\mathcal{D}} \, ds
$$

Gradient Discretisation - (GD) for mixed BCs (homo. Dirichlet)

For $\delta > 0$, find $p_\mathcal{D} \in X_\mathcal{D}$ such that $\forall q_\mathcal{D} \in X_\mathcal{D}$,

$$\int_\Omega \delta \frac{(H\nabla_\mathcal{D} p_\mathcal{D} + \Pi_\mathcal{D} p_\mathcal{D} \nabla\zeta)\cdot(H\nabla_\mathcal{D} q_\mathcal{D} + \Pi_\mathcal{D} q_\mathcal{D} \nabla\zeta) + \alpha^2 \Pi_\mathcal{D} p_\mathcal{D} \Pi_\mathcal{D} q_\mathcal{D}}{H}\, dx$$
$$= \int_\Omega f \Pi_\mathcal{D} q_\mathcal{D} + \frac{\mathbf{g_1}\cdot\nabla\zeta - \alpha g_2}{H}\Pi_\mathcal{D} q_\mathcal{D}\, dx + \int_\Omega \mathbf{g_1}\cdot\nabla_\mathcal{D} q_\mathcal{D}\, dx - \int_{\Gamma_n} \phi \mathbb{T}_{\mathcal{D},\Gamma_n} q_\mathcal{D}\, ds$$

$$\mathcal{D} = (X_\mathcal{D}\ ,\ \Pi_\mathcal{D}\ ,\ \nabla_\mathcal{D}\ ,\ \mathbb{T}_{\mathcal{D},\Gamma_n})$$

- discrete space $X_\mathcal{D} = X_{\mathcal{D},\Gamma_d} \oplus X_{\mathcal{D},\Omega,\Gamma_n} = \mathbb{R}^{\{d.o.f.\}}$
- reconstruction of function $\Pi_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)$ linear mapping
- reconstruction of gradient $\nabla_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)^d$ linear mapping, such that $\|\nabla_\mathcal{D} \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},\Omega,\Gamma_n}$
- reconstruction of trace $\mathbb{T}_{\mathcal{D},\Gamma_n} : X_\mathcal{D} \to L^2(\Gamma_n)$ linear mapping

GD-coercivity, GD-consistency, GD-limit conformity and GD-trace consistency.

Conforming formulation - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

For $\delta > 0$, find $p_{nh} \ : \ \Omega \to \mathbb{R}$ s.t.

$$-\operatorname{div}^{\alpha}_{sw}\left(\frac{\delta}{H}\nabla^{\alpha}_{sw}\, p_{nh}\right) = f - \operatorname{div}^{\alpha}_{sw}\left(\frac{\mathbf{g}}{H}\right) \qquad \text{on } \Omega$$

$$\delta\, \nabla^{\alpha}_{sw}\, p_{nh} \cdot \boldsymbol{n}_s = \boldsymbol{g} \cdot \boldsymbol{n}_s - \phi \qquad \text{on } \Gamma_n$$

$$p_{nh} = 0 \qquad \text{on } \Gamma_d$$

Weak formulation :

For $\delta > 0$, find $p \in H^1_{0,\Gamma_d}(\Omega)$ such that $\forall q \in H^1_{0,\Gamma_d}(\Omega)$,

$$\int_\Omega \frac{\delta}{H}\nabla^{\alpha}_{sw}\, p \cdot \nabla^{\alpha}_{sw}\, q = \int_\Omega \frac{\mathbf{g}}{H} \cdot \nabla^{\alpha}_{sw}\, q - \int_{\Gamma_n} \phi\, \gamma(q)$$

Elliptic part: Abstract GDM

Conforming formulation - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

For $\delta > 0$, find $p_{nh} : \Omega \to \mathbb{R}$ s.t.

$$-\operatorname{div}_{sw}^{\alpha}\left(\frac{\delta}{H}\nabla_{sw}^{\alpha} p_{nh}\right) = f - \operatorname{div}_{sw}^{\alpha}\left(\frac{\mathbf{g}}{H}\right) \qquad \text{on } \Omega$$

$$\delta\,\nabla_{sw}^{\alpha} p_{nh}\cdot \boldsymbol{n}_s = \boldsymbol{g}\cdot\boldsymbol{n}_s - \phi \qquad \text{on } \Gamma_n$$

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Gradient scheme :

For $\delta > 0$, find $p_{\mathcal{D}} \in X_{\mathcal{D}}$ s.t. $\forall q_{\mathcal{D}} \in X_{\mathcal{D}}$

$$\int_{\Omega}\frac{\delta}{H}\mathsf{G}_{\mathcal{D}}p_{\mathcal{D}}\cdot\mathsf{G}_{\mathcal{D}}q_{\mathcal{D}} = \int_{\Omega}\frac{\mathbf{g}}{H}\cdot\mathsf{G}_{\mathcal{D}}q_{\mathcal{D}} - \int_{\Gamma_n}\phi\,\mathbb{T}_{\mathcal{D}}q_{\mathcal{D}}$$

Abstract Gradient Discretisation - (AGD) for any BCs

For $\delta > 0$, find $p_{\mathcal{D}} \in X_{\mathcal{D}}$ s.t. $\forall q_{\mathcal{D}} \in X_{\mathcal{D}}$

$$\int_\Omega \frac{\delta}{H} \mathsf{G}_{\mathcal{D}} p_{\mathcal{D}} \cdot \mathsf{G}_{\mathcal{D}} q_{\mathcal{D}} = \int_\Omega \frac{\mathbf{g}}{H} \cdot \mathsf{G}_{\mathcal{D}} q_{\mathcal{D}} - \int_{\Gamma_n} \phi \, \mathbb{T}_{\mathcal{D}} q_{\mathcal{D}}$$

$$\mathcal{D} = (X_{\mathcal{D}}\ ,\ \mathbb{T}_{\mathcal{D}}\ ,\ \Pi_{\mathcal{D}}\ ,\ \mathsf{G}_{\mathcal{D}})$$

- discrete space $X_{\mathcal{D}} = \mathbb{R}^{\{d.o.f.\}}$ ($X_{\mathcal{D}}$ suited to boundary conditions)
- reconstruction of trace operator $\mathbb{T}_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Gamma_n)$ linear mapping
- reconstruction of function $\Pi_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Omega)$ linear mapping
- reconstruction of gradient SW $\mathsf{G}_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Omega)^{d+1}$ linear mapping

  s.t. $\|\cdot\|_{\mathcal{D}}^2 := \|\mathbb{T}_{\mathcal{D}} \cdot\|_{L^2(\Gamma_n)}^2 + \|\Pi_{\mathcal{D}} \cdot\|_{L^2(\Omega)}^2 + \|\mathsf{G}_{\mathcal{D}} \cdot\|_{L^2(\Omega)^{d+1}}^2$ is a norm on $X_{\mathcal{D}}$

AGD-coercivity, AGD-consistency, AGD-limit conformity.

Exemple: Conforming $\mathbb{P}_1$ Finite Elements

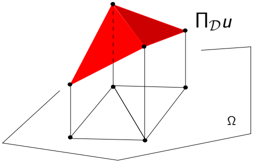
On a triangular/tetrahedral mesh, $\mathcal{V}$ = set of vertices of the mesh

(Abstract) Gradient discretisation :

- $X_\mathcal{D} := \{ \ (u_s)_{s\in\mathcal{V}} \ : \ u_s = 0 \text{ if } s \in \Gamma_d \ \}$
- $\Pi_\mathcal{D} : X_\mathcal{D} \to C(\Omega) \ ; \ u_\mathcal{D} \mapsto u_h = \sum_{s\in\mathcal{V}} u_s \varphi_s$ (φ_s $\mathbb{P}_1$ FE shape functions)
- $\mathbb{T}_\mathcal{D} : X_\mathcal{D} \to L^2(\Gamma_n) \ ; \ u_\mathcal{D} \mapsto \gamma(u_h)$
- GDM ▶ $\nabla_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)^d \ ; \ u_\mathcal{D} \mapsto \nabla_\mathcal{D} u_\mathcal{D} = \nabla u_h$

  AGDM ▶ $\mathrm{G}_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)^{d+1} \ ; \ u_\mathcal{D} \mapsto \mathrm{G}_\mathcal{D} u_\mathcal{D} = (H\nabla u_h + u_h \nabla\zeta \ , \ -\alpha u_h)$

Poincaré inequality	**(A)GD-coercivity**
$S_\mathcal{D}(\varphi) \leq C\,h$	**(A)GD-consistency**
$W_\mathcal{D}(\varphi) = 0$	**(A)GD-limit conformity**

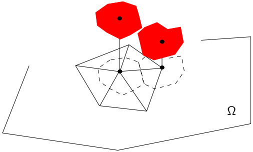


On a triangular/tetrahedral mesh, $\mathcal{V}$ = set of vertices of the mesh

Abstract Gradient discretisation :

- $X_{\mathcal{D}} := \{ (u_s)_{s\in\mathcal{V}} \ : \ u_s = 0 \text{ if } s \in \Gamma_d \}$
- $\Pi_{\mathcal{D}} : X_{\mathcal{D}} \to C(\Omega) \ ; \ u_{\mathcal{D}} \mapsto u_h = \sum_{s\in\mathcal{V}} u_s \varphi_s$ (φ_s $\mathbb{P}_1$ FE shape functions)
- $\mathbb{T}_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Gamma_n) \ ; \ u_{\mathcal{D}} \mapsto \gamma(u_h)$
- AGDM
  - $\mathrm{G}_{\mathcal{D}} : X_{\mathcal{D}} \to L^2(\Omega)^{d+1} \ ; \ u_{\mathcal{D}} \mapsto \mathrm{G}_{\mathcal{D}} u_{\mathcal{D}} = (H\nabla u_h + u_h \nabla\zeta \ , \ -\alpha\widetilde{\Pi}_{\mathcal{D}} u_{\mathcal{D}})$

Poincaré inequality	**(A)GD-coercivity**
$S_{\mathcal{D}}(\varphi) \leq C\,h$	**(A)GD-consistency**
$W_{\mathcal{D}}(\varphi) = 0$	**(A)GD-limit conformity**



Error estimates

GDM:

$$\|p - \Pi_{\mathcal{D}} p_{\mathcal{D}}\|_{L^2} + \|\nabla p - \nabla_{\mathcal{D}} p_{\mathcal{D}}\|_{L^2} \leq S_{\mathcal{D}}(p) + \frac{W_{\mathcal{D}}(\delta(H\nabla p + p\nabla\zeta) - \boldsymbol{g}_1) + S_{\mathcal{D}}(p)\, C_1}{\delta C_2}$$

$$\|\mathbb{T}_{\mathcal{D},\Gamma_n} p_{\mathcal{D}} - \gamma(p)\|_{L^2} \leq C_{\mathcal{D}} \frac{W_{\mathcal{D}}(\delta(H\nabla p + p\nabla\zeta) - \boldsymbol{g}_1) + \overline{S}_{\mathcal{D}}(p)\, C_1}{\delta\, C_2} + \overline{S}_{\mathcal{D}}(p)$$

AGDM:

$$\|\, \nabla^{\alpha}_{sw}\, p - \mathrm{G}_{\mathcal{D}} p_{\mathcal{D}}\|_{L^2} \leq \frac{C_3}{\delta} \underbrace{\left[\, \widehat{W}_{\mathcal{D}} \left(\frac{\delta}{H} \nabla^{\alpha}_{sw}\, p - \frac{\boldsymbol{g}}{H} \right) + \widehat{S}_{\mathcal{D}}(\delta\, p) \,\right]}_{\widehat{WS}_{\mathcal{D}}(p)}$$

$$\|p - \Pi_{\mathcal{D}} p_{\mathcal{D}}\|_{L^2} + \|\gamma(p) - \mathbb{T}_{\mathcal{D}} p_{\mathcal{D}}\|_{L^2} \leq \frac{C_3}{\delta} \left[\, \widehat{C}_{\mathcal{D}}\, \widehat{WS}_{\mathcal{D}}(p) + \widehat{S}_{\mathcal{D}}(\delta\, p) \,\right]$$

Conclusion

GDM

- a framework to study convergence analyses
- replace the continuous spaces and operators by discrete ones
- choose $\mathcal{D} = (X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathbb{T}_{\mathcal{D},\Gamma_n})$ and ensure **coercivity**, **consistency** and **limit-conformity** properties
- Error estimates: $\|\Pi_{\mathcal{D}} p_{\mathcal{D}} - p\|_{L^2}$, $\|\nabla_{\mathcal{D}} p_{\mathcal{D}} - \nabla p\|_{L^2}$ and $\|\gamma(p) - \mathbb{T}_{\mathcal{D},\Gamma_n} p_{\mathcal{D}}\|_{L^2}$

Abstract GDM

- Same processing of different BCs
- Non-classic operators ∇_{sw}^{α} and $\mathrm{div}_{sw}^{\alpha}$ related by a Stokes-type formula
- choose $\mathcal{D} = (X_{\mathcal{D}}, P_{\mathcal{D}}, G_{\mathcal{D}})$ and ensure **coercivity**, **consistency** and **limit-conformity** properties
- Can use a quadrature formula for $-\alpha p$ in $\nabla_{sw}^{\alpha} p$ ($\neq -\alpha \Pi_{\mathcal{D}} p_{\mathcal{D}}$)

GD-Coercivity: $C_{\mathcal{D}_m}$ **remains bounded**

$$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D},\Omega,\Gamma_n}\setminus\{0\}} \left(\max\left\{ \frac{\|\Pi_{\mathcal{D}} v\|_{L^2}}{\|\nabla_{\mathcal{D}} v\|_{L^2}}, \frac{\|\mathbb{T}_{\mathcal{D},\Gamma_n} v\|_{L^2}}{\|\nabla_{\mathcal{D}} v\|_{L^2}} \right\}\right)$$

GD-Consistency: $S_{\mathcal{D}_m} \to 0$

$$\forall \varphi \in H^1_{\Gamma_d}(\Omega),\ S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},\Omega,\Gamma_n}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2} \right)$$

GD-Trace Consistency: $\overline{S}_{\mathcal{D}_m} \to 0$

$$\forall \varphi \in H^1_{\Gamma_d}(\Omega),$$
$$\overline{S}_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},\Omega,\Gamma_n}} \left\{ \|\Pi_{\mathcal{D}} v - \varphi\|_{L^2} + \|\mathbb{T}_{\mathcal{D},\Gamma_n} v - \gamma(\varphi)\|_{L^2} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2} \right\}$$

GD-Limit Conformity: $W_{\mathcal{D}_m} \to 0$

$$\forall \boldsymbol{\varphi} \in H_{div,\Gamma_n}(\Omega),$$
$$W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{v\in X_{\mathcal{D},\Omega,\Gamma_n}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v\|_{L^2}} \left| \int_{\Omega} (\nabla_{\mathcal{D}} v \cdot \boldsymbol{\varphi} + \Pi_{\mathcal{D}} v\, \mathrm{div}\boldsymbol{\varphi}) - \int_{\Gamma_n} \mathbb{T}_{\mathcal{D},\Gamma_n} v\ \gamma_{\mathbf{n}}(\boldsymbol{\varphi}) \right|$$

AGD-Coercivity: $C_{\mathcal{D}_m}$ **remains bounded**

$$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D}}\setminus\{0\}} \frac{\|\mathbb{T}_{\mathcal{D}} v\|_{L^2} + \|\Pi_{\mathcal{D}} v\|_{L^2}}{\|v\|_{\mathcal{D}}}$$

AGD-Consistency: $S_{\mathcal{D}_m} \to 0$

$$\forall \varphi \in H^1_{\Gamma_d}(\Omega)\,,\ \widehat{S}_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D}}} \left(\|\mathbb{T}_{\mathcal{D}} v - \gamma(\varphi)\|_{L^2} + \|\Pi_{\mathcal{D}} v - \varphi\|_{L^2} + \|\mathsf{G}_{\mathcal{D}} v - \nabla^{\alpha}_{sw}\varphi\|_{L^2} \right)$$

AGD-Limit Conformity: $W_{\mathcal{D}_m} \to 0$

$$\forall \boldsymbol{\varphi} \in H_{\mathrm{div}^{\alpha}_{\mathrm{sw}}}(\Omega)\,,\ \widehat{W}_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u\in X_{\mathcal{D}}\setminus\{0\}} \frac{1}{\|u\|_{\mathcal{D}}} \left| \int_{\Omega} \mathsf{G}_{\mathcal{D}} u \cdot \boldsymbol{\varphi} + \Pi_{\mathcal{D}} u \ \mathrm{div}^{\alpha}_{sw}\,\boldsymbol{\varphi} - \int_{\Gamma_n} \mathbb{T}_{\mathcal{D}} u \ H\boldsymbol{\varphi}\cdot \boldsymbol{n}_{sw} \right|$$

Additional properties for non linear problems

Compactness

A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is compact if, for any sequence $u_m \in X_{D_m}$ such that $(||u_m||_{D_m})_{m\in\mathbb{N}}$ is bounded, the sequences $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$ and $(\mathbb{T}_{D_m} u_m)_{m\in\mathbb{N}}$ are relatively compact in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

Piecewise constant reconstruction

Let $D = (X_D, \Pi_D, \nabla_D)$ be a gradient discretisation. The operator $\Pi_D : X_D \mapsto L^p(\Omega)$ is a piecewise constant reconstruction if there exists a basis $(e_i)_{i\in B}$ of X_D and a family of disjoint subsets $(\Omega_i)_{i\in B}$ of Ω such that $\Pi_D u = \sum_{i\in B} u_i \mathbb{1}_{\Omega_i}$ for all $u = \sum_{i\in B} u_i e_i \in X_D$. In other words, $\Pi_D u$ is the piecewise constant function equal to u_i on Ω_i , for all $i \in B$.