Time-periodic parabolic equations

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Abstract parabolic evolution equation:

$$(1.1) \qquad \begin{cases} y'(t) = Ay(t) + f(t), \ t > 0, \\ y(0) = y^0. \end{cases}$$

Hypothesis:

- Hilbertian framework H.
- A is the infinitesimal generator of an analytic semigroup of operators $S(t)$.
- The resolvent of A is compact. $(\sigma(A) = \sigma_p(A))$

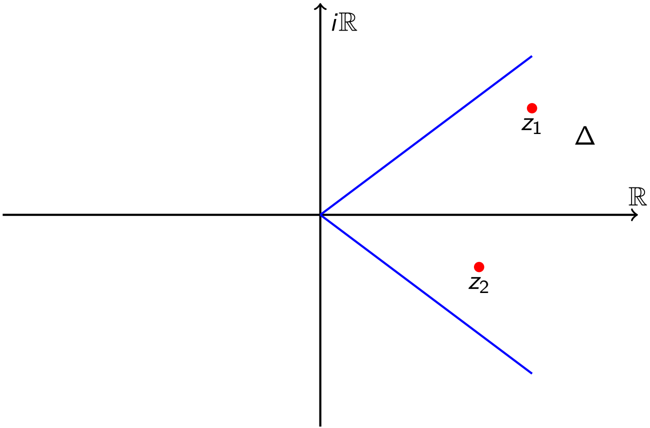
Definition of a semigroup of operators $S(t) \in \mathcal{L}(H)$, $t \geq 0$:
(i) $S(0) = Id$ on H
(ii)$S(t+s) = S(t) \circ S(s)$ for every $t, s \geq 0$.

A trivial example: $y'(t) = ay(t) \Rightarrow y(t) = \mathrm{e}^{at} y^0 = S(t) y^0$.

Analytic semigroups

($\mathcal{C}^0$-) Analytic semigroup:

- $S(0) = I$, $\lim_{z \to 0, z \in \Delta} S(z)x = x$ for all $x \in H$.
- $z \mapsto S(z)$ is analytic in a sector Δ.
- $S(z_1 + z_2) = S(z_1) \circ S(z_2)$ for all $z_1, z_2 \in \Delta$.



Another definition/property

A is the infinitesimal generator of an analytic semigroup $\Leftrightarrow$

- The resolvent set $\rho(A)$ contains a sector $\Sigma = \{\lambda \in \mathbb{C} \mid \lambda \neq \omega \text{ and } |\arg(\lambda - \omega)| < \theta\}$ with $\omega \in \mathbb{R}$ and $\theta > \frac{\pi}{2}$.
- $\|R(\lambda, A)\|_{\mathcal{L}(H)} \leq \dfrac{M}{|\lambda - \omega|}, \forall \lambda \in \Sigma$ with $M > 0$.

Dunford integral:

$$e^{tA} := S(t) = \frac{1}{2i\pi} \int_\gamma e^{\lambda t} R(\lambda, A) d\lambda, \; t > 0.$$

Why are analytic semigroups so important?

Formally, in Fourier:

1. $ik\hat{y}(k) = A\hat{y}(k) + \hat{f}(k)$,
2. $\hat{y}(k) = R(ik, A)\hat{f}(k)$,
3. $\Rightarrow |A\hat{y}(k)| \leq (M+1)|\hat{f}(k)|$,

Using that: $AR(\lambda, A) = -Id + \lambda R(\lambda, A)$.

y', Ay and f have the same regularity $\Rightarrow$ “Maximal regularity property”

Theorem 1

Assume that S is an analytic semigroup, then for each $T > 0$, the map

$$Iso : \begin{cases} L^2(0,T;\mathcal{D}(A)) \cap H^1(0,T;H) \to L^2(0,T;H) \times [\mathcal{D}(A), H]_{1/2} \\ y \mapsto (y' - Ay, y(0)) \end{cases}$$

is an isomorphism.

A concrete example

Consider the heat equation:

$$(1.2) \qquad \begin{cases} y'(t) - \Delta y(t) = f(t), t > 0, \\ y(0) = y^0 \end{cases}$$

with :

- Ω a smooth bounded domain.
- $\mathcal{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ (Dirichlet boundary condition).
- $f \in L^2(0, +\infty; L^2(\Omega))$.
- $y^0 \in [H^2(\Omega) \cap H_0^1(\Omega), L^2(\Omega)]_{1/2} = H_0^1(\Omega)$.

Theorem 1:

$\Rightarrow \exists ! y \in L^2(0, +\infty; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, +\infty; L^2(\Omega))$ solution to (1.2).

And with a nonlinear term $y\Delta y$?

The periodic problem

Periodic evolution equation:

$$\begin{cases} y'(t) = Ay(t) + f(t)\,, \text{ for all } t \in [0, T], \\ y(0) = y(T). \end{cases} \tag{2.1}$$

From the Duhamel formula: $y(0) = y(T) = S(T)y(0) + \int_0^T S(T-s)f(s)ds$.

Existence of time-periodic solutions $\Longleftrightarrow$ Existence of a solution z to

$$(I - S(T))z = \int_0^T S(T-s)f(s)ds. \tag{2.2}$$

We need some spectral assumptions on (A, T) to invert $(I - S(T))$.

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Assumptions on A:

- A is the infinitesimal generator of an analytic semigroup and its resolvent is compact.

Spectral theorem: $\sigma_p(S(T)) = e^{T\sigma_p(A)}$.

$1 \in \sigma_p(S(T)) \Leftrightarrow 0 \in \sigma_p(A)$ or A has a complex eigenvalue $\dfrac{2ik\pi}{T}$ with $k \in \mathbb{Z}^*$.

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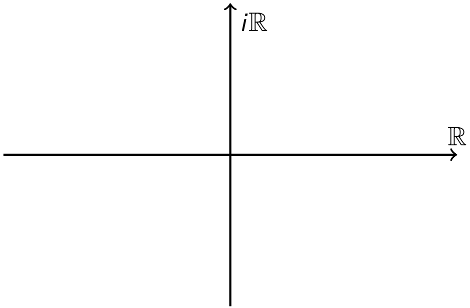

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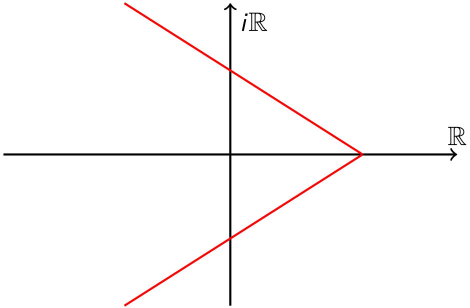

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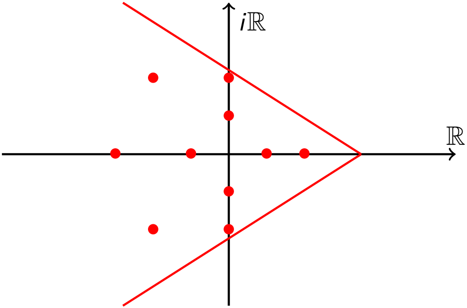

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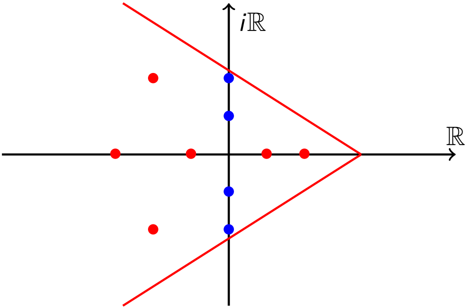


Denote by $\{ib_j\}_{0\leq j\leq N_A}$ the (finite) number of eigenvalue of A on the imaginary axis $i\mathbb{R}$ (• in the previous example).
Assumption on the period T:

$$T \in \mathbb{R}^+ \setminus \{\frac{2k\pi}{b_j} \mid k \in \mathbb{Z},\, 0 \leq j \leq N_A\} \tag{2.3}$$

Under the previous assumptions on (A, T) we have
$y(0) = (I - S(T))^{-1} \int_0^T S(T-s)f(s)ds \in [\mathcal{D}(A), H]_{1/2}$ and we obtain:

Theorem 2

For $f \in L^2(0,T;H)$, the periodic evolution equation (2.1) admits a unique strict solution $y \in L^2(0,T;\mathcal{D}(A)) \cap H^1_\sharp(0,T;H)$ in $L^2(0,T;H)$. The following estimate holds

$$\|y\|_{L^2(0,T;\mathcal{D}(A))\cap H^1_\sharp(0,T;H)} \leq C\,\|f\|_{L^2(0,T;H)}\,.$$

Hölder regularity in time

When the source term f is Hölder continuous in time:

Theorem 3

For $f \in \mathcal{C}^{\rho}_{\sharp}([0,T];H)$ with $\rho \in (0,1)$ the periodic evolution equation (2.1) admits a unique strict solution y in $\mathcal{C}([0,T];H)$. Moreover

$$y \in \mathcal{C}^{\rho}([0,T];\mathcal{D}(A)) \cap \mathcal{C}^{\rho+1}([0,T];H),$$

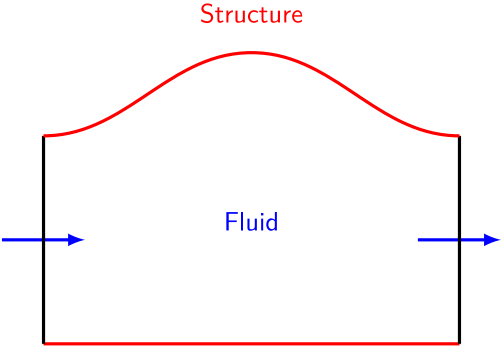
and the following estimate holds

$$\|y\|_{\mathcal{C}^{\rho}([0,T];\mathcal{D}(A))\cap\mathcal{C}^{\rho+1}([0,T];H)} \leq C\,\|f\|_{\mathcal{C}^{\rho}([0,T];H)}\,. \tag{2.4}$$

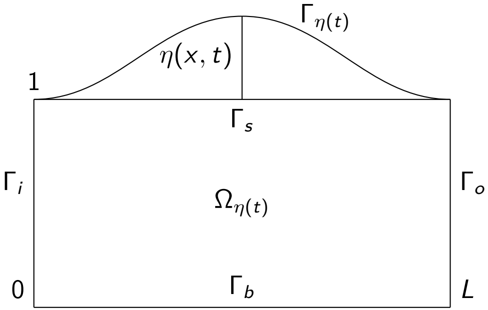
Remark 4

Very specific result for parabolic equation $\Rightarrow$ Not true in the non-periodic framework.

Simplified model of blood flow through arteries



- Incompressible fluid, viscous, Newtonian : Incompressible Navier–Stokes equations.
- Viscoelastic structure : Damped Euler–Bernoulli beam equation.



- Eulerian-Lagrangian formulation.
- Structure displacement $\eta : \Gamma_s \times (0, T) \to (-1, +\infty)$.
- Fluid domain $\Omega_{\eta(t)}$: Unknown of the problem.

Fluid–structure interaction system

Fluid : $2D$ Incompressible Navier–Stokes equation

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0 \text{ et div } \mathbf{u} = 0 \text{ in } \Omega_{\eta(t)},\ t > 0.$$

Structure : Damped Euler–Bernoulli beam equation

$$\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{txx} + \alpha\eta_{xxxx} = F(\mathbf{u}, p, \eta) \text{ on } \Gamma_s,\ t > 0.$$

Kinematic coupling :

$$\mathbf{u} = \eta_t\mathbf{e}_2 \text{ on } \Gamma_{\eta(t)},\ t > 0.$$

Boundary conditions and time-periodic forcing term :

$$\begin{aligned}
&\mathbf{u} = \boldsymbol{\omega}_1 \text{ on } \Gamma_i,\ u_2 = 0 \text{ and } p + (1/2)|\mathbf{u}|^2 = \omega_2 \text{ on } \Sigma^o,\\
&\mathbf{u} = 0 \text{ on } \Gamma_b,\\
&\eta(0,t) = \eta(L,t) = \eta_x(0,t) = \eta_x(L,t) = 0,\ t > 0.
\end{aligned}$$

Periodic solutions: $(\mathbf{u}(0), \eta(0), \eta_t(0)) = (\mathbf{u}(T), \eta(T), \eta_t(T)).$

Theorem 5 (C, 19)

Fix $\theta \in (0,1)$ and $T > 0$. There exists $R > 0$ such that, for all T-periodic source terms

$$(\boldsymbol{\omega}_1, \omega_2) \in \left(\mathcal{C}^{\theta}_{\sharp}([0,T]; \mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_{\sharp}([0,T]; \mathbf{H}^{-1/2}(\Gamma_i))\right) \times \mathcal{C}^{\theta}_{\sharp}([0,T]; H^{1/2}(\Gamma_o)),$$

satisfying

$$\|\boldsymbol{\omega}_1\|_{\mathcal{C}^{\theta}_{\sharp}([0,T];\mathbf{H}^{3/2}_0(\Gamma_i)) \cap \mathcal{C}^{1+\theta}_{\sharp}([0,T];\mathbf{H}^{-1/2}(\Gamma_i))} + \|\omega_2\|_{\mathcal{C}^{\theta}_{\sharp}([0,T];H^{1/2}(\Gamma_o))} \leq R,$$

the fluid–structure system admits a T-periodic strict solution $(\boldsymbol{u}, p, \eta)$ belonging to (after a change of variables mapping $\Omega_{\eta(t)}$ into Ω)

- $\boldsymbol{u} \in \mathcal{C}^{\theta}_{\sharp}([0,T]; \boldsymbol{H}^2(\Omega)) \cap \mathcal{C}^{1+\theta}_{\sharp}([0,T]; \boldsymbol{L}^2(\Omega))$.
- $p \in \mathcal{C}^{\theta}_{\sharp}([0,T]; H^1(\Omega))$.
- $\eta \in \mathcal{C}^{\theta}_{\sharp}([0,T]; H^4(\Gamma_s) \cap H^2_0(\Gamma_s)) \cap \mathcal{C}^{1+\theta}_{\sharp}([0,T]; H^2_0(\Gamma_s)) \cap \mathcal{C}^{2+\theta}_{\sharp}([0,T]; L^2(\Gamma_s))$.

Main steps

- Change of variables to fix the domain $\Omega_{\eta(t)} \to \Omega$. Linearization.
- Introduce the Leray projector Π adapted to the mixed boundary conditions.
- Use Π to remove the pressure in the fluid equations.
- The pressure appears in the right-hand side of the beam equation (in the term $F(\mathbf{u}, p, \eta)$).
- Apply $(I - \Pi)$ to the fluid equations to express p in terms of $\Pi\mathbf{u}$ and η.
- Rewrite the linear system associated to (FS) as an evolution equation on $(\Pi\mathbf{u}, \eta, \eta_t)$,

$$\frac{d}{dt}\begin{pmatrix}\Pi\mathbf{u}\\ \eta\\ \eta_t\end{pmatrix} = \mathcal{A}\begin{pmatrix}\Pi\mathbf{u}\\ \eta\\ \eta_t\end{pmatrix} + \boldsymbol{f}, \quad \begin{pmatrix}\Pi\mathbf{u}(0)\\ \eta(0)\\ \eta_t(0)\end{pmatrix} = \begin{pmatrix}\Pi\mathbf{u}(T)\\ \eta(T)\\ \eta_t(T)\end{pmatrix}. \tag{3.1}$$

Here:

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + N_s)^{-1} \end{pmatrix} \begin{pmatrix} A_s & 0 & -A_s \Pi L_1 \\ 0 & 0 & I \\ N_v & A_{\alpha,\beta} & \gamma \Delta_s \end{pmatrix}, \tag{3.2}$$

Energy identity for the eigenvalue of $\mathcal{A}$:

$$\lambda \left[\int_\Omega |\mathbf{u}|^2 + \int_{\Gamma_s} |\eta_2|^2 \right] + \overline{\lambda} \left[\beta \int_{\Gamma_s} |\eta_{1,x}|^2 + \alpha \int_{\Gamma_s} |\eta_{1,xx}|^2 \right]$$
$$+ \nu \int_\Omega |\nabla \mathbf{u}|^2 + \gamma \int_{\Gamma_s} |\eta_{2,x}|^2 = 0.$$

Re $\lambda < 0 \Rightarrow$ no restriction on the period T.