Heaving buoys in axisymmetric shallow water and the return to equilibrium problem

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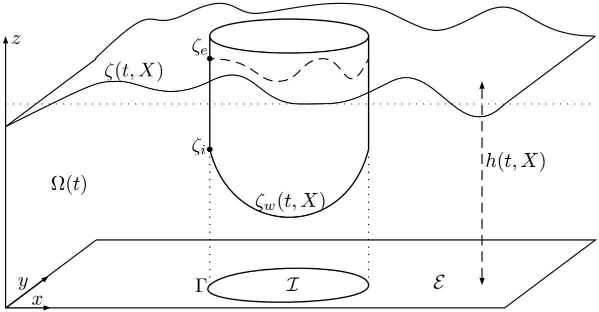
6 August, 2019

CEMRACS 2019, CIRM, Luminy



Assumptions on the solid:

- Vertical side-walls
- Only vertical motion

The contact line Γ does not depend on time

$\Rightarrow$ One free boundary problem: surface elevation $\zeta(t, X)$

Equations in the fluid domain $\Omega(t)$ for $\mathbf{U}$:

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e_z} \quad \text{in} \quad \Omega(t) \tag{1}$$

$$\text{div } \mathbf{U} = 0 \tag{2}$$

$$\text{curl } \mathbf{U} = 0 \tag{3}$$

Boundary conditions at the surface and the bottom:

$$z = \zeta, \qquad \partial_t \zeta - \mathbf{U} \cdot N = 0 \quad \text{with } N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix} \tag{4}$$

$$z = -h_0, \qquad \mathbf{U} \cdot \mathbf{e_z} = 0 \tag{5}$$

Pressure in $\mathcal{E}$:

$$\underline{P}_e = P_{atm} \tag{6}$$

Constraint in $\mathcal{I}$:

$$\zeta_i(t, X) = \zeta_w(t, X) \tag{7}$$

Jump at Γ:

$$\zeta_e(t, \cdot) \neq \zeta_i(t, \cdot) \tag{8}$$

$$\underline{P}_i(t, \cdot) = P_{atm} + \rho g(\zeta_e - \zeta_i) + P_{NH} \tag{9}$$

Continuity of the normal velocity at the vertical walls:

$$V \cdot \nu = V_C \cdot \nu \tag{10}$$

Shallow water approximation

Regime: the wavelength L is larger than the depth h_0, *i.e.* $\mu = \frac{{h_0}^2}{L^2} \ll 1$

Nonlinear shallow water equations

At precision $O(\mu)$, h and $Q = \int_{-h_0}^{\zeta} V\,dz$ solve

$$\begin{cases} \partial_t h_e + \nabla \cdot Q_e = 0 \\ \partial_t Q_e + \nabla \cdot (\frac{1}{h_e} Q_e \otimes Q_e) + g h_e \nabla h_e = -\frac{h_e}{\rho} \nabla \underline{P}_e = 0 \\ \underline{P}_e = P_{atm} \end{cases} \qquad \text{in} \quad \mathcal{E}$$

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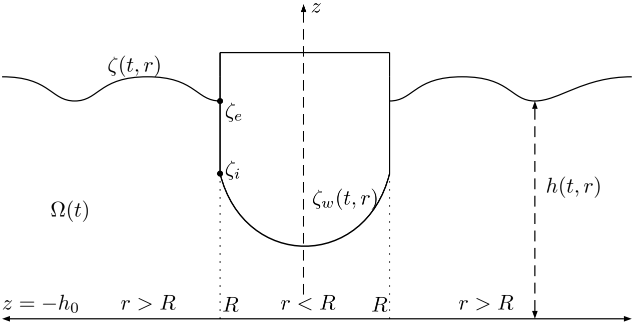
B.C. at Γ : $\quad \underline{P}_{i|_\Gamma} = P_{atm} + \rho g (\zeta_e - \zeta_i)_{|_\Gamma} + P_{cor}, \quad Q_e \cdot \nu_{|_\Gamma} = Q_i \cdot \nu_{|_\Gamma}.$

Axisymmetric case

Cylindrical coordinates:

$\mathbf{U} = \mathbf{U}(t, r, \theta, z)$, $\mathbf{U} = (u_r, u_\theta, u_z) \Longrightarrow Q = Q(t, r, \theta)$, $Q = (q_r, q_\theta)$



We assume that the flow is axisymmetric without swirl, which means that the flow has no dependence on the angular variable θ and $u_\theta = 0$.

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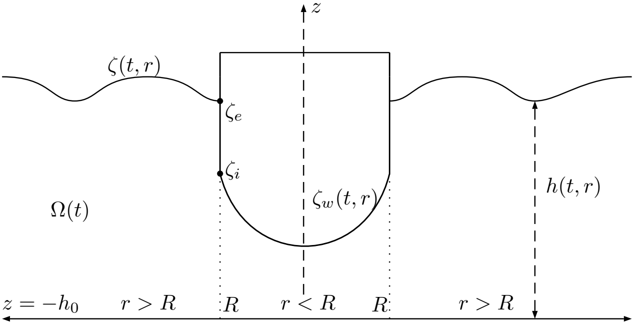

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$$\Longrightarrow Q(t, r) = (q_r, 0)$$

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$$\text{B.C.} \quad \begin{cases} \underline{P}_{i|_\Gamma} = P_{atm} + \rho g (\zeta_e - \zeta_i)_{|_\Gamma} + P_{cor}, \\ Q_e \cdot \nu_{|_\Gamma} = Q_i \cdot \nu_{|_\Gamma}. \end{cases} \quad \text{at} \quad \Gamma$$

$\rightsquigarrow$ Pressure eq: $-\nabla \cdot (\frac{h_w}{\rho} \nabla \underline{P}_i) = -\partial_t^2 h_w + \dots$

Axisymmetric nonlinear shallow water equations

$$\begin{cases} \partial_t h_e + \quad \partial_r q_e + \dfrac{q_e}{r} = 0, \\ \partial_t q_e + \partial_r \left(\dfrac{q_e^2}{h_e} \right) + \dfrac{q_e^2}{r h_e} + g h_e \partial_r h_e = 0, \\ \underline{P}_e = P_{atm} \end{cases} \qquad \text{in} \quad (R, +\infty)$$

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$$\text{B.C.} \quad \begin{cases} \underline{P}_i = P_{atm} + \rho g (\zeta_e - \zeta_i) + P_{cor}, \\ q_e = q_i. \end{cases} \qquad \text{at} \quad r = R$$

$\rightsquigarrow$ for $P_{cor} \sim {q_i}^2_{|_{r=R}} \left(\frac{1}{h_e^2} - \frac{1}{h_i^2} \right)_{|_{r=R}}$ conservation of the fluid-solid energy!

Solid motion

Solid: $G(t) = (0, 0, z_G(t))$, $\mathbf{U}_G(t) = (0, 0, \dot{z}_G(t))$, $\omega = 0$

Define the displacement $\delta_G(t) := z_G(t) - z_{G,eq}$

From the assumptions on the solid: $h_w(t, r) = h_{w,eq}(r) + \delta_G(t)$

By the interior constraint $h_w = h_i$ we have also

$$q_i(t, r) = -\frac{r}{2}\dot{\delta}_G(t)$$

Newton's law for the conservation of the linear momentum

$$m\ddot{\delta}_G = -mg + \int_0^R (\underline{P}_i - P_{atm})$$

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Using the elliptic equation on $\underline{P}_i$

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathfrak{c}\delta_G(t) + \mathfrak{c}\zeta_e(t, R) + \left(\frac{\mathfrak{b}}{h_e^2(t, R)} + \beta(\delta_G)\right)\dot{\delta}_G^2(t)$$

Writing $u = (\zeta_e, q_e)$:

Fluid part (Hyperbolic IBVP)

$$\begin{cases} \partial_t u + A(u)\partial_r u + B(u,r)u = 0, \qquad r \in (R, +\infty) \\ \mathbf{e}_2 \cdot u_{|_{r=R}} = -\dfrac{R}{2}\dot{\delta}_G(t), \\ u(t=0) = u_0. \end{cases} \tag{11}$$

Solid part (Nonlinear ODE)

$$\begin{cases} (m + m_a(\delta_G))\ddot{\delta}_G = -\mathfrak{c}\delta_G + \mathfrak{c}(\mathbf{e}_1 \cdot u_{|_{r=R}} - h_0) + (\mathfrak{b}(u) + \beta(\delta_G))\,\dot{\delta}_G^2, \\ \delta_G(0) = \delta_0, \\ \dot{\delta}_G(0) = \delta_1. \end{cases} \tag{12}$$

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Theorem (E.B. '18)

Local well-posedness of the coupled system (11) - (12) *for compatible initial data* u_0, δ_0, δ_1 *and* $u_0 \in H^k_r((R, +\infty))$ *with* $k \geqslant 2$.

Return to equilibrium

It consists in dropping the solid with no initial velocity from a non-equilibrium position into a fluid initially at rest.

Initial data

Solid: $\delta_G(0) = \delta_0 \neq 0$, $\dot{\delta}_G(0) = 0$

Fluid: $h_e(0,r) \equiv h_0$, $\zeta_e(0,r) \equiv 0$, $q_e(0,r) \equiv 0$

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$\Rightarrow$ Compatibility conditions are NOT satisfied

Different approach:

- linearized equations in the exterior domain
- nonlinear equations in the interior domain

Hydrodynamical linear-nonlinear model (L-NL)

- $r \in (R, +\infty)$

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \dfrac{q_e}{r} = 0 \\ \partial_t q_e + g h_0 \partial_r \zeta_e = 0 \end{cases}$$

- $r \in (0, R)$

$$\begin{cases} \partial_t h_i + \partial_r q_i + \dfrac{q_i}{r} = 0 \\ \partial_t q_i + \partial_r \left(\dfrac{q_i^2}{h_i} \right) + \dfrac{q_i^2}{r h_i} + g h_i \partial_r h_i = -\dfrac{h_i}{\rho} \partial_r \underline{P}_i \end{cases}$$

- $r = R$

$$q_{e|_{r=R}} = -\frac{R}{2} \dot{\delta}_G(t), \qquad \underline{P}_{i|_{r=R}} = P_{atm} + \rho g (\zeta_e - \zeta_i)_{|_{r=R}} + P_{cor}$$

Focus on the solid equation

$$(m + m_a(\delta_G))\ddot{\delta}_G(t) = -\mathfrak{c}\delta_G(t) + \mathfrak{c}\zeta_e(t, R) + \left(\frac{\mathfrak{b}}{h_e^2(t, R)} + \beta(\delta_G)\right)\dot{\delta}_G^2(t)$$

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Exterior problem:

$$\begin{cases} \partial_t \zeta_e + \partial_r q_e + \dfrac{q_e}{r} = 0 \\ \partial_t q_e + v_0^2 \partial_r \zeta_e = 0, \end{cases} \qquad \text{B.C.} \quad q_{e|_{r=R}} = -\frac{R}{2}\dot{\delta}_G(t).$$

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$$\partial_{tt}\zeta_e - v_0^2 \Delta_r \zeta_e = 0$$

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Linear wave equation

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Helmholtz equation with complex coefficients ($\mathcal{L}$ Laplace tranform):

$$\begin{cases} s^2 \mathcal{L}(\zeta_e) - v_0^2 \Delta_r \mathcal{L}(\zeta_e) = 0. \\ \partial_r \mathcal{L}(\zeta_e)_{|_{r=R}} = \dfrac{sR}{2v_0^2}\mathcal{L}(\dot{\delta}_G)(s) \end{cases}$$

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Theorem (E. B. '19)

Considering the linear-nonlinear hydrodynamical model (L-NL), the solid motion is governed by

$$(m + m_a(\delta_G))\ddot{\delta}_G = -\,\mathfrak{c}\delta_G - \nu\dot{\delta}_G + \mathfrak{c}\int_0^t F(s)\dot{\delta}_G(t-s)ds + \Big(\mathfrak{b}(\dot{\delta}_G) + \beta(\delta_G)\Big)\,\dot{\delta}_G^2\ , \tag{14}$$

The Cauchy problem for (14) *with* $\delta_0 \neq 0$ *and* $\delta_1 = 0$ *admits a unique solution* $\delta_G \in C^2([0,+\infty),\mathbb{R})$ *provided some admissibility condition on the initial datum* δ_0*.*

Assumption (numerical justification at this moment)

The impulse response function $|F(t)| \leqslant C\,t^{-2}$ *for* $t \geqslant t_0$*.*

Linearizing (14) around the equilibrium state, we get

$$(m + m_a(0))\,\ddot{\delta}_G(t) = -\mathfrak{c}\delta_G(t) - \nu\dot{\delta}_G(t) + \mathfrak{c}\int_0^t F(t-s)\dot{\delta}_G(s)ds \qquad (15)$$

which is a Cummins-type equation for the vertical motion.

Cummins equation[1] for the heave

$$\sum_{j=1}^{6}\left[(m_j\,\delta_{jk} + m_{jk})\,\ddot{x}_j + c_{jk}\,x_j + \int_{-\infty}^{t} K_{jk}(t-\tau)\,\dot{x}_j(\tau)\,d\tau\right] = f_k(t)$$

is implemented in naval architecture and hydrodynamical engineering.

[1] Cummins, W.E., *The Impulse Response Function and Ship Motions*, Navy Department, David Taylor Model Basin, 1962.

Numerical results

$h_0 = 15$ m, $R = 10$ m, $H = 10$ m, $\rho = 1000$ kg/m^3, $\rho_m = 0.5\ \rho$.

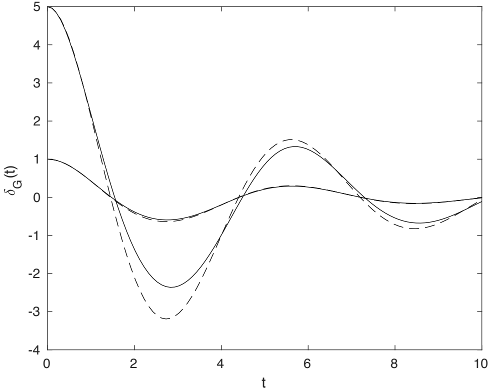

Figure: Time evolution of δ_G given by the nonlinear integro-differential (14) (full) and by the linear Cummins equation (15) (dash) for $\delta_0 = 1$ m and $\delta_0 = 5$ m.

Summary

- We do take into account nonlinear terms
- Validation of the shallow water approach to the floating body problem: several experimental data with an axisymmetric geometry
- Validation and improvement of the Cummins equation

Perspectives

- Add horizontal motion and rotation: evolution of the contact line + no axisymmetric flow (Iguchi-Lannes '18 in 1d)
- Large time behavior of the solution (almost done by Kai Koike: t^{-2}-decay)
- Study the nonlinear-nonlinear system for the return to equilibrium

THANK YOU FOR THE ATTENTION!