Functional inequalities and applications

B. Al TAKI

Sorbonne Université and INRIA-Paris

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Geophysical Fluids, Gravity Flows

Outline

- Part I: Weighted Sobolev inequalities
  - Applications to geophysics models: Lake equations
  - Difficulties...
- Part II: Logarithmic Sobolev inequalities
  - Application to gas dynamic system.
  - Diffusive capillary models of Korteweg type.

Importance of weighted spaces

Poincaré-Wirtinger's inequality: let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth, compactly supported

$$\int_\Omega |u - \bar{u}|^p \, dx \leq C \int_\Omega |\nabla u|^p \, dx \qquad \bar{u} = \frac{1}{|\Omega|} \int_\Omega |u| \, dx.$$

For $p = 2$, this inequality is the key tool of solving

$$(\mathcal{P}_1) \begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{cases}$$

for Ω smooth domain and f, g such that

$$f \in H^{-1}(\Omega), \qquad g \in H^{1/2}(\partial\Omega).$$

But: What happens when $f \notin H^{-1}(\Omega)$??

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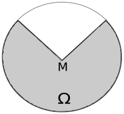
Example 1:

$$(\mathcal{P}_2)\begin{cases} -\Delta u &= -\operatorname{div} f \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{cases}$$



For $x \in \Omega$, we define f and d_M as follows

$$f(x) = |x|^{-N/p'} \quad x \in \Omega \qquad d_M := \operatorname{dist}(x, M) = |x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}.$$

Notice that

$$\|f\|^{p'}_{L^{p'}(\Omega)} = \int_\Omega |f(x)|^{p'}\,dx = \int_\Omega |x|^{-N}\,dx = \text{const}\int_0^1 r^{-1}\,dr = \infty,$$

However

$$\|f\|^{p'}_{L^{p'}(\Omega, d_M^\varepsilon)} = \int_\Omega |f(x)|^{p'}|x|^\varepsilon\,dx = \int_\Omega |x|^{-N+\varepsilon}\,dx = \text{const}\int_0^1 r^{-1+\varepsilon}\,dr < \infty.$$

$f \notin L^{p'}(\Omega)$ but $f \in L^{p'}(\Omega, d_M^\varepsilon) \Longrightarrow$ possibility of solution in $W^{1,p}(\Omega, d_m^\epsilon)$?

Remark

Details about Laplacian equation in weighted spaces can be found in the works of Farwig, Sohr,.

For Ω smooth domain, consider

$$\begin{cases} -\mathrm{div}(b\nabla u) & = b\,f \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Variational problem: find u such that

$$a(u,v) := \int_\Omega \nabla u \nabla v\, b\, dx = \int_\Omega f\cdot v\, b\, dx := L(v) \qquad v \in H^1_0(\Omega)$$

- If $0 < c_1 \le b(x) \le c_2 < \infty$

$$a(u,u) = \int_\Omega |\nabla u|^2 b\, dx \ge c_1 \|u\|_{H^1_0(\Omega)} \quad \Rightarrow \quad \text{Coercivity on Sob. space}$$

$$L(v) = \int_\Omega f\cdot v\, b\, dx \le c_2 \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \quad \Rightarrow \quad \text{Continuity on Sob. spaces}$$

Lax Milgram's Theo. $\Rightarrow$ existence. of sol. in $H^1_0(\Omega)$.

- When $b(x) \to 0$ or $b(x) \to \infty$ when $x \to \partial\Omega$,

We lose the coercivity and continuity in Sobolev spaces

What about existence of sol. in weighted Sob. spaces.??

If we define "formally" V_b as follows

$$\mathrm{V_b} = \left\{ v \text{ measurable s.t } \int_\Omega |\nabla v|^2 \, b \, dx < \infty \right\}$$

and suppose that

$$\|\nabla \cdot \|_{L^2_b(\Omega)} := \int_\Omega |\nabla \cdot |^2 \, b \, dx$$

define a norm. Then we can verify that

- $a(\cdot,\cdot)$ is bilinear and coercive,i.e.,

$$a(u,u) = \int_\Omega |\nabla u|^2 \, b \, dx \geq \|u\|^2_{V_b}$$

- $L(\cdot)$ is linear and continuous, i.e.,

$$|L(u)| = \left| \int_\Omega f \cdot v \, b \, dx \right| \leq C \|b^{1/2} f\|_{L^2(\Omega)} \|b^{1/2} u\|_{L^2(\Omega)}$$

- To finish the proof, it remains to verify that
  - $\|\nabla \cdot \|_{L^2_b(\Omega)}$ is a norm
  - V_b endowed with the "norm" $\|\nabla \cdot \|_{L^2_b(\Omega)}$ is a Hilbert space

In other words, the question is:

What conditions on the weight function b guarantee the validity of these two statements

Among the most known weights, we found the Muckenhoupt weight. That means

$$b \in \mathcal{A}_q \Longleftrightarrow \sup_Q \Big(\frac{1}{|Q|}\int_Q b\,dx\Big)\Big(\frac{1}{|Q|}\int_Q b^{-1/q-1}\,dx\Big)^{q-1} < \infty, \quad b \in L^1_{\mathrm{loc}}(\Omega)$$

This family of weights ensure that $W^{n,p}_b(\Omega)$ admits similar properties as $W^{n,p}(\Omega)$.

- Some examples in $\mathcal{A}_q$:

$$b(x) = |x - x_0|^\alpha \quad \text{or} \quad b(x) = \rho(x)^\alpha := \mathrm{dist}(x, \partial\Omega)^\alpha, \quad -(n-1) < \alpha < (n-1)(q-1)$$

Ref.

B. O. Turessson, Nonlinear Potential Theory and Weighted Sobolev Spaces (2000).

Importance of Muckenhoupt weight

$\Omega =]0,1[$ and u smooth s.t. $u|_{\partial\Omega} = 0$. Then if $b \in \mathcal{A}_q$, then we have

$$\int_0^1 |u|^q\, b\, dx \leq \int_0^1 |u'|^q\, b\, dx.$$

Proof. Since $u|_{\partial\Omega} = 0$, then we can write

$$u(x) = \int_0^x u'(y)\, dy \quad \text{hence} \quad |u|^q\, b = b \left| \int_0^x u'(y)\, dy \right|^q.$$

Hence

$$\begin{aligned}
\int_0^1 |u|^q\, b\, dx &= \int_0^1 \left(b \left| \int_0^x u'(y)\, dy \right|^q \right) dx \\
&\leq C \int_0^1 \left(b \left| \int_0^1 |u'(y)|\, dy \right|^q \right) dx \\
&\leq C \Big(\int_0^1 b\, dx \Big) \left| \int_0^1 |u'(y)|\, b^{1/q}\, b^{-1/q}\, dy \right|^q \\
&\leq C \Big(\int_0^1 b\, dx \Big) \Big(\int_0^1 b^{-1/q-1}\, dx \Big)^{q-1} \Big(\int_0^1 |u'(y)|^q\, b\, dx \Big).
\end{aligned}$$

Difficulties

♣ Trace: Characterization when $b = \rho^\alpha(x) = \mathrm{dist}^\alpha(x, \partial\Omega), \ -1 < \alpha < q - 1$ ($\Rightarrow b \in \mathcal{A}_q$)

$$T_b^{1,q} : W_b^{1,q}(\Omega) \longrightarrow W^{1-\frac{1+\alpha}{q},q}(\partial\Omega) \text{ bounded linear operator}$$

♣ Regular solution: $\Omega =]0,1[$, $b = \rho^\alpha$, $-1 < \alpha < 1$ and $\rho(x) = \mathrm{dist}(x, \partial\Omega)$

$$\begin{cases} -\partial_x(b(x)\partial_x u) & = f, \qquad \text{in } \Omega, \\ u & = 0, \qquad \text{on } \partial\Omega, \end{cases}$$

Straightforward computation yields to

$$\partial_x^2 u = -\frac{b'(x)}{(b(x))^2} \int_0^x f(z)\,dz + \frac{1}{b(x)} f(x) - \frac{b'(x)}{(b(x))^2}\, c.$$

However

$$\rho\partial_x^2 u \in L^2(\Omega) \Longleftrightarrow \int_0^1 \left(\frac{1}{\rho^\alpha(x)}\right)^2 dx < \infty \Longrightarrow \alpha < \frac{1}{2}.$$

♣ Higher dimension? Big issue!

Tangential regularity $\Longrightarrow$ Normal regularity?

Lake equations

Theo.

$$\begin{cases} \partial_t(bu^\mu) + \text{div}(bu^\mu \otimes u^\mu) - 2\mu\, \text{div}(bD(u^\mu) + b\,\text{div}\, u^\mu\, \mathbb{I}) + b\nabla p^\mu = 0 \\ \text{div}(bu^\mu) = 0 \end{cases}$$

Navier boundary condition

$$bu^\mu \cdot n = 0 \quad (bD(u^\mu) \cdot n + b\,\text{div}\, u^\mu\, \mathbb{I} \cdot n) \cdot \tau = \eta u \cdot \tau.$$

- If $b \in W^{1,\infty}_{\text{loc}}(\Omega)$ with $b = \rho^\alpha \;\; 0 < \alpha < 1$ ($\Rightarrow b \in \mathcal{A}_2$)

$$\Longrightarrow \exists!\; u^\mu \text{ sol. with } u^\mu \in L^\infty(0, T; L^2_b(\Omega)) \cap L^2(0, T; V_b) \cap C((0, T), H_b - \text{weak})$$

$$V_b = \{v \in H^1_b(\Omega),\; bv \cdot n = 0,\; \text{div}(bv) = 0\}.$$

- Moreover if $b = \rho(x)^\alpha$, $0 < \alpha < 1/2$, ($\Rightarrow b \in A_{3/2}$) for $x \in V(\partial\Omega)$,

$$\Longrightarrow \text{we can find } p^\mu \text{ with } \nabla p^\mu \in W^{-1,\infty}(0, T; H^{-1}_b(\Omega))$$

Work in progress with C. Lacave (Institute of Fourier)

- Regular solutions?
- What happen when μ tends to zero?

Logarithmic Sobolev inequalities

Sobolev inequality: let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth, compactly supported

$$\left(\int_{\mathbb{R}^n} |f|^p \, dx \right)^{2/p} \leq C \int_{\mathbb{R}^n} |\nabla f|^2 \, dx, \qquad p = \frac{2n}{n-2} (> 2) \quad n \geq 3$$

$$\Longrightarrow$$

$$\frac{2}{p} \log \left(\int_{\mathbb{R}^n} |f|^p \, dx \right) \leq C \log \left(\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right)$$

Assume $\int_{\mathbb{R}^n} f^2 \, dx = 1$, Jensen's inequality for $f^2 \, dx$

$$\log \left(\int_{\mathbb{R}^n} |f|^p \, dx \right) = \log \left(\int_{\mathbb{R}^n} |f|^{p-2} f^2 \, dx \right) \geq \left(\int_{\mathbb{R}^n} \log \left(|f|^{p-2} \right) f^2 \, dx \right)$$

$$\frac{p-2}{p} \int_{\mathbb{R}^b} f^2 \log f^2 \, dx \leq \log \left(\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right)$$

Form of logarithmic sobolev inequality

Different forms and applications

(Euclidean) Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right) \qquad \int_{\mathbb{R}^n} f^2 \, dx = 1$$

$$dx \longrightarrow d\mu(x) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

μ standard Gaussian probability measure on $\mathbb{R}^n$

change f^2 into $f^2 e^{-|x|^2/2}$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \text{ smooth s.t. } \int_{\mathbb{R}^n} f^2 \, d\mu = 1$$

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu$$

(Gaussian) Logarithmic Sobolev inequality

Different forms and applications

(Euclidean) Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \right) \qquad \int_{\mathbb{R}^n} f^2 \, dx = 1$$

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μ standard Gaussian probability measure on $\mathbb{R}^n$

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$f : \mathbb{R}^n \longrightarrow \mathbb{R}$, smooth s.t. $\int_{\mathbb{R}^n} f^2 \, d\mu = 1$

(entropy) $\int_{\mathbb{R}^n} f \log f \, d\mu \leq 2 \int_{\mathbb{R}^n} f |\nabla \log f|^2 \, d\mu$ (Fisher information)

(Gaussian) Logarithmic Sobolev inequality

Interest of Logarithmic Sobolev Inequalities

Many applications:

- Quantum Navier-Stokes

  Superfluids, quantum semiconductors, weakly interacting Bose gases,...

- Navier-Stokes-Korteweg

  liquid-vapour flows including phase transitions

- Compressible Navier-Stokes equations

  Gas dynamics

- Derrida-Lebowitz-Speer-Spohn equations:

  quantum semiconductor

- Dispersive N-S eq.([Sone, Birkhäuser Boston, Inc., '02])

  In some situation, Asymp. analysis of the Boltz. eq. for $\not\Rightarrow$ NS and Euler eqs.

- Probability (work by M. Ledoux, D. Bakry)

- ...

Quantum Navier-Stokes

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma - \mathrm{div}(\rho D(u)) = -\varepsilon \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right),$$

Bohm's identity

$$\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \mathrm{div}(\rho \nabla^2 \log \rho)$$

Need to estimates

$$-\int_\Omega \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) \cdot (u + \nabla \log \rho)\, dx = \frac{d}{dt} \int_\Omega |\nabla \sqrt{\rho}|^2 + \int_\Omega \rho |\nabla^2 \log \rho|^2\, dx.$$

Question: Information from

$$\int_\Omega \rho |\nabla^2 \log \rho|^2\, dx \geq \int_\Omega |\nabla \rho^{1/4}|^4\, dx$$

Global weak sol. of Quantum Navier-Stokes, Jungel (SIAM, 2010)

$\Downarrow$

Compressible Navier-Stokes ($\varepsilon = 0$)

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma - \mathrm{div}(\rho D(u)) = 0,$$

Global weak sol. of compressible Navier-Stokes, A. Vasseur, C. Yu (Invent. Math. 2016)

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Navier-Stokes Korteweg

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma - \mathrm{div}(\mu(\rho)D(u)) + \nabla(\lambda(\rho)\,\mathrm{div}\, u) = \varepsilon \rho \nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}ds)),$$

Generalized Bohm's identity (Bresch et al (2016))

$$\rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}ds)) = \mathrm{div}(F(\rho)\nabla\nabla\psi(\rho)) + \nabla\Big((F'(\rho)\rho - F(\rho))\Delta\psi(\rho)\Big)$$

with

$$\sqrt{\rho}\psi'(\rho) = \sqrt{K(\rho)},\ F'(\rho) = \sqrt{K(\rho)}\rho.$$

Difficulty: find an estimate/sign of ($s'(\rho) = \mu'(\rho)/\rho$)

$$\int_\Omega \rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}ds)) \cdot (u + \nabla s(\rho))\, dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\sqrt{K(\rho)}|^2\, dx + \int_\Omega F(\rho)\nabla\nabla\psi(\rho) : \nabla\nabla s(\rho)\, dx + \int_\Omega (F'(\rho)\rho - F(\rho))\Delta\psi(\rho))\, \Delta s(\rho)\, dx$$

Global weak sol. for $s(\rho) =?$ $\quad \psi(\rho) =?$

$\Downarrow$

Compressible Navier-Stokes ($\varepsilon = 0$)

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \rho^\gamma - \mathrm{div}(\mu(\rho)D(u)) + \nabla(\lambda(\rho)\,\mathrm{div}\, u) = 0,$$

Global weak sol. for $\mu(\rho) =?$ $\lambda(\rho) =?$ (In progress)

Question: Sign of

$$I := \int_\Omega F(\rho)\nabla\nabla\psi(\rho) : \nabla\nabla s(\rho)\, dx + \int_\Omega (F'(\rho)\rho - F(\rho))\Delta\psi(\rho))\, \Delta s(\rho)\, dx$$

We take particular case

$$\psi(\rho) = \rho^n \quad s(\rho) = \rho^m \quad \Longrightarrow \quad F(\rho) = \rho^{n+1} \quad (F'(\rho)\rho - F(\rho))\Delta\psi(\rho)) = n\rho^{n+1}$$

Thus

$$I = \int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\rho^m\, dx + n\int_\Omega \rho^{n+1}\Delta\rho^n\Delta\rho^m\, dx$$

Particular cases:

- Jungel, Matthes ($n = m = 0$)

$$\int_\Omega \rho|\nabla^2 \log\rho|^2\, dx \geq c\int_\Omega |\nabla\rho^{1/4}|^4\, dx$$

- Bresch, Vasseur, Yu, ($n = m$)

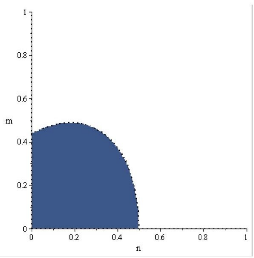
$$\int_\Omega \rho^{n+1}|\nabla\nabla\rho^n|^2\, dx \geq c\int_\Omega |\nabla\nabla\rho^{\frac{3n+1}{2}}|^2\, dx + \int_\Omega |\nabla\rho^{\frac{3n+1}{4}}|^4\, dx \quad \left(-1 + \frac{2}{d} < n < 1\right)$$

- A., (Key estimate of a publication about ghost effect system)

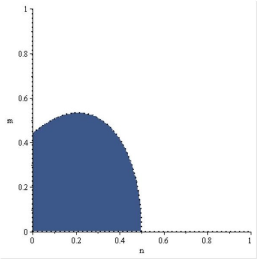
$$\int_\Omega \rho^{n+1}\nabla\nabla\rho^n : \nabla\nabla\log\rho\, dx + n\int_\Omega \rho^{n+1}\Delta\rho^n\Delta\log\rho\, dx \geq \int_\Omega \left(\Delta\rho^{\frac{2n+1}{2}}\right)^2 dx \quad (0 \leq n \leq 1/2)$$

Lemma. $\Omega \subset \mathbb{R}^d, d = 2, 3$, periodic domain and ρ positive smooth and n, m given in Figure 1, then $\exists\ C_1 > 0, C_2 > 0$ such that

$$\mathcal{I} = \int_\Omega \rho^{n+1} \nabla\nabla\rho^n : \nabla\nabla\rho^m \, dx + n \int_\Omega \rho^{n+1} \Delta\rho^n \Delta\rho^m \, dx$$
$$\geq C_1 \int_\Omega \left(\nabla\nabla\rho^{\frac{2n+m+1}{2}}\right)^2 + C_2 \int_\Omega |\nabla\rho^{\frac{2n+m+1}{4}}|^4 \, dx.$$



dimension 2



dimension 3

Figure:

Proof. Simple computations, we obtain

$$
\begin{aligned}
J_1 :&= \int_\Omega \rho^{n+1} \nabla\nabla\rho^n : \nabla\nabla\rho^m \, dx \\
&= \int_\Omega \rho^{n+1} \nabla\Big(\frac{n}{\theta}\,\rho^{n-\theta}\nabla\rho^\theta\Big) : \nabla\big(\frac{m}{\theta}\,\rho^{m-\theta}\nabla\rho^\theta\big)\, dx \\
&= \frac{n\,m}{\theta^2}\Big[\int_\Omega \rho^{2n+m-2\theta+1}|\nabla\nabla\rho^\theta|^2\,dx - \gamma_1\int_\Omega \rho^{2n+m+1-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx \\
&\quad -\gamma_2\int_\Omega \rho^{2n+1+m-\theta}\nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx + \gamma_1\,\gamma_2\int_\Omega \rho^{2n+m+1-\theta}(\nabla\rho^{\theta/2})^4\,dx\Big]
\end{aligned}
$$

with

$$\gamma_1 = \frac{4(\theta-m)}{\theta} \quad \gamma_2 = \frac{4(\theta-n)}{\theta}.$$

Now, let us choose θ such that

$$\theta = \frac{2n+m+1}{2}.$$

Thus the integral J_1 becomes

$$J_1 := \frac{n\,m}{\theta^2}\Big[\int_\Omega |\nabla\nabla\rho^\theta|^2\,dx - (\gamma_1+\gamma_2)\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx + \gamma_1\gamma_2\int_\Omega (\nabla\rho^{\theta/2})^4\,dx\Big]$$

Similar computation give us

$$J_2 = \int_\Omega \rho^{n+1} \Delta\rho^n \Delta\rho^m \, dx$$
$$= \frac{n\,m}{\theta^2}\Big[\int_\Omega |\Delta\rho^\theta|^2 \, dx - (\gamma_1+\gamma_2)\int_\Omega \Delta\rho^\theta \, (\nabla\rho^{\theta/2})^2 \, dx + \gamma_1\gamma_2 \int_\Omega (\nabla\rho^{\theta/2})^4 \, dx\Big].$$

Gathering J_1 and J_2 together and take in mind that

$$\int_\Omega |\nabla\nabla\rho^\theta|^2 \, dx = \int_\Omega |\Delta\rho^\theta|^2 \, dx$$

We infer with

$$\mathcal{I} = \frac{n\,m}{\theta^2}\Big[(1+n)\int_\Omega |\nabla\nabla\rho^\theta|^2 \, dx + \gamma_1\gamma_2(1+n)\int_\Omega (\nabla\rho^{\theta/2})^4 \, dx$$
$$- (\gamma_1+\gamma_2)\Big(\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2} \otimes \nabla\rho^{\theta/2} \, dx + n\int_\Omega \Delta\rho^\theta \, (\nabla\rho^{\theta/2})^2 \, dx\Big)\Big].$$

We want to establish an estimate on

$$-(\gamma_1+\gamma_2)\Big(\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2} \otimes \nabla\rho^{\theta/2} \, dx + n\int_\Omega \Delta\rho^\theta \, (\nabla\rho^{\theta/2})^2 \, dx\Big).$$

Remark that

$$\rho\nabla\nabla\log\rho = \nabla\nabla\rho - 4\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}, \qquad \rho\Delta\log\rho = \Delta\rho - 4(\nabla\sqrt{\rho})^2$$

This implies that

$$\int_\Omega \rho^2|\nabla\nabla\log\rho|^2\,dx = \int_\Omega |\nabla\nabla\rho|^2\,dx + \int_\Omega |2\nabla\sqrt{\rho}|^4\,dx - 8\int_\Omega \nabla\nabla\rho : \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}\,dx$$

$$\int_\Omega \rho^2|\Delta\log\rho|^2\,dx = \int_\Omega |\Delta\rho|^2\,dx + \int_\Omega |2\nabla\sqrt{\rho}|^4\,dx - 8\int_\Omega \Delta\rho\,(\nabla\sqrt{\rho})^2\,dx$$

Therefore

$$\begin{aligned}
&-(\gamma_1+\gamma_2)\Big(\int_\Omega \nabla\nabla\rho^\theta : \nabla\rho^{\theta/2}\otimes\nabla\rho^{\theta/2}\,dx + n\int_\Omega \Delta\rho^\theta\,(\nabla\rho^{\theta/2})^2\,dx\Big)\\
&= \frac{(\gamma_1+\gamma_2)}{8}\int_\Omega \rho^{2\theta}|\nabla\nabla\log\rho^\theta|^2\,dx + \frac{n(\gamma_1+\gamma_2)}{8}\int_\Omega \rho^{2\theta}|\Delta\log\rho^\theta|^2\,dx\\
&\quad - \frac{(1+n)(\gamma_1+\gamma_2)}{8}\int_\Omega |\nabla\nabla\rho^\theta|^2\,dx - 2(1+n)(\gamma_1+\gamma_2)\int_\Omega |\nabla\rho^{\theta/2}|^4\,dx
\end{aligned}$$

And remember that

$$\int_\Omega \rho^{2\theta}|\nabla^2\log\rho^\theta|^2\,dx + n\int_\Omega \rho^{2\theta}|\Delta\log\rho^\theta|^2\,dx \geq c\int_\Omega |2\nabla\rho^{\theta/2}|^4\,dx.$$

Thank you for your attention