TOLOSA
Stability of Numerical scheme for some systems connected with Shallow Water equations

1 Modified Equation for a Numerical FD scheme and its associated PDE.

The aim is to Understand the concept of Modified Equation for a Numerical FD scheme and its associated PDE.

1.1 Formal Construction of the modified equation

Once an explicit scheme is linearized,

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=\sum_{l}a_l(\Delta x)\,u(x+l\Delta x,t)$$

Via Fourier transform we get

$$\frac{\hat{u}(k,t+\Delta t)-\hat{u}(k,t)}{\Delta t}=\left(\sum_{l}a_l(\Delta x)\,e^{ikl\Delta x}\right)\hat{u}(k,t)=g_{\Delta x}(k\Delta x)\,\hat{u}(k,t)$$

The Von Neumann stabilty condition is

$$\sup_{|k|<\frac{1}{\Delta x}}\|1+\Delta t g_{\Delta x}(k\Delta x)\|\leq 1$$

We search for a modified equation

$$u_t=\sum_{p}\gamma_p u_{(px)}$$

where $\gamma_p(\Delta t,\Delta x)$

$$\hat{u}(k,t)_t=\left(\sum_{p}\gamma_p(-ik)^p\right)\hat{u}(k,t)$$

$$\hat{u}(k,t)=\left(\exp\left(\sum_{p}\gamma_p(-ik)^p\right)t\right)\hat{u}(k,0)$$

$$\hat{u}(k,t+\Delta t)=\left(\exp\left(\sum_{p}\gamma_p(-ik)^p\right)\Delta t\right)\hat{u}(k,t)$$

$$G(k,\Delta t)=\exp\left(\sum_{p}\gamma_p(-ik)^p\right)\Delta t$$

is the symbol of the modified equation.

Practically we troncate the $\sum_p \gamma_p\left(\Delta t, \Delta x\right)(-ik)^p$ by sorting the above sum with respect to Δt or Δx and making an additionnal assumption such as $\frac{\Delta t}{\Delta x} = cst = r_1$or $\frac{\Delta t}{(\Delta x)^2} = cst = r_2$ in order to cope with expected stability condition of the associated scheme.

The problem is the link of stability of the scheme :

$$\sup_{|k|<\frac{\pi}{\Delta x}} \left\|1 + \Delta t g_{\Delta x}\left(k\Delta x\right)\right\| \leq 1$$

and some well posedness of the Modified truncated equation:

Classically the condition is

$$sup_{k\in\mathbb{R}}\mathcal{R}_e\left(\sum_{p\leq N_T}\gamma_p\left(-ik\right)^p\right) < C$$

and the condition is given by the sign of the higher order even term.

Here we need to analyse further this requirement by considering that we cannot see short wave length (smaller than the size of the mesh) and require that

$$|k| < \frac{\pi}{\Delta x}$$

and thus use as a new criterium

$$sup_{|k|<\frac{\pi}{\Delta x}}\mathcal{R}_e\left(\sum_{p\leq N_T}\gamma_p\left(-ik\right)^p\right) < C$$

where C is independent of Δx

We define the complete modified equation as the one such as its symbol is equal to the one of the scheme

$$\left(1 + \Delta t g_{\Delta x}\left(k\Delta x\right)\right) = \exp\left(\sum_p \gamma_p\left(-ik\right)^p\right)\Delta t$$

Under the additional condition $\sup_{|k|<\frac{1}{\Delta x}}\left\|\Delta t g_{\Delta x}\left(k\right)\right\| < 1$

$$\ln\left(1 + \Delta t g_{\Delta x}\left(k\Delta x\right)\right) = \Delta t\sum_{p\geq 0}(-1)^p\left(\Delta t\right)^p\frac{\left(g_{\Delta x}\left(k\Delta x\right)\right)^{p+1}}{p+1} = \sum_p \gamma_p\left(-ik\right)^p\Delta t$$

$$\sum_p(-1)^p\left(\Delta t\right)^p\frac{\left(g_{\Delta x}\left(k\Delta x\right)\right)^{p+1}}{p+1} = \sum_p \gamma_p\left(-ik\right)^p$$

Usually $g_{\Delta x}\left(k\Delta x\right)$ is a serie in $k\Delta x$ of convergence radius equal to ∞ and we can identify the coefficients of the rhs in an unique way

1.2 Examples and Counter Examples

1.2.1 Upwind Scheme for Convection equation

1.2.2 Central scheme for Diffusion equation

2 Linear Stability of SW equation with Surface Tension

The aim is to review and clarify some results in the paper by Noble and Vila "Stability Theory for Difference Approximations of Euler--Korteweg Equations and Application to Thin Film Flows. Augmented formulation for Shallow water with surface tension" SIAM Journal on Numerical Analysis Volume 52 Numéro 6 Pages 2770-2791

Linear unstability of classical upwind scheme of Godunov type is established both for explicit and implicit schemes. Lax Friedrichs type scheme are shown to be linerly stable under time step restriction.

An augmented version is shown to be stable both for implicit and explicit scheme (under time step restriction) with numerical viscosity compatible with enytropy variables.

We ask to study more prcisely linear stability conditions in some situation:

- original formulation with low froude scheme in

Martin Parisot and Jean-Paul Vila

Centered-Potential Regularization for the Advection Upstream Splitting Method SIAM J. Numer. Anal., 54(5), 3083–3104.

F. Couderc, A. Duran, J.-P. Vila An explicit asymptotic preserving low Froude scheme for the multilayer shallow water model with density stratification Journal of Computational Physics 343 (2017) 235–

Arnaud Duran, Jean-Paul Vila, Rémy Baraille "Semi-implicit staggered mesh scheme for the multi-layer shallow water system" Comptes Rendus Mathematique Volume 355 Numéro 12 Pages 1298-1306 Éditeur Elsevier Masson

Augmented formulation in the general case

Finally we aim to analyse modified equation associated to these schmes and try to understand stability vs well posedness of such modified equation.

3 Linear Stability of Low Mach scheme with Coriolis effect

The strucure is

$$U_t + AU_x = MU$$

with Mantisymetric ....

To be completed

R.FWarming B.JHyett The modified equation approach to the stability and accuracy analysis of finite-difference methods,Journal of Computational Physics Volume 14, Issue 2, February 1974, Pages 159-179

D. F. GRIFFITHS, J. M. SANZ-SERNA On the Scope of the Method of Modified Equations, SIAM J. ScI. STAT. COMPUT. Vol. 7, No. 3, July 1986

F.R. Villatoro *, J.I. Ramos On the method of modi®ed equations. I to VI: Asymptotic analysis of the Euler forward difference method Applied Mathematics and Computation 103 (1999) 111±139