Numerical methods for the resolution of variational Mean Field Games

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1 Background

In this project we will consider the following time-dependent MFG system

$$
\begin{array}{rcl}
\partial_t u - \nu \Delta u + H(x,t,\nabla u) & = & f(x,t,m(x,t)) \quad \text{in } \mathbb{T}^d \times]0,T[, \\
\partial_t m - \nu \Delta m - \operatorname{div}\left(\partial_p H(x,t,\nabla u) m\right) & = & 0 \quad \text{in } \mathbb{T}^d \times]0,T[, \\
u(\cdot,T) = g(\cdot, m(T)) & & m(\cdot,0) = m_0(\cdot), \\
\int_{\mathbb{T}^n} m(x,t) \mathrm{d}x = 1 \quad \forall\, t \in [0,T], & & m \geq 0.
\end{array}
\tag{MFG}
$$

In the system above $\nu \geq 0$, $\mathbb{T}^d$ is the d-dimensional torus, $(x,t,p) \in \mathbb{T}^d \times]0,T[\times \mathbb{R}^d \mapsto H(x,t,p) \in \mathbb{R}$ is the given *Hamiltonian*, which is convex w.r.t. p, and $f : \mathbb{T}^d \times]0,T[\times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ are given and usually called *coupling functions*.

System (MFG) has been introduced by J.-M. Lasry and P.-L. Lions in [25, 26] with the aim of simplifying the theoretical and numerical analysis of stochastic differential games with a large number of *small* and *indistinguishable* players. In the first equation in (MFG) the unknown function u corresponds to the optimal cost of a typical player who solves an optimal control problem whose cost depends on the statistical distribution m of all the agents. The time evolution of m is characterized by the second equation in (MFG), when the agents act optimally. We refer the reader to [14, 21] for surveys on MFG theory and a more detailed description of the model behind system (MFG).

The numerical resolution of (MFG), using finite-difference schemes, has been studied in [2, 1, 4, 5]. The finite dimensional analogue of (MFG) is solved by using Newton's method (see also [13]). As it is well known, Newton's method is very fast and, when convergence is guaranteed, only few iterations are needed in order to obtain a sharp approximation of the solution. On the other hand, the method ensures the convergence only if the initial guess is sufficiently near the exact solution and, in the context of MFGs, the stability of the methods depends heavily on the viscosity parameter ν, as $\nu \downarrow 0$. In particular, several difficulties are encountered in the first order case $\nu = 0$.

If f and g are increasing w.r.t. to m, (MFG) can be seen, at least formally, as the optimality condition of the following *infinite-dimensional* variational problem (see [26])

$$
\begin{array}{l}
\inf_{(m,w)} \int_0^T \int_{\mathbb{T}^d} \left[b(x,t,m(x,t),w(x,t)) + F(x,t,m(x,t)) \right] \mathrm{d}x + \int_{\mathbb{T}^d} G(x,m(x,T)) \mathrm{d}x, \\
\text{subject to} \quad \begin{array}{rcl} \partial_t m - \nu \Delta m + \operatorname{div}(w) & = & 0 \;\text{ in } \mathbb{T}^d \times]0,T[, \\ m(\cdot,0) & = & m_0(\cdot) \;\text{ in } \mathbb{T}^d, \end{array}
\end{array}
\tag{P}
$$

where, denoting by $H^*(x,t,\cdot)$ the Fenchel conjugate of $H(x,t,\cdot)$ for a.a. $x\in\mathbb{T}^d$, $t\in]0,T[$,

$$F(x,t,m):=\begin{cases}\int_0^m f(x,t,m')\mathrm{d}m' & \text{if } m\geq 0,\\ +\infty & \text{otherwise,}\end{cases}\qquad G(x,m):=\begin{cases}\int_0^m g(x,m')\mathrm{d}m' & \text{if } m\geq 0,\\ +\infty & \text{otherwise,}\end{cases}$$

$$b(x,t,m,w):=\begin{cases} mH^*\left(x,t,-\frac{w}{m}\right) & \text{if } m>0,\\ 0, & \text{if } (m,w)=(0,0),\\ +\infty, & \text{otherwise.}\end{cases}$$

If we consider the finite-difference discretization of (MFG), introduced in [2], then the previous procedure can be rigorously justified, at this discrete level, and thus the resolution of the finite-difference scheme is *equivalent* to the resolution of a *finite-dimensional convex optimization problem*, that we will call (P_h). In this framework, several algorithms can be implemented in order to solve (P_h) (see [7] for detailed account of a large class of efficient first order methods). The major features of these algorithms and their advantages compared with Newton's method, mentioned above, are that they are *global*, i.e. they converge independently of the initial guess, and, in the context of MFGs, they are robust as $\nu\downarrow 0$. In this context, the the Alternating Direction Method of Multipliers (ADMM), introduced in [20, 19, 18], which is a variation of the classical Augmented Lagrangian method (see [22, 23, 28]), has been applied to solve MFG system in [8, 9] when $\nu=0$. Also in the case of null viscosity, and following [12], the test of several first order methods to solve Mean Field Type Control problems (see [10]) is the subject of current research [11]. If $\nu>0$, the ADMM method has been used to solve (MFG) in [6]. The main difficulty here is that, at each iteration of the method, we need to invert a matrix that corresponds to the discretization of a differential operator of second order in time and of fourth order in space. In [6] the problem is solved by using suitable preconditioners which allow to decouple the time variable and so to simplify considerably the computations.

2 Project objectives

In this project we will focus on the second order case, i.e. $\nu>0$, which can be small, justifying the use of variational techniques instead of Newton's method. The following problems will be addressed:

1.- The test of several first order methods based on *proximal operators* (see [7, 12]) to solve the discretized version of (MFG). In particular, following the ideas in [4, 6], we will explore the implementation of several preconditioners to solve the linear problems which appear in the iterations.

2.- In the context of Mean Field Type Control problems with positive viscosity (see [10]), we will also consider problems involving the so-called *soft congestion* effects, in which the cost on the velocity of the agents at point $x\in\mathbb{T}^d$ at time t increases w.r.t. the density $m(x,t)$ of the agents at x at time t. We will compare the numerical resolution of these problems using proximal methods with the numerical resolution using the ADMM method (see [3]).

3.- We will also consider problems involving *strong congestion effects* on which the density of the agents is constrained to not exceed a certain bound at each point in the space. We refer the reader to [27] and [12] for theoretical and numerical studies in the case of stationary problems.

4.- Another class of interesting problems is the so-called "planing problems" on which the final distribution of the agents is prescribed. We will apply our techniques to solve numerically these problems, presenting a variational (and stable) alternative to the analysis in [1], where the numerical solution is found by using Newton's method.

5.- Finally, by using probabilistic techniques and the numerical solutions of system (MFG), we will simulate the stochastic trajectories for the typical agents and we will analyse their long time behaviour, making the connection with the stationary version of (MFG) (see [24, 15, 16, 17]).

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