PROJECT ON MFG

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1. General Background

Mean field game theory was initiated a decade ago in independent contributions by Lasry and Lions [29, 30, 31] and by Huang, Caines and Malhamé [26, 27].

The general purpose is to address stochastic differential games with a large number of players subject to mean field interactions. Numerous works on the theory have been dedicated to the analysis of the asymptotic formulation of the game, which is precisely referred to as a mean field game. In his lectures at Collège de France, see also the lecture notes by Cardaliaguet [9], Lions [32] exposed most of the background of the analytical approach. Since then, alternative strategies, including probabilistic ones, have been suggested, see for instance [11] or the forthcoming monograph [12, 13] together with the textbook [8].

The rationale for regarding the limit under the number of players is pretty clear: Generally speaking, games with a large number of players are known to be of a high complexity; because of the underlying mean field structure, equilibria are expected to be of a somewhat simpler structure in the asymptotic regime. This is indeed a key feature of mean field particle systems that, asymptotically, particles are not only statistically identical but become also independent, this latter fact being usually known as *propagation of chaos*, see the seminal lecture notes by Sznitman [34]. Equivalently, the limiting behavior of a mean field particle system may be summed up through the dynamics of a sole representative particle interacting with its own distribution. When recast within games with a large number of players, propagation of chaos yields the following picture. First, an equilibrium (or a solution) of the asymptotic mean field game should consist of a flow of marginal distributions $(\mu_t)_{0\le t\le T}$ accounting for the statistical states of the population (when in equilibrium) at any time $t \in [0, T]$, where T is the time duration of the game. Second, it should suffice to focus on a typical player (or particle) interacting with the flow $(\mu_t)_{0\le t\le T}$ instead of regarding the whole collection of players. Given $(\mu_t)_{0\le t\le T}$, the player (or particle) aims at optimizing some strategy (say for instance its velocity if speaking of a particle) in order to minimize its own energy, say for instance its kinetic energy (which directly depends on the velocity) together with some potential energy (which may include the interaction with the environment). Last, the Nash condition underpinning the notion of equilibrium says that the environment $(\mu_t)_{0\le t\le T}$ forms an equilibrium if the best response under the flow $(\mu_t)_{0\le t\le T}$ fits $(\mu_t)_{0\le t\le T}$ exactly, namely if the collection of marginal laws of the optimal state is $(\mu_t)_{0\le t\le T}$ itself. In short, an equilibrium is a fixed point for an application mapping a flow of statistical distributions onto another flow of statistical distributions.

From the analytical point of view, fixed points may be characterized as solutions of a forward-backward system made of two equations, a forward Fokker-Planck equation and a backward Hamilton-Jacobi-Bellman equation. The second one accounts for the optimization part in the search of an equilibrium, whilst the first one is used to solve for the Nash condition. The probabilistic approach obeys more or less the same principle, as it also relies on a forward-backward system, but of a somewhat different structure. Precisely, this forward-backward system reads as a forward-backward stochastic differential equation of the McKean-Vlasov type. The forward component provides the form of the optimal trajectories of the stochastic optimization problem whilst the

McKean-Vlasov condition enforces the fixed point constraint following from the Nash condition. Whatever the approach, the key problem is to sort out the forward-backward structure arising in the characterization. It is indeed known that the Cauchy-Lipschitz theory for forward-backward systems only applies when T is small enough. Rephrased in our setting, this says that, when T is fixed, equilibria of mean field games cannot be systematically constructed by a straightforward contraction argument. Most of the time, it requires another method; for instance, it may be based on a fixed point theorem without uniqueness. Subsequently, uniqueness for mean field games is known in very few cases; for example, it holds true when the coefficients satisfy specific monotonicity conditions, which we shall illustrate below.

Although there is no specific reason for expecting it to hold true in full generality, uniqueness is however an important question. When it fails, it may be a very difficult question to select one of the equilibria. Also, uniqueness may be very useful for justifying the convergence of equilibria of games with finitely many players towards solutions of mean field games. Proving the convergence of finite player games is indeed a difficult problem. When the finite player equilibria are taken over open loop strategies, compactness arguments, without any need for asymptotic uniqueness, may be used, see for instance [22, 28]; however, this strategy fails when equilibria are computed over strategies in closed loop form. In the latter case, the only strategy that has been known so far for tackling the convergence problem requires uniqueness, see [10]. The idea for doing so goes back to another key object due to Lions, which is known as the *master equation*. Whenever uniqueness holds, the forward backward system used to characterize the solution of the mean field game (whatever the approach is analytical or probabilistic) may be regarded as the characteristics of a nonlinear PDE set on the space of probability measures. This latter PDE is precisely the master equation. Whenever the master equation has a classical solution, say $\mathcal{U}$, see for instance [10, 16] for solvability results in that direction, convergence may be proved by letting $\mathcal{U}$ act onto the empirical distribution of the equilibria of the finite player game.

2. Numerical Aspects

Generally speaking, the goal of the project is to address numerical approximation of solutions to mean field games by means of a probabilistic approach.

2.1. A first example.

In its PhD dissertation, Alanko [4] developed a numerical method for mean field games based upon a Picard iteration: Given the proxy for the equilibrium distribution of the population (which is represented by the mean field component in an FBSDE of the aforementioned form), one solves for the value function by approximating the solution of the (standard) BSDE associated with the control problem; given the solution of the BSDE, we then get a new proxy for the equilibrium distribution and so on... Up to a Girsanov transformation, the BSDE associated with the control problem coincides with the backward equation in the above FBSDEs. In [4], the Girsanov transformation is indeed used to decouple the forward and backward equations and it is the keystone of the paper to address the numerical impact of the change of measure onto the mean field component. Loosely speaking, this method consists more or less in solving for the backward equation given a proxy for the forward equation and then in iterating, which is what we call below the *Picard method* for the FBSDE system. Unfortunately, convergence of the Picard iterations is a difficult issue, as the convergence is known in small time only. It is indeed well-known that Picard theorem only applies in small time for fully coupled problems.

2.2. A second example.

In [17], the authors proposed another algorithm based on BSDEs, but, in contrast with the previous one, its convergence is known on any interval of a given length, provided that the underlying mean field game has a unique solution. This algorithm may be regarded as a numerical scheme for solving the aforementioned *master equation* for the underlying mean-field game. In the case when the master equation reduces to a classical nonlinear PDE, a similar question has been addressed by several authors, among which [19, 20] and [7], but all these methods rely on the Markov structure of the problem. In mean field games, the Markov property is true but at the price of regarding the entire $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ as state space: The fact that the second component is infinite dimensional makes (at least at first sight) intractable the complexity of these approaches. To avoid any similar problem, the authors in [17] use a pathwise approach for the forward component; it consists in iterating successively the *Picard method* on small intervals, all the Picard iterations being implemented with a tree approximation of the Brownian motion. This strategy is inspired from the method of continuation, the parameter in the continuation argument being the time length T itself.

2.3. **Other methods.** We refer to the following papers for other numerical methods, based upon finite differences or variational approaches, for mean field games: [1, 2, 3] and [6, 25, 24]. Recently, a Markov chain approximation method was also suggested in [5].

3. Objectives of the Project

The objectives of the project are twofold. The first one is to implement the existing methods on $1d$, $2d$ or $3d$ benchmark examples and to discuss their numerical accuracy on these models. The second one is address refinements or generalizations of the aforementioned methods:

(1) The Girsanov transformation used by Alanko [4] may be regarded as a "Feynman-Kac formula". This suggests to implement, as an alternative method, a particle method, see for instance the monograph by Del Moral [21].
(2) As already said, the fact that the state space is infinite dimensional makes the problem of a high complexity. It would be interesting to discuss cases when the state space can be reduced to a space of finite dimension or examples that can be approximated by finite dimensional problems, see for instance the last chapter in [13].
(3) The solutions of some types of mean field games may be directly constructed by iterative methods, see for instance [15]. These iterative methods could be addressed from the numerical point of view.
(4) It might be interesting to think of extensions to MFG with a common noise, see for instance [14]. This might be helpful in some simple benchmark cases when uniqueness fails, see for instance [23].

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