Optimal Vector Quantization: from signal processing to clustering and numerical probability

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CEMRACS 2017

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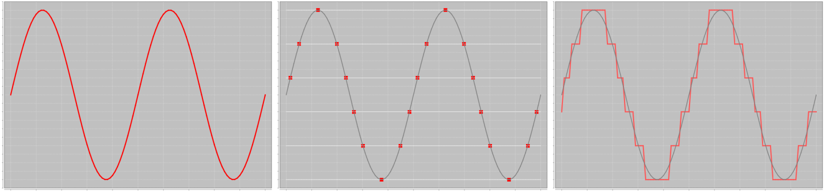
19th July 2017

What is Vector Quantization?

- Has its origin in the fields of signal processing in the late 1940's
- Describes the discretization of a random signal and analyses its recovery/reconstruction from the discretized one.



- Examples: Pulse-Code-Modulation (PCM), JPEG-Compression
- Signal: *Learning Vector Quantization* Extensive Survey about the IEEE-History: Gersho & Gray [GN98], 1998.
- Probability Theory: *Foundation of Quantization for Probability Distributions*: S. Graf & H. Luschgy in [GL00], 2000.
- and (survey, G.P.) *Optimal Vector Quantization and Application to Numerics*, in ESAIM Proc&Survey ([Pag15]), 2015.
- Statistics: unsupervised learning, clustering (k-means, nuées dynamiques), Mc Queen (*CLVQ* [Mac67], 1967), S.P. Lloyd (Lloyd I [Llo82], 1982 but. . .)

At the beginning was rough quantization

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▷ Let $X : (\Omega, \mathcal{S}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d), |\cdot|)$ be a random vector such that

$$\mathbb{E}|X|^p < +\infty \qquad \text{for some } p \in (0, +\infty).$$

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$$\widehat{X} = q(X)$$

is called a quantization of X.

▷ Example: if X is $[0,1]$-valued, one may choose a mid-point quantization

$$q(x) = \frac{2k-1}{2N}, \;\; \text{if } \frac{k-1}{N} \le x \le \frac{k}{N}, \; x \in [0,1].$$

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▷ L^p-mean quantization error induced by q:

$$e_{p,N}(X; q) = \big\|X - q(X)\big\|_p = [\mathbb{E}|X - q(X)|^p]^{\frac{1}{p}}$$

Voronoi Quantization (from Signal transmission to Numerical probability)

▷ Geometric optimization: For a *fixed* grid Γ,

$$|X - q(X)| \geq \mathrm{dist}\big(X, \Gamma\big).$$

Can this inequality hold as an equality for an appropriate $q : \mathbb{R}^d \to \Gamma$?

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▷ Given a (finite) "grid" $\Gamma = \{x_1, x_2, \ldots, x_N\} \subset \mathbb{R}^d$, we define a (Borel) Nearest Neighbor projection.

- Let $\big(C_i(\Gamma)\big)_{1\leq i\leq N}$ be a *Voronoi partition* of $\mathbb{R}^d$ generated by Γ, *i.e.* such that

$$C_i(\Gamma) \subset \Big\{z \in \mathbb{R}^d : |z - x_i| \leq \min_{1\leq j\leq N} |z - x_j|\Big\}.$$

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- Let $\pi_\Gamma : \mathbb{R}^d \to \Gamma$ the induced Γ-Nearest Neighbor projection,

$$\xi \mapsto \sum_{i=1}^{N} x_i \mathbf{1}_{C_i(\Gamma)}(\xi).$$

so that

$$|\xi - \pi_\Gamma(\xi)| = \mathrm{dist}(\xi, \Gamma)$$

⇒ We define the *Voronoi Quantization* of the random vector X as

$$\widehat{X}^{\Gamma} = \pi_{\Gamma}(X) = \sum_{i=1}^{N} x_i \mathbf{1}_{C_i(\Gamma)}(X).$$

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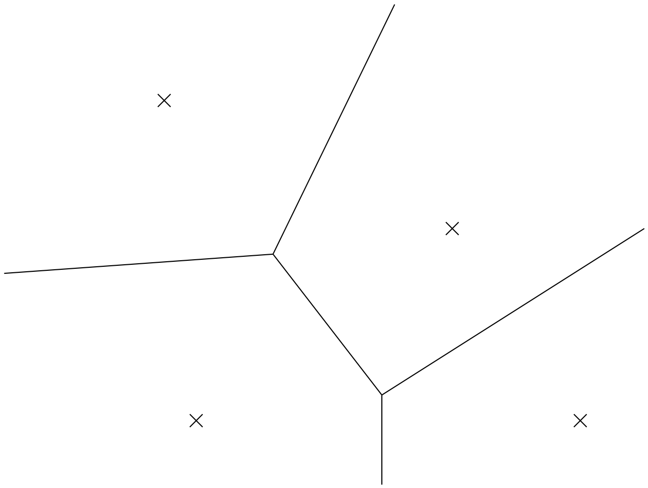

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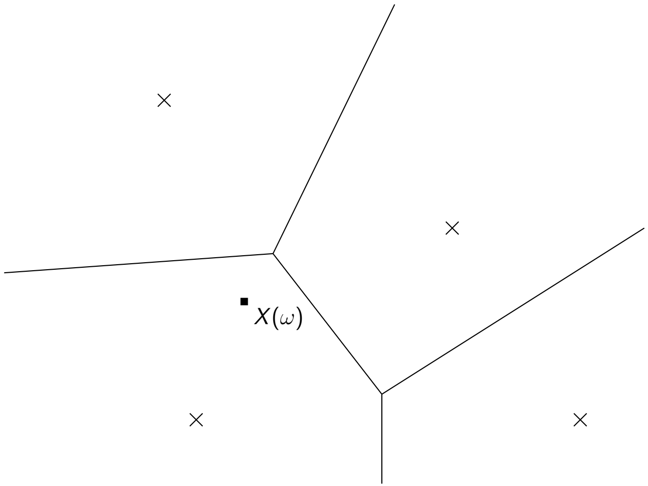

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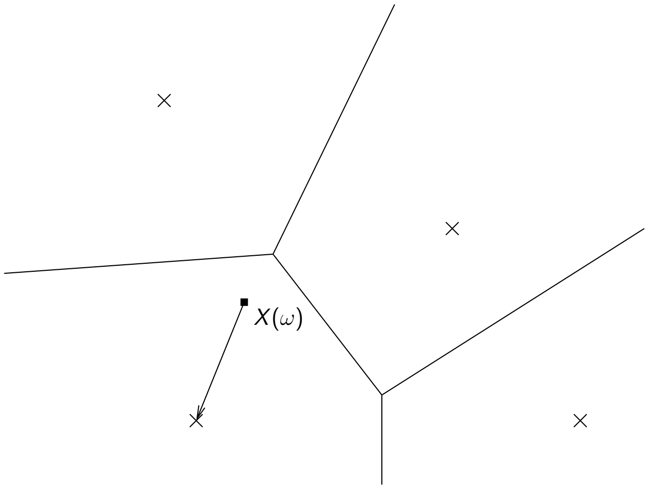

Starting with (optimal) quantization theory (Signal/probability)

▷ Quantization Theory starts when getting interested to the L^p-mean of this pointwise error

$$\left\|\mathrm{dist}(X,\Gamma)\right\|_1 = \mathbb{E}\,\mathrm{dist}(X,\Gamma) \quad \text{or} \quad \left\|\mathrm{dist}(X,\Gamma)\right\|_2 = \left[\mathbb{E}\,\mathrm{dist}(X,\Gamma)^2\right]^{\frac{1}{2}}.$$

▷ Why? If F is Lipschitz continuous

$$\left|\mathbb{E}F(X) - \mathbb{E}F(\widehat{X}^{\Gamma})\right| \le [F]_{\mathrm{Lip}}\left\|X - \widehat{X}^{\Gamma}\right\|_1 = \|\mathrm{dist}(X,\Gamma)\|_1$$

and, since $\xi \mapsto \mathrm{dist}(\xi,\Gamma)$ is 1-Lipschitz, one has

$$\sup_{[F]_{\mathrm{Lip}}\le 1}\left|\mathbb{E}F(X) - \mathbb{E}F(\widehat{X}^{\Gamma})\right| = \left\|X - \widehat{X}^{\Gamma}\right\|_1 = \left\|\mathrm{dist}(X,\Gamma)\right\|_1.$$

hence

$$\left\|\mathrm{dist}(X,\Gamma)\right\|_1 = \mathcal{W}_1\big(\mathcal{L}(X),\mathcal{P}_{\Gamma}\big)$$

i.e. the L^1-Wasserstein distance between $\mathcal{L}(X)$ and the set $\mathcal{P}_{\Gamma}$ of Γ-supported distributions.

▷ Signal Transmission: $\left\|\mathrm{dist}(X,\Gamma)\right\|_{1-2}$ measures the mean error transmission of the signal.

Classification point of view (Clustering/Unsupervised learning)

- Dataset $(\xi_k)_{k=1,\ldots,n}$.
- The random variable X models the sampling of one data uniformly at random in the dataset *i.e.*

$$\mathbb{P}_X = \frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k}$$

- Γ is a set of *prototypes* (codewords, elementary quantizers, ...) of size $N \ll n$.
- The above L^1-mean error the reads

$$\big\|\mathrm{dist}(X,\Gamma)\big\|_1 = \frac{1}{n}\sum_{k=1}^{n} \min_{1\le i\le N} \big|\xi_k - x_i\big|$$

as a measure of how the set of prototypes Γ "sums up" $(\xi_k)_{k=1,\ldots,n}$.

- Idem in the quadratic sense with

$$\big\|\mathrm{dist}(X,\Gamma)\big\|_2^2 = \frac{1}{n}\sum_{k=1}^{n} \min_{1\le i\le N} \big|\xi_k - x_i\big|^2$$

Clustering of a (small) dataset

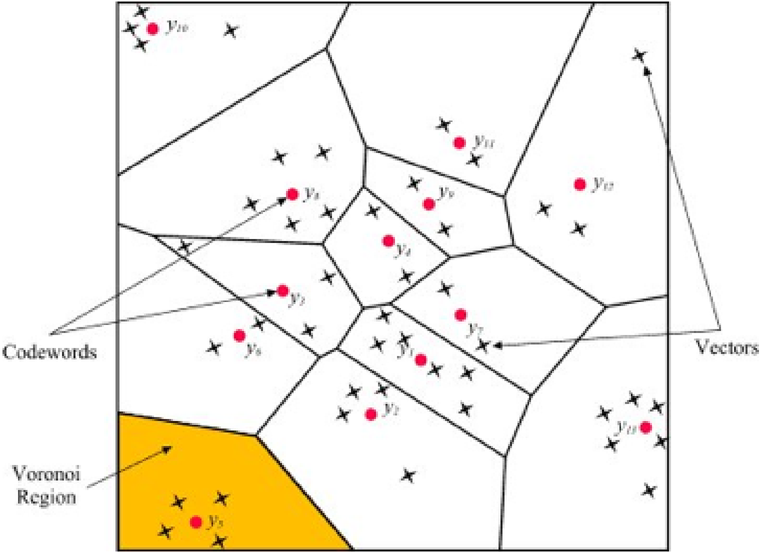

Figure: • Codewords/prototypes/elementary quantizers × data.

L^p-mean quantization error

▷ What about "Optimal"? Is there an optimal way to select the grid/N-quantizer to classify the data? In data analysis optimal clustering ?

▷ The L^p-*mean quantization error*

Definition

The L^p-mean quantization error induced by a grid $\Gamma \subset \mathbb{R}^d$ with size $|\Gamma| \leq N$, $N \in \mathbb{N}$

$$e_p(X;\Gamma) = \big\|\mathrm{dist}(X,\Gamma)\big\|_p = \Big\|\min_{x\in\Gamma}|X-x|\Big\|_p \tag{1}$$

(only depends on the distribution $\mu = \mathbb{P}_X$ of X).

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▷ The optimal L^p-mean quantization problem consists in minimizing (1) over all grids of size $|\Gamma| \le N$.

We define the L^p*-optimal mean quantization error* at level N as

$$e_{p,N}(X) := \inf\Big\{\Big\|\min_{x\in\Gamma}|X-x|\Big\|_p : \Gamma\subset\mathbb{R}^d,\ |\Gamma|\le N\Big\}.$$

Voronoi Quantization

▷ Noting that

$$|X(\omega) - \Xi(\omega)| \geq \operatorname{dist}\big(X(\omega), \Xi(\Omega)\big) = |X(\omega) - \widehat{X}^{\Xi(\Omega)}|$$

one derives the more general optimality result

$$e_{p,N}(X) = \inf\big\{\|X - \Xi\|_p : \; \Xi \in L^p(\mathbb{R}^d), \, \mathrm{Card}(\Xi(\Omega)) \leq N\big\} = \mathcal{W}_p(\mathbb{P}_X, \mathcal{P}_N).$$

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⇒ Voronoi Quantization $\widehat{X}^{\Gamma}$ provides an optimal L^p-mean discretization of X by Γ-valued random variables for every $p\in(0,+\infty)$.

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⇒ Voronoi Quantization $\widehat{X}^{\Gamma}$ provides an optimal L^p-mean discretization of X by Γ-valued random variables for every $p\in(0,+\infty)$.

⇒ The Nearest Neighbor projection is the coding rule, which yields the smallest L^p-mean approximation error for X.

Theorem (Kieffer, Cuesta-Albertos, (P.), Graf-Luschgy)

(a) *Let $p\in(0,+\infty)$, $X\in L^p$. For every level $N\geq 1$, there exists (at least) one L^p-optimal quantization grid $\Gamma^{*,N}$ at level N and*

$$N\longmapsto e_{p,N}(X)\downarrow 0 \qquad \textit{(vanishes if } supp(X) \textit{ is finite, } \downarrow\downarrow 0 \textit{ otherwise)}$$

(b) *If $p=2$,* $\mathbb{E}\Big(X\,|\,\widehat{X}^{\Gamma^{N,*}}\Big)=\widehat{X}^{\Gamma^{N,*}}$ *a.s. (stationarity/self-consistency).*

Sketch of proof ($p \geq 1$)

(a) We proceed by induction

- $N = 1$: $\xi \mapsto \|X - \xi\|_p$ is convex and coercive and atteins its minmum at an L^p-median.
- $N \Longrightarrow N+1$: Let $\xi \in \mathrm{supp}(X) \setminus \Gamma^{*,N}$, $\Gamma^{*,N}$ L^p-optimal at level$N$.

$$\ell^*_{N+1} := e_p(X, \Gamma^{*,N} \cup \{\xi\})^p < e_p(X, \Gamma^{*,N})^p = e_{p,N}(X)^p$$

  so that

$$K^* = \Big\{\Gamma \subset \mathbb{R}^d : |\Gamma| = N+1,\ e_p(X, \Gamma)^p \leq \ell^*_{N+1}\Big\} \neq \emptyset, \text{closed} \ldots$$

  ...and bounded (send one component or more to infinity and use Fatou's Lemma).
- Then $\Gamma \longmapsto e_p(X, \Gamma)$ attains a global minimum over K^*.

(b) The random variable $\widehat{X}^{\Gamma^{N,*}} - \mathbb{E}\Big(X \,|\, \widehat{X}^{\Gamma^{N,*}}\Big) \perp L^2\big(\sigma(\widehat{X}^{\Gamma^{N,*}})\big)$. Hence

$$\|X - \widehat{X}^{\Gamma^{N,*}}\|_2^2 = \|X - \mathbb{E}\Big(X \,|\, \widehat{X}^{\Gamma^{N,*}}\Big)\|_2^2 + \|\widehat{X}^{\Gamma^{N,*}} - \mathbb{E}\Big(X \,|\, \widehat{X}^{\Gamma^{N,*}}\Big)\|_2^2.$$

Hence, uniqueness of conditional expectation yields

$$\mathbb{E}\Big(X \,|\, \widehat{X}^{\Gamma^{N,*}}\Big) = \widehat{X}^{\Gamma^{N,*}} \quad a.s.$$

Applications

- Signal transmission: Let $\Gamma^{*,N} = \{x_1^*, \ldots, x_N^*\}$
  - Pre-processing I : re-ordering the labels i so that $i \mapsto p_i^* := \mathbb{P}(\widehat{X}^{\Gamma^{*,N}} = x_i^*)$ is decreasing.
  - Pre-processing II : encoding $i \rightsquigarrow \mathrm{Code}(i)$ see [CT06].
  - A who emits and B who receives both share the one-to-one bible.

$$x_i^* \leftrightarrow \mathrm{Code}(i)$$

  - X is encoded, Code(i) is transmitted, then decoded.
  - Naive encoding : dyadic coding of the labels i

$$\mathit{Complexity} = \sum_{i=1}^{N} p_i^*(1 + \lfloor \log_2 i \rfloor) \leq 1 + \lfloor \log_2 N \rfloor.$$

  - Uniform signal $X \sim U([0,1])$ then $\Gamma^{*,N} = \{\frac{2i-1}{2N}, \, i = 1:N\}$ and $p_i^* = \frac{1}{N}$ so that

$$\mathrm{Complexity} = 1 + \frac{1}{N}\sum_{i=1}^{N} \lfloor \log_2 i \rfloor \sim \log_2(N/e).$$

  - On the way to Shannon's Source coding theorem (see *e.g.* [Dembo-Zeitouni])...

- Quantization for (Probability and) Numerics:
  - What for? Cubature formulas for the computation of expectations.

$$\mathbb{E}\,F(X) \approx \mathbb{E}\big(F(\widehat{X}^{\Gamma^{*,N}})\big) = \sum_{i=1}^{N} p_i^* F(x_i^*).$$

  - What is needed? The distribution $(x_i^*, p_i^*)_{i=1,\dots,N}$ of $\widehat{X}^{\Gamma^{*,N}}$.
  - How to perform grid optimization? Lloyd I (Lloyd, 1982) and CLVQ (Mc Queen, further on).
  - Conditional expectation approximation:

$$\mathbb{E}(F(X)\,|\,Y) \approx \mathbb{E}\big(F(\widehat{X}^{\Gamma_X}\,|\,\widehat{Y}^{\Gamma_Y}\big).$$

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- Clustering (unsupervised learning):
  - What for? Unsupervised classification Mc Queen, 1957; (up to improvements like Self-Organizing Kohonen Maps, Cottrell-Fort-P. 1998, among others).
  - How to perform? Lloyd I (Lloyd, 1982) and CLVQ (Mc Queen, 1967, further on).
  - A typical problem in progress:
    - Distribution $\mu_n(\omega, d\xi) = \frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k(\omega)}$, $(\xi_k)_{k\geq 1}$ i.i.d.
    - L^2-Optimal quantization grid $\Gamma_n^*(\omega)$ at a fixed level $N \geq 1$.
    - One has $\lim_{n\to+\infty} \Gamma_n^*(\omega) = \Gamma^{*,N}$ optimal grid at level N for $\mu = \mathcal{L}(\xi_1)$.
    - At which rate?

Extension and. . .

Extension and...

▷ Generalization to infinite dimension

Still true in:

- a separable Hilbert space,
- even in a reflexive Banach space E (Cuesta-Albertos, *PTRF*, 1997) for a tight r.v.

$$(x_1,\ldots,x_N)\longmapsto \Big\|\min_{1\le i\le N}|X-x_i|_E\Big\|_p \quad \text{is l.s.c. fro the product weak topology on } E^N$$

- or even in a L^1 space (Graf-Luschgy-P., *J. of Approx.*, 2005) using τ-topology...
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▷ Convergence to 0

$$e_{p,N}(X) \downarrow 0 \quad \text{as} \quad N \to +\infty.$$

Let $(z_n)_{n\ge1}$ be an everywhere dense sequence in $\mathbb{R}^d$

$$e_{p,N}(X)^p \le e_p\big(X, \{z_1, \ldots, z_N\}\big)^p = \mathbb{E}\left[\min_{1 \le i \le N} |X - z_i|^p\right] \downarrow 0 \quad \text{as} \quad N \to +\infty.$$

by the Lebesgue dominated convergence theorem.

▷ But... at which rate? At least for the finite dimensional vector space.

Theorem (Zador's Theorem, from 1963 (PhD) to 2000)

(a) SHARP ASYMPTOTIC *(Zador, Kieffer, Bucklew & Wise, Graf & Luschgy in [GL00]):*

Let $X \in L^{p+}(\mathbb{R}^d)$ *with distribution* $\mathbb{P}_X = \varphi.\lambda^d \overset{\perp}{+} \nu$. *Then*

$$\lim_{N\to\infty} N^{\frac{1}{d}} \cdot e_{p,N}(X) = Q_{p,|\cdot|}.\left(\int_{\mathbb{R}^d} \varphi^{d/(d+p)}\, d\lambda_d\right)^{(d+p)/d}$$

where $Q_{p,|\cdot|} = \inf_{N\geq 1} N^{\frac{1}{d}}.e_{p,N}\big(U([0,1]^d)\big)$.

(b) NON-ASYMPTOTIC *(Pierce, Graf & Luschgy in [GL00], Luschgy-P. [LP08]):*

Let $p' > p$. *There exists* $C_{p,p',d} \in (0,+\infty)$ *such that, for every* $\mathbb{R}^d$*-valued* X *r.v.*

$$\forall\, N \geq 1, \quad e_{p,N}(X) \leq C_{p,p',d}\, \sigma_{p'}(X).\, N^{-\frac{1}{d}}.$$

Remarks. • $\sigma_{p'}(X) := \inf_{a\in\mathbb{R}^d} \|X - a\|_{p'} \leq +\infty$ is the $L^{p'}$-(pseudo-)standard deviation.

• The rate $N^{-\frac{1}{d}}$ is known as the *curse of dimensionality*.

Theorem (Zador's Theorem, 2016)

(a) SHARP ASYMPTOTIC *(Zador, Kieffer, Bucklew & Wise, Graf & Luschgy in [GL00], Luschgy-P., 2016):*

Let $X \in L^p(\mathbb{R}^d)$ *with distribution* $\mathbb{P}_X = \varphi.\lambda^d \overset{\perp}{+} \nu$ *such that* φ *is essentially* L^p*-radial and non-increasing [e.g.* $\varphi(\xi) \asymp g(|\xi|_0)$, $g \downarrow$ *on* $(a_0, +\infty)$&...*]*
Then

$$\lim_{N\to\infty} N^{\frac{1}{d}} \cdot e_{p,N}(X) = Q_{p,|\cdot|} \cdot \left(\int_{\mathbb{R}^d} \varphi^{d/(d+p)}\, d\lambda_d\right)^{(d+p)/d}$$

where $Q_{p,|\cdot|} = \inf_N N^{\frac{1}{d}} \cdot e_{p,N}\big(U([0,1]^d)\big)$.

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Numerical computation of quantizers

▷ Stationary quantizers Optimal grids Γ^* at level satisfy

$$\widehat{X}^{\Gamma^*} = \mathbb{E}\big(X \,|\, \widehat{X}^{\Gamma^*}\big)$$

or equivalently if $\Gamma^* = \{x_1^*, \ldots, x_N^*$

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(Nearly) optimal grids can be computed by optimization algorithms :

▷ LLOYD'S I ALGORITHM (Randomized) fixed-point method.

- $n = 0$ Initial grid $\Gamma^{[0]} = \{x_1^{[0]}, \ldots, x_N^{[0]}\}$
- $k \Longrightarrow k+1$ Standard step : Let $\Gamma^{[k]}$ the current grid.

$$x_i^{[k+1]} = \mathbb{E}\big(X \,|\, X \in C_i(\Gamma^{[k]})\big) = \mathbb{E}\big(X \,|\, \widehat{X}^{\Gamma^{[k]}} = x_i^{[k]}\big)$$

and set $\Gamma_i^{[k+1]} = \{x_i^{[k+1]}, i = 1 : N\}$.

Proposition (Lloyd I always makes the quantization error decrease)

$$\big\|X - \widehat{X}^{\Gamma^{(k+1)}}\big\|_2 \le \big\|X - \underbrace{\mathbb{E}\big(X \,|\, \widehat{X}^{\Gamma^{(k)}}\big)}_{\Gamma^{(k+1)}\text{- valued}}\big\|_2 \le \big\|X - \widehat{X}^{\Gamma^{(k)}}\big\|_2$$

- When $d = 1$ and $\mathcal{L}(X)$ is log-concave: exponetially fast convergence (Kieffer, 1982). Renewal of interest for 1-D quantization for quadrature formulas [Callegaro et al., 2017].
- However ... no general proof of convergence when $\mathcal{L}(X)$ has a non compact support and $d \geq 2$.
- Splitting method : initialize Lloyd's I procedure inductively on the size N by

$$\Gamma^{N,(0)} = \Gamma^{N-1,(\infty)} \cup \{\xi_N\}, \ \xi_N \in \mathrm{supp}\big(\mathcal{L}(X)\big).$$

(see P. -Yu, SICON, 2016). Then

$$\Gamma^{N+1,(k)} \to \Gamma^{N,(\infty)} \text{ (stationary quantizer of full size } N\ldots) \quad \text{as} \quad k \to +\infty$$

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- Practical implementation based on Monte Carlo simulations (or a dataset).

$$\mathbb{E}\big(g(X)\,|\,\widehat{X}^{\Gamma} = x_i\big) = \lim_{M\to+\infty} \frac{\sum_{m=1}^{M} g(X^m)\mathbf{1}_{\{X^m\in C_i(\Gamma)\}}}{\sum_{m=1}^{M}\mathbf{1}_{\{X^m\in C_i(\Gamma)\}}},\ (X^m)_{m\geq 1} \text{ i.i.d.} \sim X.$$

▷ Competitive Learning Vector Quantization algorithm ($p = 2$)

"Simply" a Stochastic gradient descent

- Let $D_N : (\mathbb{R}^d)^N \to \mathbb{R}_+$ be the (quadratic) distortion function

$$D_N(x) := \mathbb{E} \min_{1 \le i \le N} \|X - x_i\|^2 \;\to \min_{x \in (\mathbb{R}^d)^N}.$$

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- As soon as $|\cdot|$ is smooth enough $\Rightarrow D_N$ is differentiable at grids of full size. and if $\Gamma = \{x_1, \ldots, x_N\}$,

$$\frac{\partial D_N}{\partial x_i}(\Gamma) = 2\left(\mathbb{E}\left[\left(x_i - X\right)\mathbf{1}_{\{X\in C_i(\Gamma)\}}\right]\right)_{i=1:N}$$

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- Main point : $\boxed{\nabla D_N(\Gamma) = 0 \text{ iff } \Gamma \text{ is stationary.}}$

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- Main point : $\boxed{\nabla D_N(\Gamma) = 0 \text{ iff } \Gamma \text{ is stationary.}}$

- Hence we can implement a zero search (stochastic) gradient ... known as

Competitive Learning Vector Quantization

- $d = 1$:

$$D_N(x) = \sum_{i=1}^{N} \int_{x_{i-1/2}}^{x_{i+1/2}} |\xi - x_i|^2 d\mathbb{P}_X(\xi)$$

$\Rightarrow$ Evaluation of Voronoi-Cells, Gradient and Hessian is simple if f_X, F_X & E^1_X have closed form $\rightsquigarrow$ Newton-Raphson.

- $d \geq 2$:

Stochastic Gradient Method: CLVQ

- Simulate $\xi_1, \xi_2, \ldots$ independent copies of X
- Generate step sequence $\gamma_1, \gamma_2, \ldots$
  Usually: step $\gamma_n = \frac{A}{B+n} \searrow 0$ or $\gamma_n = \eta \approx 0$
- Grid updating $n \mapsto n+1$:

  Selection: select winner index: $i^* \in \operatorname{argmin}_i |x_i^n - \xi_n|$

  Learning: $\begin{cases} x_{i^*}^{n+1} := x_{i^*}^n + \gamma_n(x_{i^*}^n - \xi_n) \equiv \mathrm{dilat}(\xi_n; 1-\gamma_n)(x_{i^*}^n) \\ x_j^{n+1} := x_j^n, \qquad \text{for } j \neq i^*. \end{cases}$

Nearest neighbour search: Computational challenge of simulation based stochastic optimization methods :

$$\mathbf{1}_{\{X \in C_i(\Gamma)\}} \equiv \text{NEAREST NEIGHBOUR SEARCH}$$

Highly challenging problem in higher dimension, say $d \geq 4$ or 5.

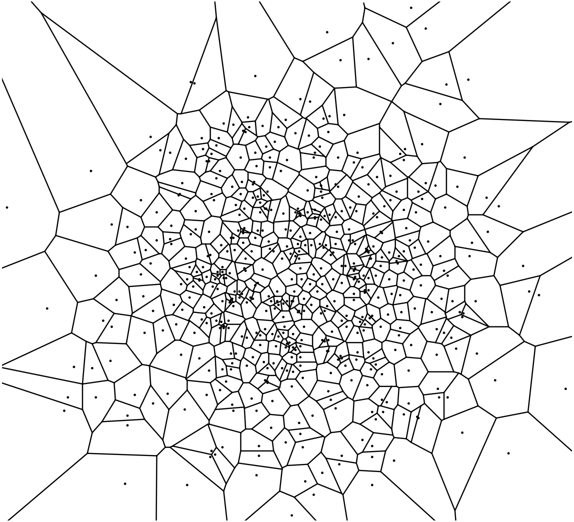
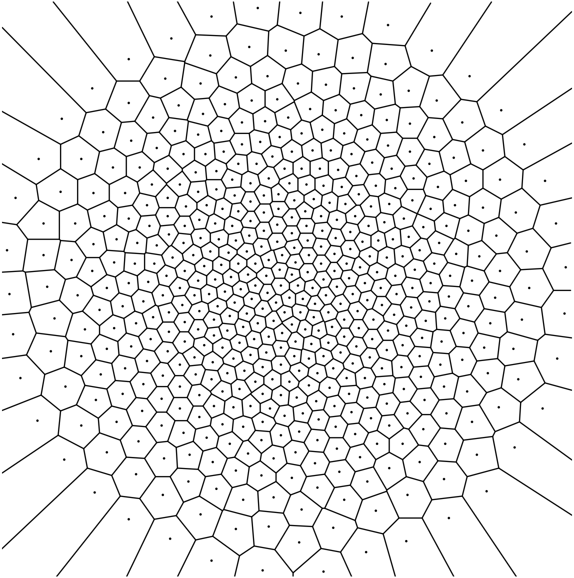


Figure: A random Quantizer for $\mathcal{N}(0, I_2)$ of size $N = 500$ in $(\mathbb{R}^2, |\cdot|_2)$.



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Benett's conjecture (1955): a coloured approach

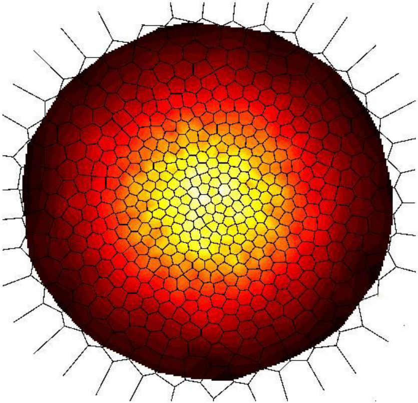

Figure: An N-quantization of $X \sim \mathcal{N}(0; I_2)$ with coloured weights: $\mathbb{P}(X \in C_i(\Gamma^{(*,N)}))$

(with J. Printems)

Toward Benett's conjecture: $\Gamma^{(*,N)} = \{x_1 \ldots, x_N\}$

$$X \sim \mathcal{N}(0; I_2).$$

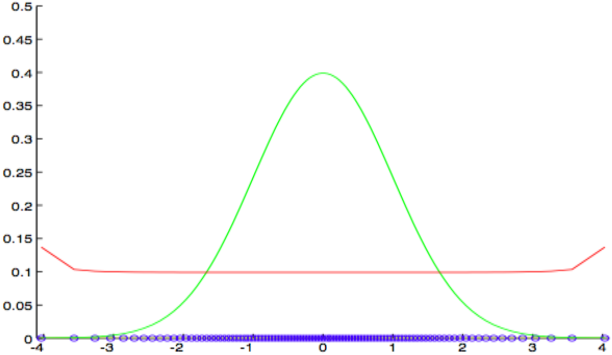

Figure: $x_i \mapsto \mathbb{P}(X \in C_i(\Gamma^{(*,N)})$ (green Gaussian line); $x_i \mapsto \mathbb{E}|X - x_i|^2 \mathbf{1}_{\{X \in C_i(\Gamma^{(*,N)})\}}$ (red flat line) (with J.C. fort)

- Local inertia: $x_i \longmapsto \mathbb{E}|X - x_i|^2 \mathbf{1}_{X \in C_i(\Gamma^{*,N})} \simeq$ Constant.
- Weights: $x_i \mapsto \mathbb{P}(X \in C_i(\Gamma^{(*,N)}) \simeq C.\left(e^{-\frac{x_i^2}{2}}\right)^{\frac{1}{3}}$ (fitting)

More on Benett's conjecture

▷ Benett's conjecture (weak form): In any dimension d, L^p-optimal quantizers satisfy

- Local inertia: $x_i \longmapsto \mathbb{E}|X - x_i|^2 \mathbf{1}_{X \in C_i(\Gamma^{*,N})} \simeq \dfrac{e_N(X)}{N}$.
- Weights: $x_i \mapsto \mathbb{P}(X \in C_i(\Gamma^{(*,N)}) \simeq C.\left(e^{-\frac{x_i^2}{2}}\right)^{\frac{d}{d+p}}$.

When $d = 1$ is holds uniformly on compacts sets ([Fort-P.], '03), when $d \geq 1$ at least in a measure sense.

▷ Strong Benett's conjecture:

- Conjecture on the geometric form of Voronoi cells go $U([0,1]^d)$ ($d = 2$: regular hexagon, $d = 3$ octaedron, $d \geq 4$????).
- Generic form of Voronoi cells for A.C. distributions.

Quantizing Non-Gaussian multivariate distributions

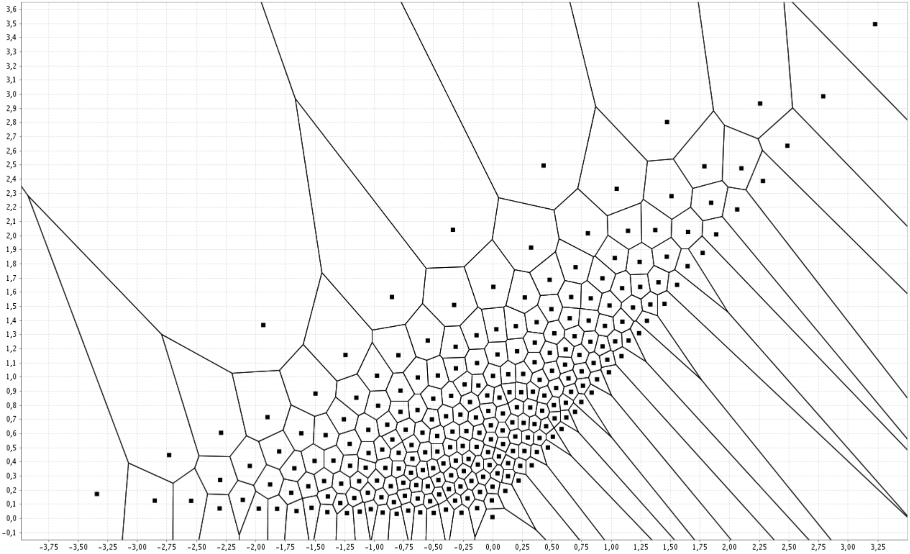

Figure: A Quantizer for $(B_1, \sup_{t\in[0,1]} B_t,\ B$ std B.M. of size $N = 500$ in $(\mathbb{R}^2, |\cdot|_2)$.

Back to clustering

- If $(\xi_k)_{k\geq}$ i.i.d.d. $\xi_1 \sim \mu = \mathcal{L}(\xi_1)$ on $\mathbb{R}^d$, consider its empirical measure

$$\mu_n(\omega, d\xi) = \frac{1}{n}\sum_{k=1}^{n}, \delta_{\xi_k}.$$

- Assume that $\mu(B(0;1)) = 1$. For every $\omega \in \Omega$, there exists (at least) an optimal quantizer $\Gamma^{(N)}(\omega, n)$ for $\mu_n(\omega, d\xi)$. Then (Biau et al., 2008, see [BDL08])

$$\mathbb{E}\Big(e_2\big(\Gamma^{(N)}(\omega, n), \mu\big)\Big) - e_{2,N}(\mu) \leq C \min\left(\sqrt{\frac{Nd}{n}}, \sqrt{\frac{d\, N^{1-\frac{2}{d}} \log n}{n}}\right)$$

where $C > 0$ is a universal real constant.

- See also (Graf-Luschgy, AoP, 2002, [GL02]) for other results on empirical measures (bounded support).

Back to numerical Probability? Quantization for Cubature

▷ Assume that we have access to $\mathcal{L}(\widehat{X}^{\Gamma})$: both the grid and the Voronoi cell weights

$$\Gamma = \{x_1, \ldots, x_N\} \text{ and } p_i^{\Gamma} = \mathbb{P}(X \in C_i(\Gamma)),\ i = 1, \ldots, N.$$

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$\Longrightarrow$ The computation of $\mathbb{E}F(\widehat{X}^\Gamma)$ for some Lipschitz continuous $F : \mathbb{R}^d \to \mathbb{R}$ becomes straightforward:

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▷ As a first error estimate, we already know that

$$|\mathbb{E}F(X) - \mathbb{E}\, F(\widehat{X}^\Gamma)| \le [F]_{\rm Lip}\, \mathbb{E}|X - \widehat{X}^\Gamma|.$$

Error Estimates

▷ First order. Moreover, if $\Gamma^{N,*}$ is L^1-optimal at level $N \geq 1$

$$\inf\Big\{\sup_{[F]_{\rm Lip}\leq 1} |\mathbb{E}F(X) - \mathbb{E}\,F(Y)|,\ \mathrm{card}(Y(\Omega)) \leq N\Big\}$$

$$= \sup_{[F]_{\rm Lip}\leq 1} |\mathbb{E}F(X) - \mathbb{E}\,F(\widehat{X}^{\Gamma^{N,*}})| = \mathbb{E}\big|X - \widehat{X}^{\Gamma^{N,*}}\big| = e_{1,N}(X)$$

i.e. Optimal Quantization is optimal for the class of Lipschitz functions or equivalently.

$$e_{1,N}(X) = \mathcal{W}_1\big(\mathcal{L}(X), \mathcal{P}_N\big).$$

with $\mathcal{P}_N = \big\{$atomic distribution with at most N atoms$\big\}$.

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▷ Second order.

Proposition

Second order cubature error bound Assume $F \in C^1_{Lip}$ and the grid Γ is stationary (e.g. because it is L^2-optimal), i.e.

$$\widehat{X}^\Gamma = \mathbb{E}(X|\widehat{X}^\Gamma).$$

Then a Taylor expansion yields

$$\begin{aligned}|\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^\Gamma)| &= |\mathbb{E}\,F(X) - \mathbb{E}\,F(\widehat{X}^\Gamma) - \mathbb{E}\big(\nabla F(\widehat{X}^\Gamma)\,\big|\,X - \widehat{X}^\Gamma\big)| \\ &\leq [DF]_{Lip} \cdot \mathbb{E}|X - \widehat{X}^\Gamma|^2.\end{aligned}$$

▷ Convexity Furthermore, if F is convex, then Jensen's inequality implies for stationary grids Γ

$$\mathbb{E}\, F(\widehat{X}^{\Gamma}) \leq \mathbb{E}\, F(X).$$

Quantization for Conditional expectation (Pythagoras' Theorem)

▷ Applications in Numerical Probability = conditional expectation approximation.

$$\widehat{X} = q_{X}(X) \qquad \widehat{Y} = q_{Y}(Y)$$

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Proposition (Pythagoras' Theorem for conditional expectation)

Let $P(y, du) = \mathcal{L}(X \mid Y = y)$ be a regular version of the conditional distribution of X given Y, so that

$$\mathbb{E}(g(X) \mid Y) = Pg(Y) \ a.s.$$

Then

$$\begin{aligned}\left\|\mathbb{E}(g(X) \mid Y) - \mathbb{E}(g(\widehat{X}) \mid \widehat{Y})\right\|_2^2 &\leq [g]_{\rm Lip}^2 \|X - \widehat{X}\|_2^2 + \|Pg(Y) - Pg(\widehat{Y})\|_2^2 \\ &\leq [g]_{\rm Lip}^2 \|X - \widehat{X}\|_2^2 + [Pg]_{\rm Lip}^2 \|Y - \widehat{Y}\|_2^2.\end{aligned}$$

If P propagates Lipschitz continuity:

$$[Pg]_{\rm Lip} \leq [P]_{\rm Lip}[g]_{\rm Lip}.$$

then quantization produces a control of the error.

Quantization for Conditional expectation

▷ Sketch of proof. As

$$Pg(Y)-\mathbb{E}\big(Pg(Y)\,|\,\widehat{Y}\big) \overset{L^2(\mathbb{P})}{\perp} \sigma(\widehat{Y})$$

and

$$\mathbb{E}\big(g(X)\,|\,Y\big)-\mathbb{E}\big(g(\widehat{X})|\widehat{Y}\big) = \Big(\mathbb{E}\big(g(X)\,|\,Y\big)-\mathbb{E}\big(Pg(Y)\,|\,\widehat{Y}\big)\Big) \overset{\perp}{+} \Big(\mathbb{E}\big(Pg(Y)\,|\,\widehat{Y}\big)-\mathbb{E}\big(g(\widehat{X})|\widehat{Y}\big)\Big)$$

so that by Pythagoras' theorem

$$\begin{aligned}\big\|\mathbb{E}\big(g(X)\,|\,Y\big)-\mathbb{E}\big(g(\widehat{X})|\widehat{Y}\big)\big\|_2^2 &= \big\|Pg(Y)-\mathbb{E}\big(Pg(Y)|\widehat{Y}\big)\big\|_2^2+\big\|\mathbb{E}\big(Pg(X)|\widehat{Y}\big)-\mathbb{E}\big(g(\widehat{X})|\widehat{Y}\big)\big\|_2^2\\ &\le \big\|Pg(Y)-Pg(\widehat{Y})\big)\big\|_2^2+\big\|g(X)-g(\widehat{X})\big\|_2^2.\\ &\le [Pg]_{\rm Lip}^2\big\|Y-\widehat{Y}\big\|_2^2+[g]_{\rm Lip}^2\big\|X-\widehat{X}\big\|_2^2.\end{aligned}$$

▷ If $p\neq 2$, a Minkowski like control is preserved

$$\begin{aligned}\big\|\mathbb{E}\big(g(X)\,|\,Y\big)-\mathbb{E}\big(g(\widehat{X})\,|\,\widehat{Y}\big)\big\|_p &\le [g]_{\rm Lip}\big\|X-\widehat{X}\big\|_p+\big\|Pg(Y)-Pg(\widehat{Y})\big\|_p\\ &\le [g]_{\rm Lip}\big\|X-\widehat{X}\big\|_p+[Pg]_{\rm Lip}\big\|Y-\widehat{Y}\big\|_p.\end{aligned}$$

A typical result (BSDE)

▷ We consider a "standard" BSDE:

$$Y_t = h(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

where the exogenous process $(X_t)_{t\in[0,T]}$ is a diffusion

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad x \in \mathbb{R}^d.$$

with b, σ, h Lipschitz continuous in x, f Lipschitz in (x, y, z) uniformly in $t \in [0, T]$...

▷ which is the probabilistic representation of the partially non-linear *PDE*

$$\partial_t u(t, x) + Lu(t, x) + f(t, x, u(t, x), (\partial_x^* u\sigma)(t, x)) = 0 \text{ on } [0, T) \times \mathbb{R}^d, \quad u(T, .) = h$$

with $Lg = (\nabla b|g) + \frac{1}{2}\mathrm{Tr}\big(\sigma^* D^2 g\sigma\big)$.

▷ ...and its time discretization scheme with step $\Delta_n = \frac{T}{n}$ recursively defined by

$$\begin{aligned}
\bar{Y}_{t_n^n} &= h(\bar{X}_{t_n^n}),\\
\bar{Y}_{t_k^n} &= \mathbb{E}(\bar{Y}_{t_{k+1}^n}|\mathcal{F}_{t_k^n}) + \Delta_n f\big(t_k^n, \bar{X}_{t_k^n}, \mathbb{E}(\bar{Y}_{t_{k+1}^n}|\mathcal{F}_{t_k^n}), \bar{\zeta}_{t_k^n}\big),\\
\bar{\zeta}_{t_k^n} &= \frac{1}{\Delta_n}\mathbb{E}\big(\bar{Y}_{t_{k+1}^n}(W_{t_{k+1}^n} - W_{t_k^n})|\mathcal{F}_{t_k}\big) = \frac{1}{\Delta_n}\mathbb{E}\big((\bar{Y}_{t_{k+1}^n} - \bar{Y}_{t_k^n})(W_{t_{k+1}^n} - W_{t_k^n})|\mathcal{F}_{t_k}\big)
\end{aligned}$$

where $\bar{X}$ is the Euler scheme of X defined by

$$\bar{X}_{t_{k+1}^n} = \bar{X}_{t_k^n} + b(_k^n, \bar{X}_{t_k^n})\Delta_n + \sigma(_k^n, \bar{X}_{t_k^n})(W_{t_{k+1}^n} - W_{t_k^n}).$$

▷ . . . spatially discretized by quantization: We “force” Markov property to write a Quantized Backward Dynamic Programming Principle

$$
\begin{aligned}
\widehat{Y}_n &= h(\widehat{X}_n)\\
\widehat{Y}_k &= \widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}) + \Delta_n f_k\big(\widehat{X}_k, \widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}), \widehat{\zeta}_k\big)\\
\widehat{\zeta}_k &= \frac{1}{\Delta_n}\widehat{\mathbb{E}}_k(\widehat{Y}_{k+1}(W_{t^n_{k+1}} - W_{t^n_k}))
\end{aligned}
$$

where

$$\widehat{\mathbb{E}}_k = \mathbb{E}(\,\cdot\,|\widehat{X}_k).$$

▷ By induction

$$\widehat{Y}_k = \hat{v}_k(\widehat{X}_k),\ k = 0, \ldots, n.$$

so that

$$\widehat{\mathbb{E}}_k\big(\widehat{Y}_{k+1}(W_{t^n_{k+1}} - W_{t^n_k})\big) = \widehat{\mathbb{E}}_k\big(\hat{v}_{k+1}(\widehat{X}_{k+1})(W_{t^n_{k+1}} - W_{t^n_k})\big).$$

Quanrization tree

▷ A Quantization tree for $(\widehat{X}_k)_{k=0,\dots,n}$: $N = N_0 + \cdots + N_n$, N_k = size of layer t_k^n.

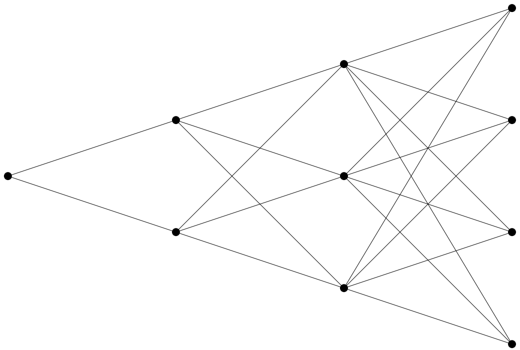

Figure: A typical (small!) 1-dimensional quantization tree

▷ At time k (i.e. t_k)

$$\widehat{X}_{t_k} = \mathrm{Proj}_{\Gamma_k}(X_{t_k}) \text{ with } \Gamma_k = \{x_1^k, \dots, x_{N_k}^k\} \text{ is a grid of size } N_k.$$

▷ What kind of tree a quantization tree is ?

- A quantization tree is not re-combining.
- But its size can designed *a priori* (and subject to possible optimization).

Calibrating the quantization tree

▷ To implement the above Quantized Backward Dynamic Programming Principled we need to compute repeatedly conditional expectations of the form

$$\mathbb{E}\left(\varphi(\widehat{X}_{k+1})\,|\,\widehat{X}_k\right) \quad \text{and} \quad \mathbb{E}\left(\varphi(\widehat{X}_{k+1})\Delta W_{t_{k+1}}\,|\,\widehat{X}_k\right)$$

▷ First, one has

$$\mathbb{E}\left(\varphi(\widehat{X}_{k+1})\mathbf{1}_{\{\widehat{X}_k=x_i^k\}}\right) = \sum_{j=1}^{N_{k+1}} \widehat{\pi}_{ij}^k \varphi(x_j^{k+1})$$

where

$$\widehat{\pi}_{ij}^k = \mathbb{P}\big(X_{k+1} \in C_j(\Gamma_{k+1})\,\&\,X_k \in C_i(\Gamma_k)\big)$$

so we need to estimate the hyper-matrix $[\hat{\pi}_{ij}^k]_{i,j,k}$.

▷ Weights for the Z term

$$\mathbb{E}\left(\varphi(\widehat{X}_{k+1})\Delta W_{t_{k+1}}\mathbf{1}_{\{\widehat{X}_k=x_i^k\}}\right) = \sum_{j=1}^{N_{k+1}} \widetilde{\pi}_{ij}^{W,k} \varphi(x_j^{k+1})$$

where

$$\widetilde{\pi}_{ij}^{W,k} = \mathbb{E}\Big(\mathbf{1}_{\{X_{k+1}\in C_j(\Gamma_{k+1})\}\cap\{,\widehat{X}_k=x_i^k\}}\Delta W_{t_{k+1}}\Big)$$

Quantized forward Kolmogorov equations (on weights)

▷ Note that by elementary Bayes formula

$$p_j^k := \mathbb{P}\big(X \in C_j(\Gamma^k)\big) = \sum_{i=1}^{N_{k-1}} \hat{\pi}_{ij}^{k-1}$$

so that we may compute

$$\mathbb{E}\left(\varphi(\widehat{X}_{k+1})\,|\,\widehat{X}_k\right) = \frac{\mathbb{E}\left(\varphi(\widehat{X}_{k+1})\mathbf{1}_{\{X_k \in C_i(\Gamma_k)\}}\right)}{\mathbb{P}(X \in C_i(\Gamma_k))}$$

▷Initialization: Quantize X_0 (often $\mathrm{X}_0 = x_0$).

Grid optimization and calibration (offline)

▷ Simulability

- Exact $X_k = X_{t_k}$ when possible.
- A discretization scheme $X_k = \bar{X}_k$.
- Let $(X_k^m, \Delta W^m_{t_{k+1}})_{0\le k\le n}$, $m = 1 : M$ be i.i.d. copies of $(X_k, \Delta W^m_{t_{k+1}})_{0\le k\le n}$.

▷ Grid Optimization: Let the sample "pass" through the quantization tree using either

- Randomized Lloyd procedure.
- or *CLVQ*.

to optimize the grids Γ_k at each time level.

▷ Calibrate $\widehat{\pi}^k_{ij}$ and $\widetilde{\pi}^k_{ij}$:

$$\widehat{\pi}^k_{ij} = \lim_{M\to+\infty} \frac{1}{M}\sum_{m=1}^{M} \mathrm{Card}\Big\{m : X_k^m \in C_i(\Gamma_k)\,\&\, X^m_{k+1} \in C_j(\Gamma_{k+1}),\, 1\le m\le M\Big\}$$

and

$$\widetilde{\pi}^k_{ij} = \lim_{M\to+\infty} \frac{1}{M}\sum_{m=1}^{M} \mathbb{E}\left[\Delta W^m_{t_{k+1}} \mathbf{1}_{\{X_k^m\in C_i(\Gamma_k)\}\cap\{X^m_{k+1}\in C_j(\Gamma_{k+1})\}}\right].$$

▷ Embedded optimal quantization: Perform optimization and calibration simultaneously.

Error estimates

Theorem (A priori error estimates (Sagna-P., SPA 2017))

Suppose that all the "Lipschitz" assumptions on b, σ, f, h are fulfilled.

(a) "Price": *Then, for every* $k = 0, \ldots, n$,

$$\big\|\bar{Y}_{t_k^n} - \widehat{Y}_k\big\|_2^2 \le [f]_{\mathrm{Lip}}^2 \sum_{i=k}^{n} e^{(1+[f]_{\mathrm{Lip}})(t_i^n - t_k^n)} K_i(b,\sigma,T,f,h)\big\|\bar{X}_{t_i^n} - \widehat{X}_{t_i^n}\big\|_2^2 = O\left(\frac{n}{N^{\frac{2}{d}}}\right).$$

(b) "Hedge":

$$\sum_{k=0}^{n-1} \Delta_n\big\|\bar{\zeta}_{t_k^n} - \widehat{\zeta}_k\big\|_2^2 \le \sum_{k=0}^{n-1} e^{(1+[f]_{\mathrm{Lip}})t_k^n}\big\|Y_{t_{k+1}^n} - \widehat{Y}_{t_{k+1}^n}\big\|_2^2 + K_k(b,\sigma,T,f,h)\big\|X_{t_k^n} - \widehat{X}_{t_k^n}\big\|_2^2$$

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(c) "RBSDE": *The same error bounds hold with Reflected BSDE (so far without Z in f) by replacing h by $h_k = h(t_k^n, .)$ where $h(t, X_t)$ is the obstacle process in the resulting quantized scheme.*

What is new (compared to Bally-P. 2003 for *reflected BSDE*)?

- $+$: Z inside the driver f for quantization error bounds.
- $+$: Squares everywhere

A new result : distortion mismatch/ L^s-rate optimality, $s > p$

▷ Let $\Gamma_N^{(p)}$, $N \geq 1$, be a sequence L^p-optimal grids.

What about $e_s(X, \Gamma_N^p)$ (L^s-mean quantization error) when $X \in L^s_{\mathbb{R}^d}(\mathbb{P})$ for $s > p$?

Theorem (L^p-L^s-distortion mismatch, Graf-Luschgy-P. 2005, Luschgy-P. 2015)

(a) *Let* $X \in L^p_{\mathbb{R}^d}(\mathbb{P})$ *and let* $(\Gamma_N^{(p)})_{N\geq 1}$ *be an* L^p*-optimal sequence for grids. Let* $s \in (p, p+d)$*. If*

$$X \in L^{\frac{sd}{d+p-s}+\delta}(\mathbb{P}),\ \delta > 0,$$

(note that $\frac{sd}{d+p-s} > s$ *and* $\lim_{s\to p+d} \frac{sd}{d+p-s} = +\infty$*), then*

$$\overline{\lim_N}\, N^{\frac{1}{d}} e_s(\Gamma_N^{(p)}, X) < +\infty.$$

(b) *If* $\mathbb{P}_X = f(|x|).\lambda_d(d\xi)$ *(radial density) then* $\delta = 0$ *is admissible.*

(c) *If* $\mathbb{E}\,|X|^{\frac{sd}{d+p-s}} = +\infty$*, then* $\underline{\lim}_N\, N^{\frac{1}{d}} e_s(\Gamma_N^{(p)}, X) = +\infty$.

▷ Possible perspectives: error bounds for quantization based numerical schemes for *BSDE* with a quadratic Z term ?

▷ So far, an application to quantized non-linear filtering.

Application to non-linear filtering

- Signal process $(X_k)_{k\geq 0}$ is an $\mathbb{R}^d$-valued Markov chain.
- The observation process $(Y_k)_{k\geq 0}$ is a sequence of $\mathbb{R}^q$-valued random vectors such that

$$(X_k, Y_k)_{k\geq 0} \text{ is a Markov chain.}$$

- The conditional distribution

$$\mathcal{L}(Y_k \,|\, X_{k-1}, Y_{k-1}, X_k) = g_k(X_{k-1}, Y_{k-1}, X_k, y)\lambda_q(dy)$$

- Aim : compute

$$\Pi_{y_{0:n},n}(dx) = \mathbb{P}(X_k \in dx \,|\, Y_1 = y_1, \cdots, Y_n = y_n)$$

- Kallianpur-Streibel formula: set $y = y_{0:n} = (y_0, \ldots, y_n)$ a vector of observations

$$\Pi_{y,n}(dx) = \Pi_{y,n} f = \frac{\pi_{y,n} f}{\pi_{y,n}\mathbf{1}}$$

with the normalized filter $\pi_{y_{0,n},n}$ defined by

$$\pi_{y_{0:n},n} f = \mathbb{E}(f(X_n) L_{y_{0:n},n}) \quad \text{with} \quad L_{y_{0:n},n} = \prod_{k=1}^{n} g_k(X_{k-1}, y_{k-1}, X_k, y_k),$$

solution to both a forward and a backward inductionsbased on the kernels

$$H_{y,k} h(x) = \mathbb{E}(h(X_k) g_k(x, y_{k-1}, X_k, y_k) | X_{k-1} = x), \quad H_{y,0} f(x) = \mathbb{E}(f(X_0)),$$

- Forward: Start from

$$\pi_{y,0} = H_{y,0}$$

and define by a forward induction

$$\pi_{y,k} f = \pi_{y,k-1} H_{y,k} f, \qquad k = 1, \ldots, n.$$

- Backward: We define by a backward induction

$$\begin{aligned} u_{y,n}(f)(x) &= f(x), \\ u_{y,k-1}(f) &= H_{y,k} u_{y,k}(f), \quad k = 0, \ldots, n. \end{aligned}$$

so that

$$\pi_{y,n} f = u_{y,-1}(f)$$

This formulation is useful in order to establish the quantization error bound.

Quantized Kallianpur-Streibel formula (P.-Pham (2005))

- Quantization of the kernel:

$$H_{y_{0:n},k}f(x) \longrightarrow \widehat{H}_{y_{0:n},k}f(x) = \mathbb{E}(f(\widehat{X}_k)g_k(x, y_{k-1}, \widehat{X}_k, y_k)|\widehat{X}_{k-1} = x)$$

- Forward quantized dynamics (I):

$$\widehat{\pi}_{y,k}f = \widehat{\pi}_{y,k-1}\widehat{H}_{y,k}f, \qquad k = 1, \ldots, n.$$

- Forward quantized dynamics (II):

$$\widehat{\Pi}_y(dx) = \widehat{\Pi}_{y,n}f = \frac{\widehat{\pi}_{y,n}f}{\pi_{y_{0:n},n}\mathbf{1}}$$

(finitely supported unnormalized filter satisfies formally the same recursions)

- Weight computation: If $\widehat{X}_n = \widehat{X}_n^{\Gamma_n}$, $\Gamma_n = \{x_1^1, \ldots, x_{N_n}^n\}$ then

$$\widehat{\Pi}_{y,n}(dx) = \sum_{i=1}^{N_n} \widehat{\Pi}_{y,n}^i \delta_{x_i^n} \quad \text{with } \widehat{\Pi}_{y,n}^i = \widehat{\Pi}_{y,n}\big(\mathbf{1}_{C_i(\Gamma_n)}\big).$$

From *Lip* to θ-Liploc assumptions

- Standard $\mathcal{H}_{\rm Lip}$assumption for the conditional densities $g_k(.,y,.,y')$: bounded by K_g and Lipschitz continuity.

$$|g_k(x,y,x',y') - g_k(\widehat{x},y,\widehat{x}',y')| \quad\leq\quad [g_k]_{\rm Lip}(y,y')\big(|x-\widehat{x}|+|x'-\widehat{x}'|\big).$$

- The kernels $P_k(x,d\xi) = \mathbb{P}(X_k \in d\xi \,|\, X_{k-1} = x)$ propagate Lipschitz continuity with coefficient $[P_k]_{\rm Lip}s$ such that

$$\max_{k=1,\dots,n}[P_k]_{\rm Lip} < +\infty$$

Aim: Switch to a θ-local Lipschitz assumption ($\theta : \mathbb{R}^d \to \mathbb{R}_+$, $\uparrow +\infty$ as $|x| \uparrow +\infty$).

$$|h(x,x') - h(\hat{x},\hat{x}')| \leq [h]_{\rm loc}\big(|x-\widehat{x}|+|x'-\widehat{x}'|\big)\big(1+\theta(x)+\theta(x')+\theta(\hat{x})+\theta(\hat{x}')\big)$$

- New $(\mathcal{H}^{\theta}_{\rm Liploc})$ assumption: the functions g_k are still bounded by K_g and θ-local Lipschitz continuous

$$|g_k(x,y,x',y')-g_k(\widehat{x},y,\widehat{x}',y')| \leq [g_k]_{\rm loc}(y,y')\big(|x-\widehat{x}|+|x'-\widehat{x}'|\big)\big(1+\theta(x)+\theta(x')+\theta(\hat{x})+\theta(\hat{x}')\big)$$

- The kernels $P_k(x,d\xi) = \mathbb{P}(X_k \in d\xi \,|\, X_{k-1} = x)$ propagate θ-local Lipschitz continuity with coefficient $[P_k]_{\rm loc} < +\infty$.
- The kernels $P_k(x,d\xi)$ propagate θ-control: $\max_{0\leq k\leq n-1} P_k(\theta)(x) \leq C\big(1+\theta(x)\big)$.

Typical example: $X_k = \bar{X}^n_{t^n_k}$ (Euler scheme with step $\Delta_n = \frac{T}{n}$), $\theta(\xi) = |\xi|^\alpha$, $\alpha > 0$.

Theorem (Sagna-P., SPA '17)

Let $s \in (1, 1+\frac{d}{2})$ *and* $\theta(x) = |x|^\alpha$, $\alpha \in (0, \frac{1}{\frac{1}{s-1}-\frac{2}{d}})$.
Assume (X_k) *and* (g_k) *satisfy* $(\mathcal{H}^\theta_{\rm Liploc})$ *(in particular* (X_k) *propagates* θ*-Lipschitz continuity) and assume* $X_k \in L^{\frac{2ds}{d+2-2s}}$, $k = 0, \ldots, n$. *Then*

$$|\Pi_{y,n}f - \widehat{\Pi}_{y,n}f|^2 \leq \frac{2(K_g^n)^2}{\phi_n^2(y) \vee \widehat{\phi}_n^2(y)} \sum_{k=0}^{n} B_k^n(f,y) \times \underbrace{\|X_k - \widehat{X}_k\|_{2s}^2}_{\asymp \|X_k - \widehat{X}_k\|_2^2 \leq c_k N_k^{-\frac{2}{d}} \ \textit{(Mismatch!!)}} \qquad (2)$$

with

$$\phi_n(y) = \pi_{y,n}\mathbf{1} \quad \textit{and} \quad \widehat{\phi}_n(y) = \widehat{\pi}_{y,n}\mathbf{1},$$

$$B_k^n(f,y) := 2[P]_{\rm loc}^{2(n-k)}[f]_{\rm loc}^2 + 2\|f\|_\infty^2 R_{n,k} + \|f\|_\infty R_{n,k}^2,$$

where

$$R_{n,k} = \frac{8^{\frac{s}{s-1}} M_s^n}{K_g^2}\left[[g_{k+1}]_{\rm loc}^2 + [g_k]_{\rm loc}^2 + \Big(\sum_{m=1}^{n-k}[P]_{\rm loc}^{m-1}(1+[P]_{\rm loc})[g_{k+m}]_{\rm loc}\Big)^2\right],$$

and

$$M_s^n := 2 \max_{k=0,\ldots,n}\Big(\mathbb{E}\big(\theta(X_k)^{\frac{2s}{s-1}}\big) + \mathbb{E}\big(\theta(\widehat{X}_k)^{\frac{2s}{s-1}}\big)\Big).$$

Numerical illustrations (3)

- Risk-neutral price under historical probability (B&S model, Euler scheme)

$$dY_t = \left(rY_t + \frac{\mu - r}{\sigma} Z\right) dt + Z_t dW_t$$

with

$$Y_T = h(X_T) = (X_T - K)_+.$$

▷ Model parameters: $r = 0.1$; $T = 0.1$; $\sigma = 0.25$; $S_0 = K = 100$.

▷ Quantization tree calibration: $7.5\,10^5$ MC and $NbLloyd = 1$.

▷ Reference $\mathrm{call}_{BS}(K, T) = 3.66$, $Z_0 = 14.148$. If $\mu \in \{0.05, 0.1, 0.15, 0.2\}$,

- $n = 10$ and $N_k = \bar{N} = 20$: Q-price $= 3.65$, $\widehat{Z}_0 = 14.06$.
- $n = 10$ and $N_k = \bar{N} = 40$, Q-price $= 3.66$, $\widehat{Z}_0 = 14.08$.

▷ Computation time :

– 5 seconds for one contract.

– Additional contracts for free (more than $10^5/s$).

▷ Romberg extrapolation price $= 2 * Q$-price(N_2)-Q-price(N_1) does improve the price (and the "hedge").

Numerical illustrations

- Bid-ask spreads on interest rates :

$$dY_t = \left(rY_t + \frac{\mu - r}{\sigma} Z_t + (R - r)\min\left(Y_t - \frac{Z_t}{\sigma}, 0\right)\right) dt + Z_t dW_t$$

with

$$Y_T = h(X_T) = (X_T - K_1)_+ - 2(X_T - K_2)_+, \quad K_1 = 95,\ K_2 = 105.$$

$$\mu = 0.05, r = 0.01,\ \sigma = 0.2,\ T = 0.25,\ R = 0.06$$

▷ Reference values: price $= 2.978$, $\widehat{Z}_0 = 0.553$.

▷ Crude Quantized prices:

- $n = 10$ and $N_k = \bar{N}_1 = 20$: Q-price $= 2.96$, $\widehat{Z}_0 = 0.515$.
- $n = 10$ and $N_k = \bar{N}_2 = 40$, Q-price $= 2.97$, $\widehat{Z}_0 = 0.531$.

▷ Romberg extrapolated price $= 2 * Q\text{-price}(\bar{N}_2)\text{-}Q\text{-price}(\bar{N}_1) \simeq 2.98$

and Romberg extrapolated hedge $\widehat{Z}_0 \approx 0.547$.

Multidimensional example (due to J.-F. Chassagneux)

▷ Let W be a d-dimensional B.M. and let

$$e_t = \exp(t + W_t^1 + \ldots + W_t^d).$$

▷ Consider the non-linear *BSDE*

$$dX_t = dW_t, \qquad -dY_t = f(t, Y_t, Z_t)dt - Z_t \cdot dW_t,\ Y_T = \frac{e_T}{1+e_T}$$

with $f(t, y, z) = (z_1 + \ldots + z_d)\big(y - \frac{2+d}{2d}\big)$.

▷ Solution:

$$Y_t = \frac{e_t}{1+e_t}, \qquad Z_t = \frac{e_t}{(1+e_t)^2}.$$

We set $d = 2, 3$ and $T = 0.5$, so that

$$Y_0 = 0.5 \quad \text{and} \quad Z_0^i = 0.24,\ i = 1, \ldots, d.$$

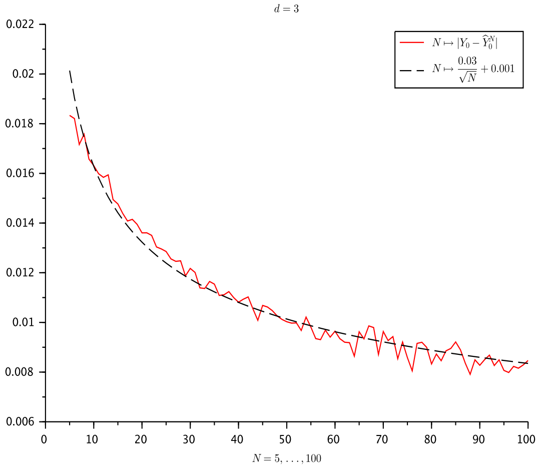

Figure: Convergence rate of the quantization error for the multidimensional example). Abscissa axis: the size $N = 5, \ldots, 100$ of the quantization. Ordinate axis: The error $|Y_0 - \widehat{Y}_0^N|$ and the graph $N \mapsto \hat{a}/N + \hat{b}$, where $\hat{a}$ and $\hat{b}$ are the regression coefficients. $d = 3$.

Local behaviour of optimal quantizers (back to Benett's conjecture)

Theorem (Local behaviour: toward Benett's conjecture, Graf-Luschgy-P. AoP, 2012)

(a) *If $\mathbb{P}_X$ is absolutely continuous on $\mathbb{R}^d$ then*

$$e^p_{N,p}(X) - e^p_{N+1,p}(X) \asymp N^{-\left(1+\frac{p}{d}\right)}.$$

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(*a*) *If* $\mathbb{P}_X$ *is absolutely continuous on* $\mathbb{R}^d$ *then*

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(*b*) Upper-bounds: *Suppose* $\mathbb{P}_X = \varphi.\lambda_d$ φ *is essentially bounded with compact support and its support is peakless*

$$\forall\, s \in (0, s_0),\ \forall\, x \in \operatorname{supp}(\mathbb{P}_X),\quad \mathbb{P}_X\big(B(x,s)\big) \geq\ c\lambda_d(B(x,s)),\ c > 0.$$

$$\exists\, c, \bar{c} \in [1,\infty)\ s.t.\ \ \forall\, N \in \mathbb{N}, \begin{cases} \displaystyle\max_{x_i \in \Gamma^{*,N}} \mathbb{P}_X\Big(C_i(\Gamma^{*,N})\Big) \leq \frac{c_1}{N}, \\ \displaystyle\max_{x_i \in \Gamma^{*,N}} \int_{C_i(\Gamma^{*,N})} \|\xi - x_i\|^p \, d\mathbb{P}_X(d\xi) \leq \bar{c} N^{-\left(1+\frac{p}{d}\right)}. \end{cases}$$

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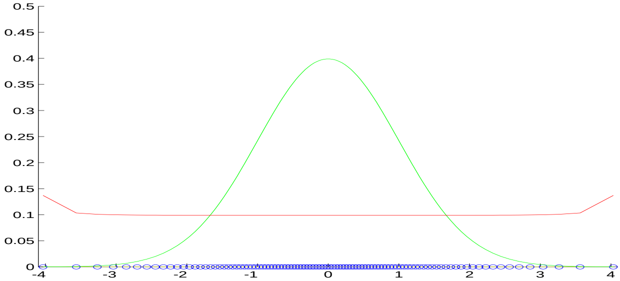
(c) Lower bounds $\forall\, n \in \mathbb{N}$, $\quad \min\limits_{a \in \Gamma^{*,N}} \displaystyle\int_{C_a(\Gamma^{*,N})} \|\xi - a\|^p\, d\mathbb{P}(\xi) \geq \underline{c}\, N^{-\left(1+\frac{p}{d}\right)}.$

- ▷ Benett's conjecture (1955): $\mathbb{P}\Big(C_a(\Gamma^{*,N})\Big) \sim c_x \dfrac{\varphi(a)^{\frac{p}{d+p}}}{N}$, $a \in \Gamma^{*,N}$, as $N \to +\infty$.
- ▷ Various extensions to unbounded r.v., including uniform results for radial decreasing

QUANTIFICATION QUADRATIQUE OPTIMALE DE TAILLE 50 DE $\mathcal{N}(0;1)$



o Le quantifieur optimal de taille 50 : $x^{(50)} = (x_1^{(50)}, \dots, x_{50}^{(50)})$,

—- Les poids : $x_i \mapsto \mathbb{P}\left(X \in C_i(x^{(50)})\right)$

—- L'inertie locale : $x_i \mapsto \displaystyle\int_{C_i(x^{(50)})} (\xi - x_i^{(50)})^2 \mathbb{P}_X(d\xi)$

Figure: $a \mapsto \mathbb{P}(X \in C(\widehat{X}_a^{*,N}),\ X \sim \mathcal{N}(0;1),\ N = 50$

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- Fast Marginal quantization [Sagna-P., 2015]

First conclusions on optimal (Voronoi) vector quantization

▷ Download free pre-computed grids of $\mathcal{N}(0; I_d)$ distributions at the URL

`www.quantize.maths-fi.com`

for $d = 1, \ldots, 10$ and $N = 1, \ldots, 10^4$ and many others items related to optimal quantization.

- Voronoi quantization is optimal for "Lipschitz approximation"
- Paradox: it does not preserve regularity
- Second order (stationarity) : (almost) only optimal grids $\Rightarrow$ lack of flexibility
- As for cubature: quantization *vs* uniformly distributed sequences? $(\xi_N)_{N\geq 1}$, $[0,1]^d$-valued sequences s.t.

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\xi_i} \stackrel{\mathbb{R}^d}{\Longrightarrow} \lambda_{|[0,1]^d}$$

1. $\mathbb{R}^d$ *vs* $[0,1]^d$ [1 − 0].
2. Lipschitz continuity *vs* Hardy & Krause finite variation on $[0,1]^d$, [2 − 0].
3. Sequences of N-tuples *vs* sequences [2 − 1] (QMC!).
4. Companion weights *vs* no weights [2 − 2].
5. Rates $n^{-\frac{1}{d}}$ *vs* $\log n \times n^{-\frac{1}{d}}$ (Stoikov, 1987, price for uniform weights!) [3 − 2].

How to "fix" (3) without affecting (4): Greedy quantization.

What greedy quantization is the name for?

▷ Switch from a sequence of N-tuples toward a sequence of points $(a_N)_{N\geq 1}$ such that

$$\forall\, N \geq 1, \quad a^{(N)} = \{a_1, \ldots, a_N\} \text{ produces "good" quantization grids.}$$

Among others, the first questions are:

- How to proceed theoretically?
- How "good"?
- How to compute them?
- How flexible can they be?

Level-by-level "greedy" optimization

Let $X \in L^p_{\mathbb{R}^d}(\Omega, \mathcal{A}, \mathbb{P})$ be a random vector with distribution $\mathbb{P}_X = \mu$.

▷ Optimal greedy quantization: We define by induction a sequence $(a_N)_{N\geq 1}$ recursively by

$$a^{(0)} = \emptyset, \quad \forall\, N \geq 0, \quad a_{N+1} \in \operatorname{argmin}_{\xi \in \mathbb{R}^d} e_p(a^{(N)} \cup \{\xi\}, X).$$

▷ It is a natural and constructive way to answer the above first question.

▷ Is it the best one? No answer so far...

▷ Note that a_1 always exists and

$$a_1 \text{ is the } L^p(\mathbb{P})\text{-median}$$

(always unique if $p > 1$).

Existence of an L^p- optimal greedy quantization sequence

Proposition (Assume $\mathrm{card}\big(\mathrm{supp}(\mu)\big) = +\infty$ and $X \in L^p(\mathbb{P})$)

(a) Existence: *There exists an L^p-optimal greedy quantization sequence $(a_N)_{N\geq 1}$ and $\big(e_p(a^{(n)}, X)\big)_{1\leq n\leq N}$ is (strictly) decreasing to 0 (and a_1 is an L^p-median).*

(b) Space filling: *Let $q > p$. If $X \in L^q_{\mathbb{R}^d}(\mathbb{P})$. Then, any L^p-optimal greedy quantization sequence $(a_N)_{N\geq 1}$ satisfies*

$$\lim_N e_q(a^{(N)}, X) = 0.$$

Greedy quantization is rate optimal

▷ *Main rate optimality result.*

Theorem (Rate optimality, Luschgy-P. '15)

Let $p\in(0,+\infty)$, $X\in L^p(\Omega,\mathcal{A},\mathbb{P})$ *and let* $\mu=\mathbb{P}_X$. *Let* $(a_N)_{N\geq 1}$ *be an* L^p*-optimal greedy quantization sequence.*

(*a*) *Let* $p'>p$. *There exists* $C_{p,p',d}\in(0,+\infty)$ *such that, for every* $\mathbb{R}^d$*-valued* X *r.v.*

$$\forall\, N\geq 1,\quad e_p(a^{(N)},X)\leq C_{p,p',d}.\sigma_{p'}(X).\,N^{-\frac{1}{d}}.$$

(*b*) *If* $\mu=\varphi(\xi)\lambda_d(d\xi)=f(|\xi|_0)\lambda_d(d\xi)$, $|.|_0$ *(any) norm on* $\mathbb{R}^d$ *and* $f=\mathbb{R}_+\longrightarrow\mathbb{R}_+$, *bounded and non-increasing outside a compact, and* X *lies in* L^p *and* $\int_{\mathbb{R}^d}f(|\xi|_0)^{\frac{d}{d+p}}d\lambda_d(\xi)<+\infty$, *then*

$$\limsup_N N^{\frac{1}{d}}e_p(a^{(N)},X)<+\infty.$$

Condition in (b) is optimal since, if $\mu=\varphi.\lambda_d$,

$$\liminf_N N^{\frac{1}{d}}e_{p,N}(X)\geq\widetilde{Q}_{p,|.|}\times\left(\int_{\mathbb{R}^d}\varphi^{d/(d+p)}\,d\lambda_d\right)^{(d+p)/d}.$$

▷ Main tool: Still micro-macro inequalities.

Flavour of proof

▷ First we not that by definition of the sequence $(a_N)_{N\geq 1}$,

$$\forall\, y\in\mathbb{R}^d,\quad \Delta^{(a)}_{N+1} := e_p(a^{(N)},X)^p - e_p(a^{(N+1)}) \geq e_p(a^{(N)},X)^p - e_p(a^{(N)}\cup\{y\},X)^p$$

So, we start from the micro-macro inequality ($0<b<\frac{1}{2}$, fixed parameter).

$$\forall\, y\in\mathbb{R}^d,\quad e_p(a^{(N)},X)^p - e_p(a^{(N)}\cup\{y\},X)^p \geq C_{p,b}d(y,a^{(N)})^p\mu\big(B(y,b\,d(y,a^{(N)}))\big).$$

Flavour of proof

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$$\forall\, y \in \mathbb{R}^d, \quad e_p(a^{(N)}, X)^p - e_p(a^{(N)} \cup \{y\}, X)^p \geq C_{p,b} d(y, a^{(N)})^p \mu\big(B(y, b\, d(y, a^{(N)}))\big).$$

▷ Let $\mu = \mathbb{P}_X$. Integrating w.r.t. a distribution $\nu(dy)$:

$$\begin{aligned}
\Delta^{(a)}_{N+1} &\geq C_{p,b} \iint \mathbf{1}_{\{|\xi - y| \leq b\, d(y, a^{(N)})\}} d(y,\ a^{(N)})^p \nu(dy)\mu(d\xi)\\
&\geq C_{p,b} \iint \mathbf{1}_{\{|\xi - y| \leq b\, d(y, a^{(N)}),\ d(y, a^{(N)}) \geq \frac{1}{b+1} d(\xi, a^{(N)})\}} d(y,\ a^{(N)})^p \nu(dy)\mu(d\xi)\\
&\geq C'_{p,b} \iint \mathbf{1}_{\{|\xi - y| \leq b\, d(y, a^{(N)}),\ d(y, a^{(N)}) \geq \frac{1}{b+1} d(\xi, a^{(N)})\}} d(\xi,\ a^{(N)})^p \nu(dy)\mu(d\xi)\\
&\geq C'_{p,b} \iint \mathbf{1}_{\{|\xi - y| \leq \frac{b}{b+1}\ d(\xi, a^{(N)})\}} d(\xi,\ a^{(N)})^p \nu(dy)\\
\Delta^{(a)}_{N+1} &= C'_{p,b} \int \nu\Big(B\Big(\xi; \frac{b}{b+1} d(\xi, a^{(N)})\Big)\Big) d(\xi, a^{(N)})^p \mu(d\xi)
\end{aligned}$$

still by Fubini's theorem.

▷ Let $b \in (0, \frac{1}{2})$ be such that $\frac{b}{b+1} = \frac{1}{4}$.

$$\nu(dx) = \frac{\kappa}{(|x-a_1|+5/4)^{d+\eta}}\lambda_d(dx).$$

Then, if $\rho \leq \frac{1}{4}|x-a_1|$,

$$\nu\big(B(\xi,\rho)\big) \geq \rho^d \times \Big[g(\xi) := \kappa' V_d \frac{1}{(|\xi-a_1|+1)^{d+\eta}}\Big].$$

Noting that $d(\xi, a^{(N)}) \leq d(x, a_1) = |x-a_1|$ yields

$$e_p(a^{(N)}, X)^p - e_p(a^{(N+1)} \geq C_p'' \int d(\xi, a^{(N)})^{p+d} g(\xi)\mu(\xi).$$

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Noting that $d(\xi, a^{(N)}) \le d(x, a_1) = |x - a_1|$ yields

$$e_p(a^{(N)}, X)^p - e_p(a^{(N+1)} \ge C''_p \int d(\xi, a^{(N)})^{p+d} g(\xi)\mu(\xi).$$

▷ Inverse Minkowski Inequality implies with $\frac{p}{p+d} < 1$ and $-\frac{p}{d} < 0$, yields

$$\Delta^{(a)}_{N+1} \ge C''_p \underbrace{\left[\int d(\xi, a^{(N)})^p \mu(d\xi)\right]^{\frac{p+d}{p}}}_{= e_p(a^{(N)}, X)^{p+d}} \left[\int g(\xi)^{-\frac{p}{d}} \mu(d\xi)\right]^{-\frac{d}{p}}.$$

Now

$$\int g(\xi)^{-\frac{p}{d}} \mu(\xi) \asymp \int |\xi - a_1|^{p+\frac{\eta}{d}} \mu(\xi) = \mathbb{E}|X|^{p+\frac{\eta}{d}} < +\infty$$

so that

$$e_p(a^{(N)}, X)^p - e_p(a^{(N+1)})^p \ge C_{p,X} \cdot e_p(a^{(N)}, X)^{p+d}.$$

▷ The sequence $(e_p(a^{(N)}, X)^p)_{N\geq 1}$ being non-negative and $\downarrow 0$, one easily derives the announced conclusion:

$$e_p(a^{(N)}, X)^p \leq \widetilde{\kappa} N^{-\frac{d}{p}}.$$

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▷ The universal bounds follows by a careful handling of the real constants and a scaling argument.

Distortion mismatch

Distortion mismatch

▷ Let $X \in L^p(\mathbb{P})$. As long as $q \in (0, p]$, any optimal greedy sequence $(a_{_N})_{N\geq 1}$ remains L^q-rate optimal for the L^q-norm (by monotony).

The *distortion mismatch problem* amounts to the following question

What happens if $s > p$?

Distortion mismatch

▷ Let $X \in L^p(\mathbb{P})$. As long as $q \in (0,p]$, any optimal greedy sequence $(a_N)_{N\geq 1}$ remains L^q-rate optimal for the L^q-norm (by monotony).

The *distortion mismatch problem* amounts to the following question

What happens if $s > p$?

▷ It was first addressed for sequences of optimal N-quantizers in joint paper with S. Graf and H. Luschgy [Graf-Luschgy-P., *ESAIM P&S*, '08].

▷ A first necessary condition to preserve the rate:

$$\liminf_N N^{\frac{1}{d}} e_{s,N}(X)^s \geq Q_{s,|.|} \left(\int f^{\frac{d}{d+p}} d\lambda_d\right)^{\frac{s}{d}} \left(\int f^{1-\frac{s}{d+p}} d\lambda_d\right).$$

Main greedy mismatch result

Theorem (Greedy Distortion mismatch, Luschgy-P. '15)

Let $X \in L^{p+}(\mathbb{P})$ an $\mathbb{R}^d$-valued random vector and let $q \in (p, p+d]$ and let $(a_N)_{N\geq 1}$ be an L^p-greedy optimal sequence. If $s \in [p, p+d)$ and

$$X \in L^{\frac{sd}{d+p-s}+\delta}(\mathbb{P}).$$

Then

$$e_q(a^{(N)}, X) \leq C_{p,d,\delta} \big\|X - a_1\big\|^{\frac{d}{p+d}}_{\frac{q(d+\delta)}{d+p-q}} \|X - a_1\|^{\frac{p}{p+d}}_{p(1+\frac{\delta}{d})} \times N^{-\frac{1}{d}}$$

Moreover if φ is essentially quadratic decreasing, it still works for $\delta = 0$ (e.g. $X \sim \mathcal{N}(m, \Sigma)$).

▷ So far, no such universal bound for optimal quantization though mismatch holds true.

▷ If X has a compact support the rate optimality (mismatch) holds for every $q > p$ (hence for every $q > 0$).

▷ Inverse Minkowski Inequality implies with Holder exponents $\frac{q}{p+d} < 1$ and $-\frac{q}{d} < 0$, yields

$$\Delta_{N+1}^{(a)} \geq C_p'' \underbrace{\left[\int d(\xi, a^{(N)})^q \mu(\xi)\right]^{\frac{p+d}{q}}}_{=e_q(a^{(N)},X)^{p+d}} \left[\int g(\xi)^{-\frac{q}{d}} \mu(\xi)\right]^{-\frac{d}{s}}.$$

$$\left[\int g(\xi)^{-\frac{q}{d}} \mu(\xi)\right]^{-\frac{d}{q}} \asymp \mathbb{E}|X|^{(1+\frac{\delta}{d}q} < +\infty.$$

▷ Inverse Minkowski Inequality implies with Holder exponents $\frac{q}{p+d} < 1$ and $-\frac{q}{d} < 0$, yields

$$\Delta^{(a)}_{N+1} \ge C''_p \underbrace{\left[\int d(\xi, a^{(N)})^q \mu(\xi)\right]^{\frac{p+d}{q}}}_{=e_q(a^{(N)},X)^{p+d}} \left[\int g(\xi)^{-\frac{q}{d}} \mu(\xi)\right]^{-\frac{d}{s}}.$$

$$\left[\int g(\xi)^{-\frac{q}{d}} \mu(\xi)\right]^{-\frac{d}{q}} \asymp \mathbb{E}|X|^{(1+\frac{\delta}{d}q} < +\infty.$$

▷ Hence

$$\Delta^{(a)}_{N+1} \ge C_{p,\delta,X} e_q(a^{(N)}, X)^{p+d}$$

so that, using that $k \mapsto e_q(a^{(k)}, X)^{p+d}$ is decreasing,

$$N e_q(a^{(2N)}, X)^{p+d} \le \sum_{k=N+1}^{2N} e_q(a^{(k)}, X)^{p+d} \le \sum_{k=N+1}^{2N} \Delta^{(a)}_k \le e_p(a^{(N)}, X)^{p+d}.$$

Finally

$$e_q(a^{(2N)}, X)^{p+d} \le \frac{1}{N} e_p(a^{(N)}, X)^{p+d} \asymp C_X N^{-1-\frac{p}{d}}.$$

Numerical computations when $d = 1$, $\mu = \mathcal{N}(0;1)$

▷ Graph $N \mapsto (2N+1)^2 e_2^2\big(a^{(2N+1)}, \mu\big)$, $N = 1, \ldots, 2^{10} = 1\,024$ where $\mu = \mathcal{N}(0;1)$.

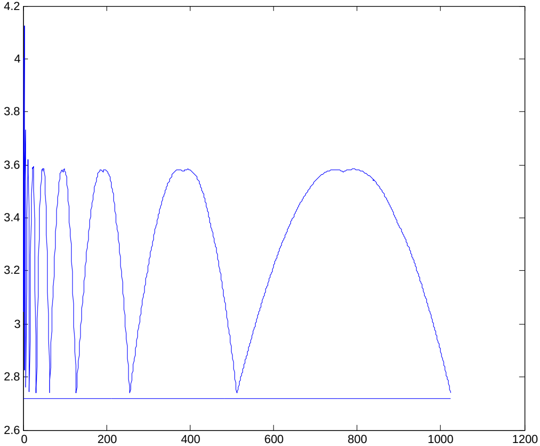


Figure: Graph $N \mapsto (2N+1)^2 e_2^2\big(a^{(2N+1)}, \mathcal{N}(0;1)\big)$, $N = 1, \ldots, 2^{10} = 1\,024$.

Unexpected (?) behavior

▷ As $\limsup_N N^2 e_2^2(a^{(N)}, \mu) = \limsup_N (2N+1)^2 e_2^2(a^{(2N+1)}, \mu)$ since $e_2^2(a^{(N)}, \mu) \downarrow 0$,

$$\liminf_N N^2 e_2^2(a^{(N)}, \mathcal{N}(0;1)) \approx 2.763\cdots > \frac{3}{2}\sqrt{\pi} = \lim_N N^2 e_2^2(\mathcal{N}(0;1))$$

since $\frac{3}{2}\sqrt{\pi} \approx 2.65868\ldots$.

▷ Hence, we cannot derive from the empirical measure theorem ([GL00], '00):

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{a_k} \xrightarrow{w} ???$$

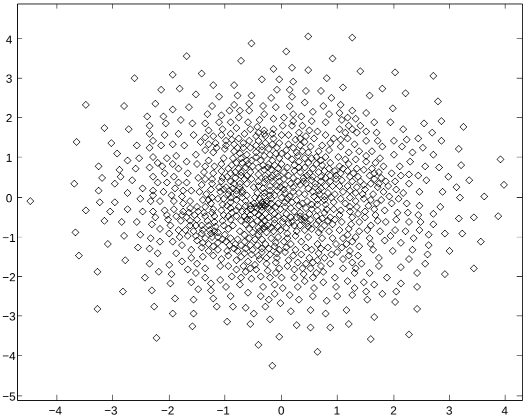
the asymptotic behavior if the empirical measure remains an open question...

Greedy prototypes, $\mu = \mathcal{N}(0, I_2)$, $N = 1000$

$a^{(1000)}$ as computed by a randomized greedy Lloyd I procedure with

$$N = 1000 \quad \text{and} \quad M = M(N) = 1000 \times N$$

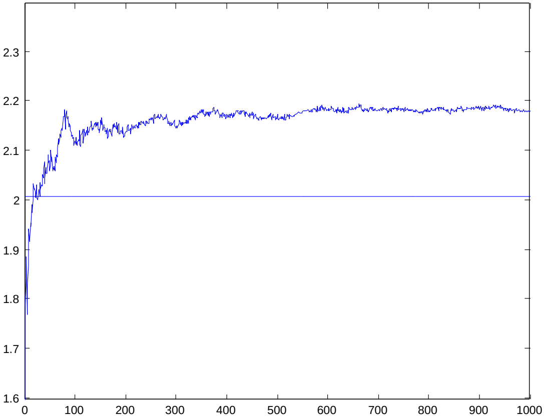
we obtain



Normalized mean Quantization error $N \mapsto \sqrt{N} e_2(a^{(N)}, \mathcal{N}(0, I_2))$, $N = 1, \ldots, 1000$

Implementing the randomized Greedy Lloyd's I algorithm with

$$M = M(N) = 1\,000 \times N, \; N = 1, \ldots, 1000.$$



Toward Functional Quantization

What remains tue when $\mathbb{R}^d \rightsquigarrow (H, |\,.\,|_H)$?

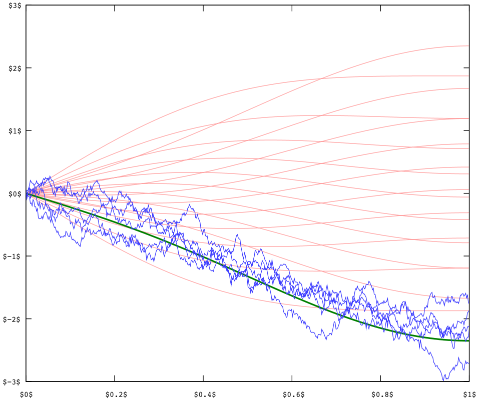


Figure: A $N = 20$-quantizers of Brownian motion *vs* some Brownian paths......

(with S. Corlay), [CP15]

W is Gaussian process with independent increments

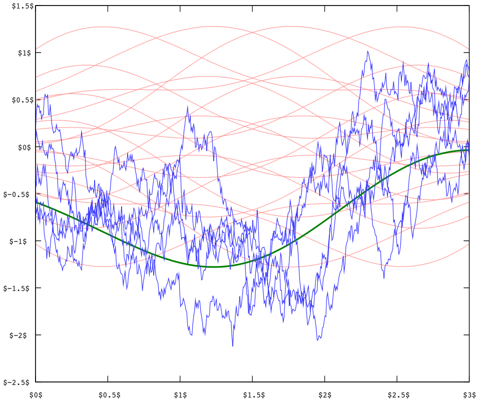


Figure: A $N = 20$-quantizers of a stationary Ornstein-Uhlenbeck process *vs* some paths. . . . . .

(with S. Corlay)

$$X_t = \int_{-\infty}^{t} e^{-(t-s)} dW_s \quad || \quad dX_t = -X_t dt + dW_t,\ X_0 \sim \mathcal{N}(0; \frac{1}{2})$$

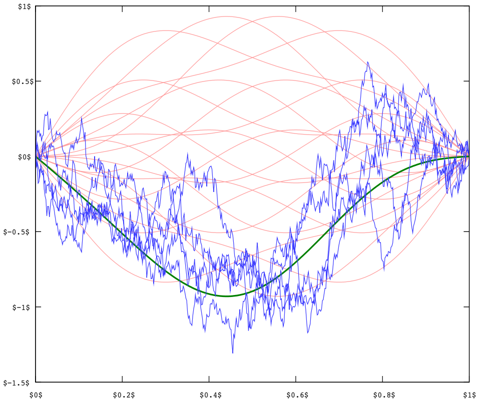


Figure: A $N = 20$-quantizers of Brownian bridge *vs* some paths. . . . . .

(with S. Corlay)

$$X_t = W_t - tW_1,\ \ t \in [0, 1]$$

non Gaussian diffusion processes? etc.

Some questions

▷ What is the connection between blue chaotic lines and pink smooth lines?

▷ How to get the pink smooth lines from the blue chaotic lines?

▷ Can we replace the blue chaotic lines by the pink smooth lines (for numerics, in a *SDE* or in a *SPDE*)?

▷ Can we take advantage of the pink smooth lines to simulate the blue chaotic lines?

Optimal Functional Quantization (of the Brownian motion)

▷ $H = L^2_T := L^2([0,T], dt)$, $(f|g) = \int_0^T f(t)g(t)dt$, $|f|_{L^2_T} = \sqrt{(f|f)}$.

▷ The Brownian motion W: centered Gaussian process with covariance operator
$C_W(f) : f \longmapsto (t \mapsto \int_{[0,T]^2} (s \wedge t) f(s) ds)$.

▷ Diagonalization of C_W yields the Karhunen-Loève system ($\equiv$ CPA of W)

$$\boxed{e_n^W(t) = \sqrt{2T}\sin\left(\left(n - \frac{1}{2}\right)\pi\frac{t}{T}\right), \qquad \lambda_n = \left(\frac{T}{\pi(n-\frac{1}{2})}\right)^2, \; n \geq 1}$$

$$W_t \overset{L^2_T}{=} \sum_{n\geq 1} (W|e_n^W)_2\, e_n^W(t) = \sum_{n\geq 1} \sqrt{\lambda_n}\,\xi_n\, e_n^W(t)$$

$$\xi_n \sim \mathcal{N}(0;1), \quad n \geq 1, \quad \text{i.i.d.}$$

Sharp (quadratic) rate

▷ THEOREM (Luschgy-P., *JFA* [LP02] (2002) and *AoP* [LP04] (2004), *EJP* [LP14](2014)) Let α^N, $N \geq 1$, be a sequence of optimal N-quantizers.

▷ $\alpha^N = (\alpha_1^N, \cdots, \alpha_N^N) \subset \text{span}\{e_1^W, \ldots, e_{d(N)}^W\}$ with

$$d(N) \gtrsim \log N/2 \quad \text{and} \quad d(N) = \lfloor \log N \rfloor \text{ is admissible}$$

▷ Conjecture: $d_{\min}(N) \sim \log N$.

▷ $$e_N(W, L_T^2) = \|W - \widehat{W}^{\alpha^N}\|_2 \sim \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{\log N}}, \qquad \left(\frac{\sqrt{2}}{\pi} = \sqrt{0.2026\ldots} = 0.4502\ldots\right).$$

▷ Reduction to finite dimension (Pythagoras)

$$(\mathcal{O}_N) \begin{cases} \|W - \widehat{W}^{\alpha^N}\|_2^2 = \|Z - \widehat{Z}^{\beta(N)}\|_2^2 + \sum_{k \geq d(N)+1} \lambda_k \\ Z = Z^{(\lambda)} \sim \bigotimes_{k=1}^{d(N)} \mathcal{N}(0, \lambda_k) \quad \& \quad \|Z - \widehat{Z}^{\beta(N)}\|_2 = e_N(Z, \mathbb{R}^{d(N)}) \end{cases}$$

Then

$$\boxed{\widehat{W}^{\alpha^N} = \sum_{k=1}^{d(N)} (\widehat{Z}^{\beta(N)})_k e_k^W.}$$

Optimal Quadratic Functional Quantization of Gaussian processes

THEOREM (Luschgy-P., *JFA* [LP02] (2002) and *AoP* [LP04] (2004), *EJP* [LP14](2014))
Let $X = (X_t)_{t\in[0,1]}$ be a Gaussian process with *K-L* eigensystem $(\lambda_n^X, e_n^X)_{n\geq 1}$. Let α^N, $N \geq 1$, be a sequence of quadratic optimal N-quantizers for X. If

$$\lambda_n^X \sim \frac{\kappa}{n^b} \quad \text{as } n \to \infty \qquad (b > 1).$$

$\triangleright$ $\alpha^N = (\alpha_1^N, \cdots, \alpha_N^N) \subset \mathrm{span}\{e_1^X, \ldots, e_{d^X(N)}^X\}$ with

$$d^X(N) \gtrsim \frac{1}{b^{1/(b-1)}} \frac{2}{b} \log N \quad \text{and} \quad d(N) = \lfloor \frac{2}{b} \log N \rfloor \text{ is admissible}$$

$\triangleright$ Conjecture: $d^X(N) \sim \dfrac{2}{b} \log N]$.

$\triangleright$ $e_N(X, L^2_{[0,1]}) = \|X - \widehat{X}^{\alpha^N}\|_2 \sim \sqrt{\kappa} \left(\dfrac{b^b}{(b-1)^{b-1}}\right)^{\frac{1}{2}} \dfrac{1}{(2\log N)^{\frac{b-1}{2}}}$.

$\triangleright$ Extensions to $\lambda_n^X \left(\begin{array}{c} \leq \\ \geq \end{array}\right) \varphi(n)$, φ regularly varying, index $-b \leq -1$.

Applications to classical (centered) Gaussian processes

▷ Applications to classical (centered) Gaussian processes

Sharp rates for $e_N(X, L^2_T)$ available for

- Brownian bridge, Ornstein-Uhlenbeck process, Gaussian diffusions (same rate).

– Fractional Brownian motion with Hurst constant $H \in (0, 1)$

$$e_N(W^H, L^2_T) \sim \frac{c_2}{(\log N)^H}.$$

– Brownian sheet, m-fold integrated Brownian motion, etc.

EXTENSIONS TO $p \neq 2$ (methods are different)

– Brownian motion and fractional Brownian motion: Dereich-Scheutzow (2005) based on self-similarity properties, random quantization, small balls

$$e_{N,r}(W^H, L^p_T) \sim \frac{c_p}{(\log N)^H}.$$

Optimal quadratic Functional Quantization (of W): numerical aspects ($T = 1$)

- ▷ Good news: $(\mathcal{O}_N)$ is a finite dimensional optimization problem.
- ▷ Bad news: $\lambda_1 = 0.40528...$ and $\lambda_2 = 0.04503... \approx \lambda_1/10$!!!
- ▷ A way out:

$$(\mathcal{O}_N) \equiv \begin{cases} N\text{-optimal quantization of } \bigotimes_{k=1}^{d(N)} \mathcal{N}(0,1) \\ \text{for the covariance norm } |(z_1, \ldots, z_{d(N)})|^2 = \sum_{k=1}^{d(N)} \lambda_k z_k^2. \end{cases}$$

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