An introduction to BSDE

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1 Utility maximization: the financial market model

(N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92)

$(\Omega, \mathcal{F}, P)$ probability space with m-dim. Brownian motion $W = (W^1, \cdots, W^n)$, T finite time horizon

price processes

trivial bond $S^0 \equiv 1$; stocks $1 \leq i \leq d$

$$\begin{aligned} \frac{dS_t^i}{S_t^i} &= \sum_{j=1}^n \sigma_t^{ij} dW_t^j + b_t^i dt \\ &= \sigma_t^i dW_t + b_t^i dt \\ &= \sigma_t^i [dW_t + \theta_t dt], \quad t \in [0, T] \end{aligned}$$

$$\theta = \sigma^* [\sigma \sigma^*]^{-1} b, \quad \underline{\varepsilon} I^d \leq [\sigma \sigma^*] \leq \overline{\varepsilon} I^d \quad \text{with} \quad \underline{\varepsilon} > 0$$

1 Utility maximization: the financial market model

investment strategies with non-convex constraints

(N. El Karoui, R. Rouge '00 for convex constraints)

$\tilde{C} \subset \mathbb{R}^d$ closed; $\tilde{\mathcal{A}}$: strategies $\pi = (\pi^1, \cdots, \pi^d)$ s. th.
$\pi \in \tilde{C}$ $\quad P \otimes \lambda$-a.s. $\quad$ (λ Lebesgue measure)
$\{\exp(-\alpha \int_0^\tau \pi_s \frac{dS_s}{S_s}) : \tau$ stopping time in $[0, T]\}$ uniformly integrable

wealth process

$$X_t^\pi = x + \sum_{i=1}^d \int_0^t \pi_s^i \frac{dS_s^i}{S_s^i} = x + \int_0^t \pi_s \sigma_s [dW_s + \theta_s ds]$$

preferences of small agent measured by **exponential utility** from terminal wealth

utility function

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}.$$

1 Utility maximization: the optimization problem

F $\mathcal{F}_T$-measurable liability at time T.

First formulation:

Find

$$V(x) = \sup_{\pi \in \tilde{\mathcal{A}}} E(U(X_T^\pi - F)) = \sup_{\pi \in \tilde{\mathcal{A}}} E(U(x + \int_0^T \pi_s \sigma_s [dW_s + \theta_s ds] - F)).$$

For simplicity:

$$\begin{aligned} p &= \pi\sigma, \\ C &= \tilde{C}\sigma, \\ \mathcal{A} &= \tilde{\mathcal{A}}\sigma. \end{aligned}$$

$$X_t^p = x + \int_0^t p_s [dW_s + \theta_s ds], \quad t \in [0, T]$$

Second formulation:

Find

$$V(x) = \sup_{p \in \mathcal{A}} E(U(X_T^p - F)) = \sup_{p \in \mathcal{A}} E(-\exp(-\alpha(x + \int_0^T p_s [dW_s + \theta_s ds] - F))).$$

1 Utility optimization: a solution method based on BSDE

Idea: Construct family of processes $R^{(p)}$ such that

form 1

$$\begin{array}{lcl} R_0^{(p)} & = & \text{constant}, \\ R_T^{(p)} & = & -\exp(-\alpha(X_T^p - F)), \\ R^{(p)} & & \text{supermartingale}, \quad p \in \mathcal{A}, \\ R^{(p^*)} & & \text{martingale, for (exactly) one} \quad p^* \in \mathcal{A}. \end{array}$$

Then

$$\begin{aligned} E(-\exp(-\alpha[X_T^p - F])) &= E(R_T^{(p)}) \\ &\leq E(R_0^p) \\ &= V(x) \\ &= E(R_0^{(p^*)}) \\ &= E(-\exp(-\alpha[X_T^{(p^*)} - F])). \end{aligned}$$

Hence p^* optimal strategy.

1 Utility optimization: a solution method based on BSDE

Introduction of BSDE into problem: find generator f of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F,$$

such that with

$$R_t^{(p)} = -\exp(-\alpha[X_T^p - Y_t]), \quad t \in [0, T],$$

we have

$$\begin{aligned} R_0^{(p)} &= -\exp(-\alpha(x - Y_0)) \\ &= \text{constant}, \end{aligned} \qquad \text{(fulfilled)}$$

form 2 $\quad R_T^{(p)} = -\exp(-\alpha(X_T^p - F))$ (fulfilled)

$R^{(p)}$ supermartingale, $p \in \mathcal{A}$,
$R^{(p^*)}$ martingale, for (exactly) one $p^* \in \mathcal{A}$.

This gives solution of valuation problem.

1 Utility optimization: construction of generator of BSDE

How to determine f:

Suppose f generator of BSDE. Then

$$
\begin{aligned}
R_t^{(p)} &= -\exp(-\alpha[X_T^p - Y_t]) \\
&= -\exp(-\alpha[x - Y_0]) \cdot \exp(-\alpha[\int_0^t (p_s - Z_s) dW_s - \int_0^t [f(s, Z_s) - p_s\theta_s] ds]) \\
&= \exp(-\alpha[x - Y_0]) \cdot \exp(-\alpha \int_0^t (p_s - Z_s) dW_s - \frac{\alpha^2}{2} \int_0^t (p_s - Z_s)^2 ds) \\
&\quad \cdot - \exp(\int_0^t [\alpha f(s, Z_s) - \alpha p_s \theta_s + \frac{\alpha^2}{2}(p_s - Z_s)^2] ds) \\
&= M_t^{(p)} \cdot A_t^{(p)},
\end{aligned}
$$

with $M^{(p)}$ nonnegative martingale. $R^{(p)}$ satisfies **(form 2)** iff for

$$q(\cdot, p, z) = f(\cdot, z) - p\theta + \frac{\alpha}{2}(p - z)^2, \quad p \in \mathcal{A}, z \in \mathbb{R},$$

we have

1 Utility optimization: construction of generator of BSDE

form 3

$$\begin{aligned} q(\cdot, p, z) &\geq 0, \quad p \in \mathcal{A} \quad \text{(supermartingale cond.)} \\ q(\cdot, p^*, z) &= 0, \quad \text{for (exactly) one} \quad p^* \in \mathcal{A} \quad \text{(martingale cond.)}. \end{aligned}$$

Now

$$\begin{aligned} q(\cdot, p, z) &= f(\cdot, z) - p\theta + \frac{\alpha}{2}(p - z)^2 \\ &= f(\cdot, z) + \frac{\alpha}{2}(p - z)^2 - (p - z) \cdot \theta + \frac{1}{2\alpha}\theta^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2 \\ &= f(\cdot, z) + \frac{\alpha}{2}[p - (z + \frac{1}{\alpha}\theta)]^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2. \end{aligned}$$

Under **non-convex constraint** $p \in C$:

$$[p - (z + \frac{1}{\alpha}\theta)]^2 \geq d^2(C, z + \frac{1}{\alpha}\theta).$$

with **equality** for at least one possible choice of p^* due to **closedness** of C.
Hence **(form 3)** is solved by the choice

1 Utility optimization: construction of generator of BSDE

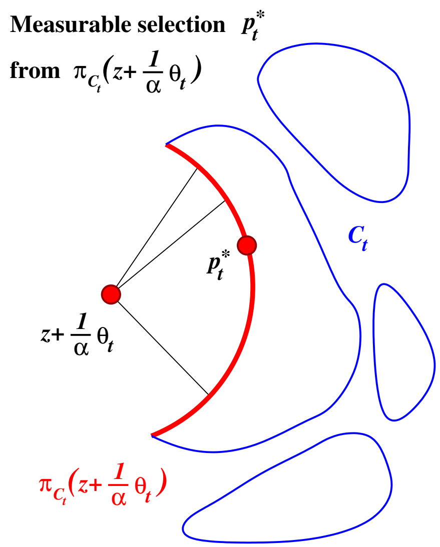
form 4

$$f(\cdot, z) = -\frac{\alpha}{2} d^2(C, z + \frac{1}{\alpha}\theta) + z \cdot \theta + \frac{1}{2\alpha}\theta^2 \quad \text{(supermartingale cond.)}$$
$$p^* \quad \text{s. th.} \quad d(C, z + \frac{1}{\alpha}\theta) = d(p^*, z + \frac{1}{\alpha}\theta) \quad \text{(martingale cond.)}.$$

Problem: Let

$$\Pi_C(v) = \{p \in \mathbb{R}^d : d(C, v) = d(p, v)\}.$$

Find measurable selection p_t^* from $\pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t)$. Solved by classical **measurable selection method**.



1 Utility optimization: main result

Thm 1
(Y, Z) unique solution of BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T],$$

with

$$f(t, Z_t) = -\frac{\alpha}{2} d^2(C_t, Z_t + \frac{1}{\alpha}\theta_t) + Z_t \cdot \theta_t + \frac{1}{2\alpha}\theta_t^2.$$

Then value function of utility optimization problem under constraint $p \in \mathcal{A}$ given by

$$V(x) = -\exp(-\alpha[x - Y_0]).$$

There exists an optimal trading strategy $p^* \in \mathcal{A}$ such that

$$p_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t), \quad t \in [0, T].$$

1 Utility optimization: methods of proof of main result

Proof:

- existence and uniqueness for BSDE locally Lipschitz in z (M. Kobylanski '00)
- measurable selection theorem for $\Pi_{C_t}(Z_t + \frac{1}{\alpha}\theta_t)$
- BMO properties of the martingales $\int Z_s dW_s, \int p^*_s dW_s$ for uniform integrability of exponentials (regularity of coefficients)$\bullet$

Aim: existence and uniqueness for BSDE with Lipschitz drivers.

2 BSDE: spaces

Time horizon $T > 0$, $m \in \mathbf{N}$; $(\Omega, \mathcal{F}, P)$ n-dimensional Wiener space, Wiener process $W = (W^1, \cdots, W^n)$; $(\mathcal{F}_t)_{t \geq 0})$ natural filtration, completed.

$L^2(\mathbf{R}^m)$ space of $\mathbf{R}^m$-valued $\mathcal{F}_T$-measurable r.v., norm $E(|X|^2)^{\frac{1}{2}}$.

$H^2(\mathbf{R}^m)$ space of $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted processes $X : \Omega \times [0, T] \to \mathbf{R}^m$, norm $||X||_2 = E(\int_0^T |X_t|^2 dt)^{\frac{1}{2}}$.

$H^1(\mathbf{R}^m)$ space of $(\mathcal{F}_t)_{0 \leq t \leq T}$-adapted processes $X : \Omega \times [0, T] \to \mathbf{R}^m$, norm $||X||_1 = E([\int_0^T |X_t|^2 dt]^{\frac{1}{2}})$.

For $\beta > 0$, $X \in H^2(\mathbf{R}^m)$ let

$$||X||_{2,\beta} = E(\int_0^T e^{\beta t} |X_t|^2 dt),$$

$H^{2,\beta}(\mathbf{R}^m)$ space $H^2(\mathbf{R}^m)$, norm $|| \cdot ||_{2,\beta}$.

2 BSDE: notions

parameters of BSDE:
terminal condition: $\xi \in L^2(\mathbf{R}^m)$.
generator: $f : \Omega \times \mathbf{R}_+ \times \mathbf{R}^m \times \mathbf{R}^{n\times m} \to \mathbf{R}^m$ product measurable, adapted in t,

$$\text{(H1)} \qquad f(\cdot, 0, 0) \in H^2(\mathbf{R}^m),$$

f *uniformly Lipschitz,* i.e. there is $C \in \mathbf{R}$ s. th. for
$(y_1, z_1), (y_2, z_2) \in \mathbf{R}^m \times \mathbf{R}^{n\times m}$, $P \otimes \lambda$-a.e. $(\omega, t) \in \Omega \times \mathbf{R}_+$

$$\text{(H2)} \qquad |f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq C[|y_1 - y_2| + |z_1 - z_2|].$$

Definition 1. (f, ξ) *fulfilling above measurability requirements, (H1), (H2):* standard parameter.

Aim: find pair of $\mathcal{F}_t)_{0\leq t\leq T}$-adapted processes $(Y_t, Z_t)_{0\leq t\leq T}$ satisfying *backward stochastic differential equation (BSDE)*

$$(*) \qquad dY_t = Z_t^* dW_t - f(\cdot, t, Y_t, Z_t)dt, \quad Y_T = \xi.$$

2 BSDE: a priori inequalities

Method: use contraction on suitable Banach space. For this purpose, need *a priori inequalities.*

Lemma 1. *For $i = 1, 2$ let (f^i, ξ^i) be standard parameters, $(Y^i, Z^i) \in L^2(\mathbf{R}^m) \times H^2(\mathbf{R}^m)$ solutions of (*) with corresponding standard parameters, where for $z \in \mathbf{R}^{n \times m}$ we denote $|z| = (tr(zz*))^{\frac{1}{2}}$. Let C be a Lipschitz constant for f^1. Define for $0 \leq t \leq T$*

$$\begin{aligned} \delta Y_t &= Y_t^1 - Y_t^2, \\ \delta_2 f_t &= f^1(\cdot, t, Y_t^1, Z_t^1) - f^2(\cdot, t, Y_t^2, Z_t^2). \end{aligned}$$

Then for any triple (λ, μ, β) with $\lambda > 0, \lambda^2 > C, \beta \geq C(2 + \lambda^2) + \mu^2$ we have

$$\begin{aligned} ||\delta Y||_{2,\beta}^2 &\leq T[e^{\beta T} E(|\delta Y_T|^2) + \frac{1}{\mu^2} ||\delta_2 f||_{2,\beta}^2], \\ ||\delta Z||_{2,\beta}^2 &\leq \frac{\lambda^2}{\lambda^2 - C}[e^{\beta T} E(|\delta Y_T|^2) + \frac{1}{\mu^2} ||\delta_2 f||_{2,\beta}^2]. \end{aligned}$$

2 Proof a priori estimate: L^2 boundedness of Y

Proof

1. $(Y, Z) \in H^2(\mathbf{R}^m) \times H^2(\mathbf{R}^{n\times m})$ solution of (*) with (ξ, f). **Show:**

$$\sup_{0\le t\le T} |Y_t| \in L^2(\mathbf{R}^m).$$

By (*)

$$\sup_{0\le t\le T} |Y_t| \le |\xi| + \int_0^T |f(\cdot, s, Y_s, Z_s)|ds + \sup_{0\le t\le T} |\int_t^T Z_s^* dW_s|,$$

and by Doob's inequality

$$E(\sup_{0\le t\le T} |\int_t^T Z_s^* dW_s|^2) \le 4E(\sup_{0\le t\le T} |\int_0^t Z_s^* dW_s|^2) \le 8E(\int_0^T |Z_s|^2 ds).$$

By (H1), (H2) $|\xi| + \int_0^T |f(\cdot, s, Y_s, Z_s)|ds \in L^2(\mathbf{R})$. Hence we get

$$E(\sup_{0\le t\le T} |Y_t|^2) < \infty.$$

2 Proof a priori estimate: preliminary bound

2. To derive **preliminary bound,** apply Itô's formula to semimartingale $(e^{\beta s}|\delta Y_s|^2)_{0\le s\le T}$. For $0 \le t \le T$

$$
\begin{aligned}
e^{\beta T}|\delta Y_T|^2 \; - \; & e^{\beta t}|\delta Y_t|^2 \\
&= \beta \int_t^T e^{\beta s}|\delta Y_s|^2 ds + 2\int_t^T e^{\beta s}\langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) \\
&\quad - f^2(\cdot, s, Y_s^2, Z_s^2)\rangle ds - 2\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s^* dW_s\rangle + \int_t^T e^{\beta s}|\delta Z_s|^2 ds.
\end{aligned}
$$

Reorder to get

$$
\begin{aligned}
e^{\beta t}|\delta Y_t|^2 \; + \; & \beta \int_t^T e^{\beta s}|\delta Y_s|^2 ds + \int_t^T e^{\beta s}|\delta Z_s|^2 ds \\
= \; & e^{\beta T}|\delta Y_T|^2 + 2\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s^* dW_s\rangle \\
& -2\int_t^T e^{\beta s}\langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2)\rangle ds.
\end{aligned}
$$

2 Proof a priori estimate: preliminary bound

3. For $0 \le t \le T$ **show**:

$$E(e^{\beta t}|\delta Y_t|^2) \le E(e^{\beta T}|\delta Y_T|^2) + \frac{1}{\mu^2} E(\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds).$$

Integrate in the inequality of 2., to get

$$\begin{aligned} E(e^{\beta t}|\delta Y_t|^2) &+ \beta E(\int_t^T e^{\beta s}|\delta Y_s|^2 ds) + E\int_t^T e^{\beta s}|\delta Z_s|^2 ds) \\ &\le E(e^{\beta T}|\delta Y_T|^2) \\ &\quad +2E(\int_t^T e^{\beta s}\langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2)\rangle ds). \end{aligned}$$

By assumptions for $0 \le s \le T$

$$\begin{aligned} |f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2)| &\le |f^1(\cdot, s, Y_s^1, Z_s^1) - f^1(\cdot, s, Y_s^2, Z_s^2)| \\ &\quad + |\delta_2 f_s| \\ &\le C[|\delta_s Y| + |\delta_s Z|] + |\delta_2 f_s|. \end{aligned}$$

2 Proof of a priori estimate: preliminary bound

The latter implies

$$\begin{aligned}
&\int_t^T E(2e^{\beta s}|\langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2)\rangle| ds \\
&\leq \int_t^T 2e^{\beta s} E(|\delta Y_s|[C(|\delta_s Y| + |\delta_s Z|) + |\delta_2 f_s|] ds \\
&= \int_t^T 2e^{\beta s}[CE(|\delta Y_s|^2) + E(|\delta_s Y|(C|\delta_s Z|) + |\delta_2 f_s|)] ds.
\end{aligned}$$

Now for $C, y, z, t > 0$ with $\mu, \lambda > 0$

$$\begin{aligned}
2y(Cz + t) &= 2Cyz + 2yt \\
&\leq C[(y\lambda)^2 + (\frac{z}{\lambda})^2] + (y\mu)^2 + (\frac{t}{\mu})^2 \\
&= C(\frac{z}{\lambda})^2 + (\frac{t}{\mu})^2 + y^2(\mu^2 + C\lambda^2).
\end{aligned}$$

2 Proof of a priori estimate: preliminary bound

With this further estimate last term:

$$\begin{aligned}
\int_t^T 2e^{\beta s}[CE(|\delta Y_s|^2) \quad &+ \quad E(|\delta_s Y|(C|\delta_s Z|)+|\delta_2 f_s|)]ds \\
&\leq \int_t^T e^{\beta s}[2CE(|\delta Y_s|^2)+\frac{C}{\lambda^2}E(|\delta_s Z|^2) \\
&\quad +\frac{1}{\mu}E(|\delta_2 f_s|^2)+(\mu^2+C\lambda^2)E(|\delta_s Y|^2]ds \\
&= \int_t^T e^{\beta s}[(\mu^2+C(2+\lambda^2))E(|\delta Y_s|^2) \\
&\quad +\frac{C}{\lambda^2}E(|\delta_s Z|^2)+\frac{1}{\mu}E(|\delta_2 f_s|^2)]ds.
\end{aligned}$$

2 Proof of a priori estimate: the inequality

Summarizing, we obtain

$$
\begin{aligned}
(**) \qquad E(e^{\beta t}|\delta Y_t|^2) &\leq E(\int_t^T e^{\beta s}|\delta Y_s|^2 ds)[-\beta + C(2+\lambda^2) + \mu^2] \\
&\quad + E(\int_t^T e^{\beta s}|\delta Z_s|^2 ds)[\frac{C}{\lambda^2} - 1] + E(e^{\beta T}|\delta Y_T|^2) \\
&\quad + \frac{1}{\mu^2} E(\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds) \\
&\leq E(e^{\beta T}|\delta Y_T|^2) + \frac{1}{\mu^2} E(\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds).
\end{aligned}
$$

This is the claimed inequality.

4. First inequality: integrate inequality from 3. in t.

5. Second inequality: take second term from right hand side to left in (**). •

2 BSDE: existence, uniqueness

Theorem 1. *Let* (ξ, f) *be standard parameters. Then there exists a uniquely determined pair* $(Y, Z) \in H^2(\mathbf{R}^m) \times H^2(\mathbf{R}^{n\times m})$ *with the property*

$$Y_t = \xi - \int_t^T Z_s^* dW_s + \int_t^T f(\cdot, s, Y_s, Z_s) ds, \quad 0 \le t \le T.$$

Proof
Consider

$$\Gamma : H_{2,\beta}(\mathbf{R}^m) \times H_{2,\beta}(\mathbf{R}^{n\times m}) \to H_{2,\beta}(\mathbf{R}^m) \times H_{2,\beta}(\mathbf{R}^{n\times m}), (y, z) \mapsto (Y, Z),$$

where (Y, Z) is solution of

$$(*) \quad Y_t = \xi - \int_t^T Z_s^* dW_s + \int_t^T f(\cdot, s, y_s, z_s) ds, \quad 0 \le t \le T.$$

2 Proof: existence, uniqueness

1. **Prove:** (Y, Z) well defined. By assumptions

$$\xi + \int_t^T f(\cdot, s, y_s, z_s)ds \in L^2(\Omega), \quad 0 \le t \le T.$$

Therefore

$$M_t = E(\xi + \int_0^T f(\cdot, s, y_s, z_s)ds|\mathcal{F}_t), \quad 0 \le t \le T,$$

is martingale. M is continuous, since we are on Wiener filtration. M square integrable. Hence martingale representation provides (unique) $Z \in H^2(\mathbf{R}^{n\times m})$ s. th.

$$M_t = M_0 + \int_0^t Z_s^* dW_s, \quad 0 \le t \le T.$$

Let

$$Y_t = M_t - \int_0^t f(\cdot, s, y_s, z_s)ds.$$

Then Y square integrable, and

$$Y_t = E(\xi + \int_t^T f(\cdot, s, y_s, z_s)ds|\mathcal{F}_t), \quad 0 \le t \le T.$$

2 Proof: existence, uniqueness

Hence

$$Y_T = \xi = M_0 + \int_0^T Z_s^* dW_s - \int_0^T f(\cdot, s, y_s, z_s) ds,$$

and thus for $0 \le t \le T$

$$\begin{aligned} Y_t &= \xi - M_0 - \int_0^T Z_s^* dW_s + \int_0^T f(\cdot, s, y_s, z_s) ds \\ &+ M_0 + \int_0^t Z_s^* dW_s - \int_0^t f(\cdot, s, y_s, z_s) ds \\ &= \xi - \int_t^T Z_s^* dW_s + \int_t^T f(\cdot, s, Y_s, Z_s) ds. \end{aligned}$$

2 Proof: existence, uniqueness

2. **Prove:** For $\beta > 2(1+T)C$ Γ is contraction.

Let $(y^1, z^1), (y^2, z^2) \in H^{2,\beta}(\mathbf{R}^m) \times H^{2,\beta}(\mathbf{R}^{n\times m})$, $(Y^1, Z^1), (Y^2, Z^2)$ solutions of (*) by 1. We apply Lemma 1 with $C = 0, \beta = \mu^2$, and $f^i = f(\cdot, y^i, z^i)$. We obtain

$$
\begin{aligned}
||\delta Y||_{2,\beta} &\leq \frac{T}{\beta} E\left(\int_0^T e^{\beta s} |f(\cdot, s, y_s^1, z_s^1) - f(\cdot, s, y_s^2, z_s^2)|^2 ds\right), \\
||\delta Z||_{2,\beta} &\leq \frac{1}{\beta} E\left(\int_0^T e^{\beta s} |f(\cdot, s, y_s^1, z_s^1) - f(\cdot, s, y_s^2, z_s^2)|^2 ds\right).
\end{aligned}
$$

By Lipschitz continuity of f

$$
\begin{aligned}
||\delta Y||_{2,\beta} &\leq \frac{2TC}{\beta} [||\delta y||_{2,\beta} + ||\delta z||_{2,\beta}], \\
||\delta Z||_{2,\beta} &\leq \frac{2C}{\beta} [||\delta y||_{2,\beta} + ||\delta z||_{2,\beta}].
\end{aligned}
$$

2 Proof: existence, uniqueness

We summarize to obtain

$$(**) \quad ||\delta Y||_{2,\beta} + ||\delta Z||_{2,\beta} \leq \frac{2C(T+1)}{\beta}[||\delta y||_{2,\beta} + ||\delta z||_{2,\beta}].$$

By choice of β, Γ is contraction.

3. Let $(\overline{Y}, \overline{Z})$ be fixed point of Γ given by 2. Let

$$Y_t = E(\xi + \int_t^T f(\cdot, s, \overline{Y}_s, \overline{Z}_s) ds | \mathcal{F}_t), \quad 0 \leq t \leq T.$$

Then Y is continuous and P-a.s. identical to $\overline{Y}$. Then $(Y, \overline{Z})$ solves BSDE.

4. Uniqueness: contractivity of Γ, uniqueness of fixed point. •

2 BSDE: recursive approximation of solution

Corollary 1. *Let $\beta > 2(1+T)C$, $((Y^k, Z^k))_{k\geq 0}$ given by $Y^0 = Z^0 = 0$,*

$$Y_t^{k+1} = \xi - \int_t^T (Z_s^{k+1})^* dW_s + \int_t^T f(\cdot, s, Y_s^k, Z_s^k) ds$$

by preceding proof. Then $((Y^k, Z^k))_{k\geq 0}$ converges in $H^{2,\beta}(\mathbf{R}^m) \times H^{2,\beta}(\mathbf{R}^{n\times m})$ to the solution (Y, Z) of (*).

Proof

(**) in proof of Theorem 1 yields

$$||Y^{k+1} - Y^k||_{2,\beta} + ||Z^{k+1} - Z^k||_{2,\beta} \leq \varepsilon^k [||Y^1 - Y^0||_{2,\beta} + ||Z^1 - Z^0||_{2,\beta}],$$

with $\varepsilon = \frac{2C(T+1)}{\beta} < 1$. Hence

$$\sum_{k\in\mathbf{N}} [||Y^{k+1} - Y^k||_{2,\beta} + ||Z^{k+1} - Z^k||_{2,\beta}] < \infty.$$

Use standard argument. •

2 Linear BSDE

Examples for explicitly solvable BSDE rare; here is one.

Proposition 1. $\beta : \Omega \times \mathbf{R}_+ \to \mathbf{R}, \gamma : \Omega \times \mathbf{R}_+ \to \mathbf{R}^n, \varphi : \Omega \times \mathbf{R}_+ \to \mathbf{R}$ *bounded, adapted, continuous,* $\varphi \in H_T^2(\mathbf{R})$, $f(\cdot, t, y, z) = \varphi_t + \beta_t y + z^* \gamma_t, t \geq 0, y \in \mathbf{R}, z \in \mathbf{R}^n$, $\xi \in L_T^2(\mathbf{R})$. *Then the linear BSDE*

$$(*) \quad dY_t = Z_t^* dW_t - f(\cdot, t, Y_t, Z_t)dt, \quad Y_t = \xi,$$

has unique solution $(Y, Z) \in H_{T,\beta}^2(\mathbf{R}) \times H_{T,\beta}^2(\mathbf{R}^n)$, *and*

$$Y_t = \mathbf{E}(\xi \Gamma_T^t + \int_t^T \Gamma_s^t \varphi_s ds | \mathcal{F}_t),$$

where for $s \geq t$ Γ_s^t *is solution to forward SDE*

$$(**) \quad d\Gamma_s^t = \Gamma_s^t (\beta_s ds + \gamma_s^* dW_s), \quad \Gamma_t^t = 1.$$

If $\xi, \varphi \geq 0$, *also* $Y \geq 0$.

2 Linear BSDE; solution

Proof

1. (ξ, f) fulfil conditions of Theorem 2; hence unique solution (Y, Z) exists.

2. Fix $t \geq 0$. Then

$$\Gamma_s^t = \exp\left(\int_t^s \gamma_u^* dW_u - \int_t^s (\frac{1}{2}|\gamma_u^*|^2 - \beta_u)du\right) = \Gamma_s^0(\Gamma_t^0)^{-1}$$

solves the linear SDE (**). Now

$$d(\Gamma_t^0 Y_t + \int_0^t \Gamma_s^0 \varphi_s ds) = d\Gamma_t^0 Y_t + \Gamma_t^0 dY_t + d\langle \Gamma_t^0, Y_t \rangle$$
$$= \Gamma_t^0 Y_t(\beta_t dt + \gamma_t^* dW_t) + \Gamma_t^0[Z_t^* dW_t - (\varphi_t + \beta_t Y_t + \gamma_t^* Z_t)dt] + \Gamma_t^0 \gamma_t^* Z_t dt + \Gamma_t^0 \varphi_t dt$$
$$= \Gamma_t^0 Y_t \gamma_t^* dW_t + \Gamma_t^0 Z_t^* dW_t,$$

hence $(\Gamma_t^0 Y_t + \int_0^t \Gamma_s^0 \varphi_s ds, t \geq 0)$ local martingale.

2 Linear BSDE: solution

Moreover,

$$\sup_{0\le t\le T}\Gamma_t^0|Y_t| \le \sup_{0\le t\le T}\Gamma_t^0 \sup_{0\le t\le T}|Y_t| \in L_T^1(\mathbf{R}), \qquad \sup_{0\le t\le T}\Gamma_t^0 \in L_T^2(\mathbf{R}).$$

So local martingale is uniformly integrable, and therefore

$$\begin{aligned} Y_t &= (\Gamma_t^0)^{-1}[\Gamma_t^0 Y_t + \int_0^t \Gamma_s^0 \varphi_s ds] - (\Gamma_t^0)^{-1}\int_0^t \Gamma_s^0\varphi_s ds \\ &= (\Gamma_t^0)^{-1}\mathbf{E}(\Gamma_T^0\xi + \int_0^T \Gamma_s^0\varphi_s ds|\mathcal{F}_t) - (\Gamma_t^0)^{-1}\int_0^t \Gamma_s^0\varphi_s ds \\ &= \mathbf{E}(\xi\Gamma_T^t + \int_t^T \Gamma_s^t\varphi_s ds|\mathcal{F}_t). \end{aligned}$$

3. If $\varphi, \xi \ge 0$, by the representation also $Y \ge 0$. •

2 BSDE: comparison principle

Important property of solutions: order given by terminal states is preserved.

Theorem 2. $(f^1, \xi^1), (f^2, \xi^2)$ *standard parameters,* $(Y^1, Z^1), (Y^2, Z^2)$ *corresponding solutions of BSDE. Assume*

$$\begin{aligned} \xi^1 &\geq \xi^2, \\ \delta_2 f_t(\omega) &= f^1(\cdot, t, Y_t^2, Z_t^2)(\omega) - f^2(\cdot, t, Y_t^2, Z_t^2)(\omega) \geq 0 \quad \text{for} \quad \mathbb{P} \otimes \lambda - \quad \text{a.a.} \quad (\omega, t). \end{aligned}$$

Then $Y_t^1 \geq Y_t^2$ *for any* $0 \leq t \leq T$*.*

Proof

For notational simplicity, assume that $n = 1$.

$$\begin{aligned} \Delta_y f^1(\cdot, t) &= \frac{f^1(\cdot, t, Y_t^1, Z_t^1) - f^1(\cdot, t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2}, \\ \Delta_z f^1(\cdot, t) &= \frac{f^1(\cdot, t, Y_t^2, Z_t^1) - f^1(\cdot, t, Y_t^2, Z_t^2)}{Z_t^1 - Z_t^2}, \end{aligned}$$

with the convention that quotient is 0 if denominator vanishes.

2 BSDE: comparison principle

$\delta Y, \delta Z$ defined as in proof of a priori inequalities; then BSDE for δY is linear:

$$\begin{aligned} d\delta Y_t &= \delta Z_t dW_t - (f^1(\cdot,t,Y_t^1,Z_t^1) - f^2(\cdot,t,Y_t^2,Z_t^2))dt \\ &= \delta Z_t dW_t - (f^1(\cdot,t,Y_t^1,Z_t^1) - f^1(\cdot,t,Y_t^2,Z_t^2))dt - \delta_2 f_t dt \\ &= \delta Z_t dW_t - [\Delta_y f^1(t)\delta Y_t + \Delta_z f^1(t)\delta Z_t + \delta_2 f_t]dt; \end{aligned}$$

initial condition $\delta\xi = \xi^1 - \xi^2$. By Lipschitz continuity, $\Delta_y f, \Delta_z f$ bounded, adapted, $\delta_2 f, \delta\xi$ square integrable. Representation formula of Proposition yields $\delta Y_t \geq 0$ for $\delta\xi, \varphi = \delta_2 f \geq 0$, all $0 \leq t \leq T$. •

Comparison principle essential in proof of existence of solutions for BSDE with coefficients fulfilling local Lipschitz conditions (as the one on the utility optimization problem of first part).

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