

# Dynamic formulations of Optimal Transportation and variational MFGs

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1. Basic Introduction to Dynamic OT
2. Time Discretization and MultiMarginal OT
3. Entropic Regularization and IPFP/Sinkhorn
4. Scaling Algorithms
5. Schrödinger bridge and system
6. Application to Stochastic VMFGs

- Source/Target Data :  $d\rho_i(x)(= \rho_i(x) dx), i = 0, 1$

$$\rho_i \geq 0, \int_D \rho_0(x) dx = \int_D \rho_1(x) dx = 1, D \subset \mathbb{R}^n$$

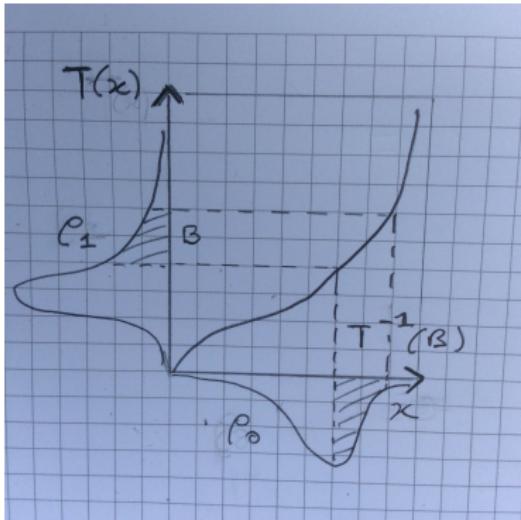
- Measure preserving Transport Maps :

$$\mathcal{M} = \{ T : D \rightarrow D, T_{\#}\rho_0 = \rho_1 \}$$

$$\forall B \subset D$$

$$T_{\#}\rho_0(B) = \rho_0(T^{-1}(B))$$

$$\det(DT)\rho_1(T(x)) = \rho_0(x)$$



- ▶ Cost Function : 
$$\mathcal{I}(T) = \int_D c(x, T(x)) \rho_0(x) dx$$
- ▶ Monge Problem : 
$$(MP) \inf_{T \in \mathcal{M}} \mathcal{I}(T)$$
- ▶ Costs : typically  $c(x, y) = \frac{1}{p} \|y - x\|^p$  (Monge  $\rightarrow p = 1$ ).
- ▶ Th. Brenier (1991) ( $p = 2$ ) :  $\exists! \nabla \varphi$ ,  $\varphi$  convex such that  $\mathcal{I}(\nabla \varphi(x)) = \min_{T \in \mathcal{M}} \mathcal{I}(T)$
- ▶ Measure preserving property yields :  
$$(MABV2) \quad \det(D^2 \varphi) \rho_1(\nabla \varphi) = \rho_0, \quad \nabla \varphi(X_0) \subset X_1$$
- ▶ Extensive Sobolev regularity theory developed since by Cafarelli and Ambrosio schools ...  
O(N) Numerical methods : Monotone FD scheme B. Froese Oberman (2014)  
B. Collino Mirebeau (2016) and B. Duval (2017) and Semi-Discrete approaches Mérigot (2011) Lévy (2015).

# Adding the dynamics

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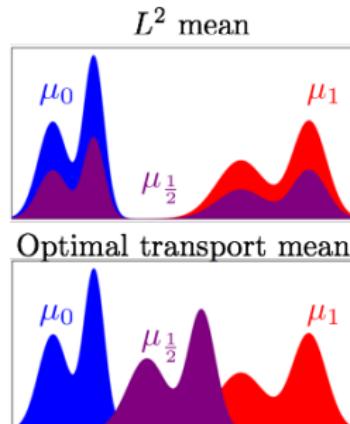
- Displacement Interpolation - McCann (1997). Def :

$$x \mapsto \Theta(t, x) = x + t(\nabla \varphi(x) - x), \quad t \in ]0, 1[$$

$$\rho^*(t, \cdot) = (\Theta(t, \cdot))_{\#} \rho_0$$

$$(t \mapsto \rho^*(t, \Theta(t, x)) = \frac{\rho_0(x)}{\det(D_x \Theta(t, x))})$$

- for all  $t$ ,  $\Theta(t, \cdot)$  solves (MP) from  $\rho_0$  to  $\rho^*(t, \cdot)$ .
- $W_2(\rho_0, \rho^*(t, \cdot)) = \sqrt{\mathcal{I}(\Theta(t, \cdot))}$  is a geodesic distance on  $\mathcal{P}(D)$ .





- ▶ Particles move in straight line at constant speed

$$\dot{\Theta}(t, x_0) = (\nabla \varphi(x_0) - x_0) \stackrel{\text{def.}}{=} v^*(t, \Theta(t, x_0))$$

- ▶ B. Brenier (2000) :  $(\rho^*, v^*)$  is the unique minimum of

$$\inf_{(\rho, v) \text{ satisfies } (CE)} \int_0^1 \int_D \frac{1}{2} \rho(t, x) \|v(t, x)\|^2 dx dt$$

$$(CE) \quad \partial_t \rho + \operatorname{div}(\rho v) = 0, \quad \partial_\nu v = 0 \text{ on } \partial D, \quad \rho(i, \cdot) = \rho_i(\cdot)$$

- ▶ This is a non-smooth convex relaxation (under  $(\rho, v) \rightarrow (\rho, \sigma) \stackrel{\text{def.}}{=} \rho v$ )  
proximal splitting methods achieve  $O(N^3)$  heuristically.
- ▶ A variational deterministic MFG - Lions Lasry (2007) : (CE) and  
 $v = \operatorname{grad} \psi$  and  $(HJ) \quad \partial_t \psi + \frac{1}{2} \|\nabla \psi\|^2 = 0$   
see B. Carlier (2015).

# Kantorovich Relaxation (1942)

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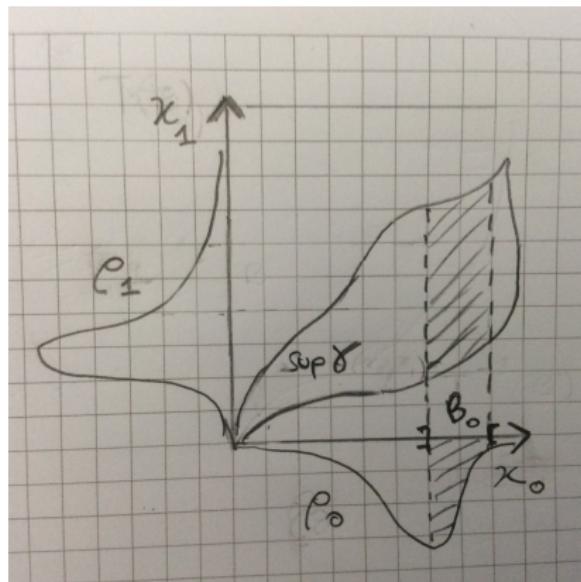
- ▶ Transport Plans :

$$\Pi(\rho_0, \rho_1) = \{\gamma \in \mathcal{P}(D_0 \times D_1), P_{\#}^{D_i} \gamma = \rho_i, i = 0, 1\}$$

$$\forall B_0 \subset D_0$$

$$P_{\#}^{D_0} \gamma(B_0) = \gamma(B_0, D_1)$$

$$\gamma(B_0, D_1) = \rho_0(B_0)$$



- ▶  $\Pi(\rho_0, \rho_1)$  is non empty :  $\rho_0 \otimes \rho_1(x_0, x_1) = \rho_0(x_0) \rho_1(x_1)$

# Kantorovich Relaxation (1942)

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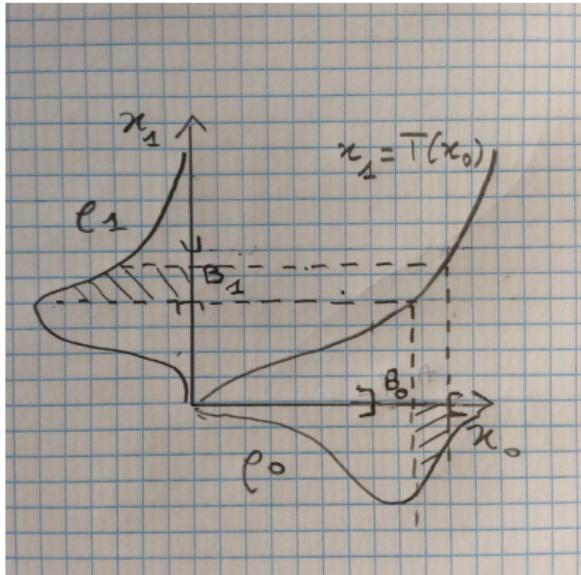
Deterministic Transport plan :

$$\gamma_T \stackrel{\text{def.}}{=} (Id, T)_\# \rho_0$$

$$\gamma_T(B_0, B_1) =$$

$$\rho_0(\{x \in B_0, \text{ s.t. } T(x) \in B_1\})$$

$$T \in \mathcal{M} \Leftrightarrow \gamma_T \in \Pi(\rho_0, \rho_1)$$



►  $\gamma_{\nabla \varphi}$  solves  $(MK)$  
$$\inf_{\gamma \in \Pi(\rho_0, \rho_1)} \int_{D_0 \times D_1} c(x_0, x_1) d\gamma(x_0, x_1)$$

- Linear program but  $N^2$  unknowns Simplex or Interior point methods stuck to  $N \simeq 100$ .

# Dynamic Kantorovich relaxation

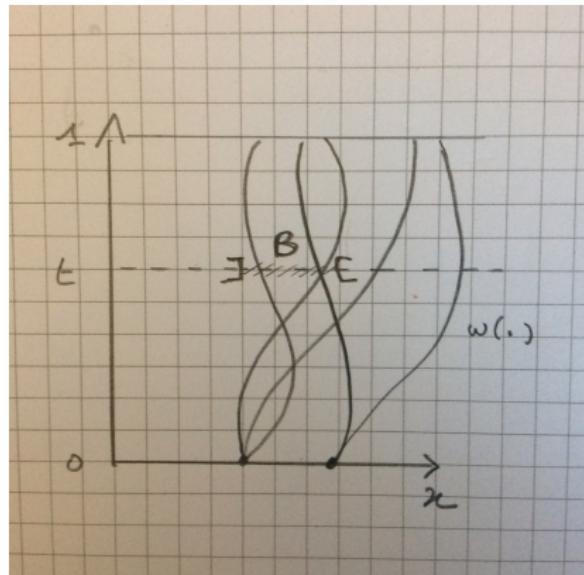
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- ▶ **Defs :**  $\Omega(D) = C([0, 1]; D)$  the set of abs. cont. path  
 $\omega : t \in [0, 1] \mapsto \omega(t) \in D$ .  
 $Q \in \mathcal{P}(\Omega(D))$  a probability measure on  $\Omega(D)$ .  
 $e_t : \Omega(D) \mapsto D$  the  $t$ -evaluation function -  $e_t(\omega) = \omega(t)$ .

$$\forall B \subset D$$

$$(e_t)_\# Q(B) =$$

$$Q(\{\omega \in \Omega(D), \omega(t) \in B\})$$



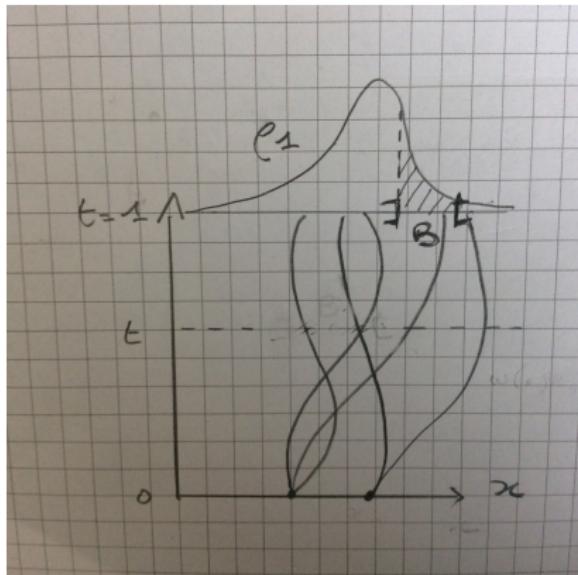
# Dynamic Kantorovich relaxation

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- ▶ **Defs :**  $\Omega(D) = C([0, 1]; D)$  the set of abs. cont. path  
 $\omega : t \in [0, 1] \mapsto \omega(t) \in D$ .  
 $Q \in \mathcal{P}(\Omega(D))$  a probability measure on  $\Omega(D)$ .  
 $e_t : \Omega(D) \mapsto D$  the  $t$ -evaluation function -  $e_t(\omega) = \omega(t)$ .

$$\forall B \subset D$$

$$(e_1)_\# Q(B) = \rho_1(B)$$



$$(DMK) \quad \inf_{\{Q \in \mathcal{P}(\Omega(D)), (e_i)_\# Q = \rho_i, i=0,1\}} \int_{\Omega(D)} \int_0^1 \|\dot{\omega}(t)\|^2 dt dQ(\omega)$$

- ▶  $\Phi_\Theta : D \rightarrow \Omega(D)$ ,  $\Phi_\Theta(x_0) = \Theta(., x_0)$ .

- ▶ The solution  $Q^* = (\Phi_\Theta)_\# \rho_0$  is deterministic.

$$\forall O \subset \Omega(D), Q^*(O) = \rho_0(\{x_0 \in D_0, s.t. \Theta(x_0, .) \in O\})$$

- ▶  $\rho(t, .) = (e_t)_\# Q^*$  is the CFD geodesic.

Analysis by Ambrosio school, see Santambrogio book  
(2015)

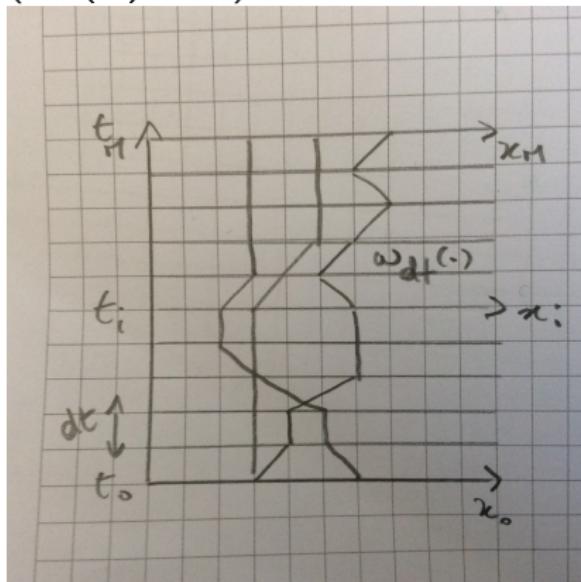
- ▶  $\mathcal{P}(\Omega(D))$  is a BIG space : next section present an efficient numerical method.

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# Time Discretization

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- Discretize time : Set  $dt = \frac{1}{M}$   $t_i = i dt$ ,  $i = 0..M$
- Restrict to piecewise linear path  $\omega_{dt} = \{x_0, x_1, \dots, x_M\}$  ( $\omega_{dt}(t_i) = x_i$ ).



- ▶ Minimize w.r.t.  $Q_{dt}(x_0, x_1, \dots, x_M) \in \mathcal{P}(\otimes_{i=0,M} D_i)$
- ▶  $(e_{t_i})_\# Q_{dt} = \rho_i$  becomes a margin condition :

$$\int_{\otimes_{j \neq i} D_j} dQ_{dt}(x_0, x_1, \dots, x_M) = \rho_i(x_i)$$

- ▶ Time integration of linear path in ( $DMK$ ) :

$$\inf_{Q \in \mathcal{E}} \int_{\otimes_{i=0,M} D_i} \left( \sum_{i=0,M-1} \frac{1}{dt} \|x_{i+1} - x_i\|^2 \right) dQ_{dt}(x_0, x_1, \dots, x_M)$$

$$\mathcal{E} = \{Q_{dt} \in \mathcal{P}(\otimes_{i=0,M} D_i), (e_{t_i})_\# Q_{dt} = \rho_i, i = 0, 1\}$$

- General Form of MMOT :

$$\inf_{Q \in \mathcal{E}} \int_{\otimes_{i=0,M} D_i} c(x_0, x_1, \dots, x_M) dQ(x_0, x_1, \dots, x_M)$$

$$\mathcal{E} = \{ Q \in \mathcal{P}(\otimes_{i=0,M} D_i), (e_{t_i})_# Q = \rho_i, i = 0, 1, \dots, M \}$$

- Ex. : Density Functional Theory (Friesecke et al, Butazzo et al (...), Pass, ... )

$$c \stackrel{\text{def.}}{=} \sum_{i < j} \frac{1}{\|x_i - x_j\|} \quad \text{Margins : } (e_i)_# Q = \bar{\rho}, i = 0, \dots, M$$

Existence of Maps open ...

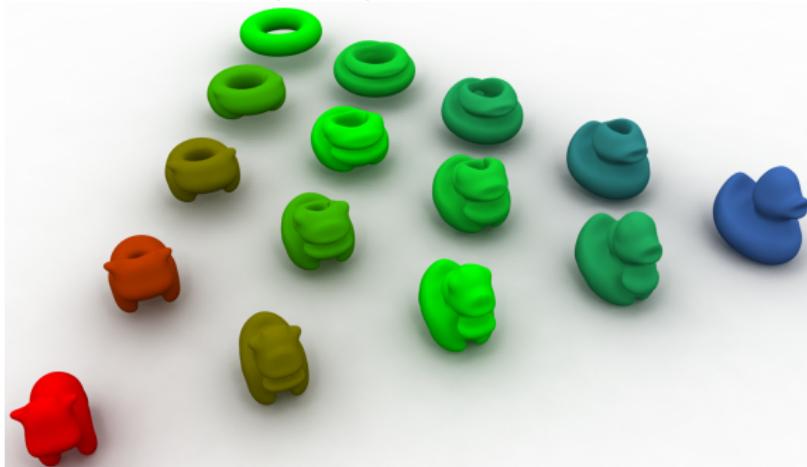
- Generalized Euler Geodesics (Brenier 89)

- Ex. : Wasserstein Barycenters (Aguech/Carlier (2011))

$$c \stackrel{\text{def.}}{=} \sum_i \lambda_i \|x_i - B(x_0, \dots, x_M)\|^2 \quad B(x_0, \dots, x_M) \stackrel{\text{def.}}{=} \sum_i \lambda_i x_i$$

Margins :  $(e_i)_\# Q = \rho_i, i = 0, \dots, M$  Barycenter :  $B_\# Q \dots$

- Solomon et al (2015)



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# Entropic regularization of OT

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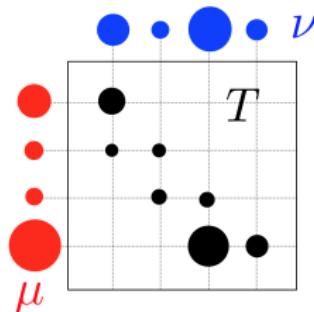
See Christian Leonard surveys for the connection with the Schrödinger problem in the continuous setting.

Discretize in space

$$D_0 : \{x_i\} \text{ and } D_1 : \{x_j\}$$

$$\rho_0 = \sum_i \mu_i \delta_{x_i} \text{ and } \rho_1 = \sum_j \nu_j \delta_{y_j}$$

$$c_{ij} = c(x_i, x_j)$$



► Entropic regularisation of MK :

$$(MK_\varepsilon) \quad \min_{\gamma \in \mathcal{G}} \sum_{ij} \gamma_{ij}^\varepsilon c_{ij} + \varepsilon \gamma_{ij}^\varepsilon (\log \gamma_{ij}^\varepsilon - 1)$$

$$\mathcal{G} = \{\gamma \in \mathbb{R}^{N \times N}, \gamma_{ij}^\varepsilon \geq 0, \sum_j \gamma_{ij}^\varepsilon = \mu_i, \sum_i \gamma_{ij}^\varepsilon = \nu_j\}$$

► Set  $\bar{\gamma}_{ij}^\varepsilon = e^{-\frac{c_{ij}}{\varepsilon}}$

$$(MK_\varepsilon) \quad \min_{\gamma \in \mathcal{G}} \sum_{ij} KL(\gamma_{ij}^\varepsilon | \bar{\gamma}_{ij}^\varepsilon)$$

$$KL(f|g) = f \left( \log\left(\frac{f}{g}\right) - 1 \right)$$

Sinkhorn (67) Ruschendorf (95) Galichon (09) Cuturi (13) ...

$$\min_{\gamma_{ij}^\varepsilon} \max_{\{\varphi_i^\varepsilon, \psi_j^\varepsilon\}} \sum_{ij} \psi_j^\varepsilon \nu_j + \varphi_i^\varepsilon \mu_i + \gamma_{ij}^\varepsilon (c_{ij} - \psi_j^\varepsilon - \varphi_i^\varepsilon + \varepsilon (\log \gamma_{ij}^\varepsilon - 1))$$

- Optimal plan is a scaling :

$$\gamma_{ij}^{\star,\varepsilon} = a_i^\varepsilon b_j^\varepsilon \bar{\gamma}_{ij}^\varepsilon$$

where  $a_i^\varepsilon = e^{\frac{\varphi_i^\varepsilon}{\varepsilon}}$  and  $b_j^\varepsilon = e^{\frac{\psi_j^\varepsilon}{\varepsilon}}$ .

- Margin constraints give :

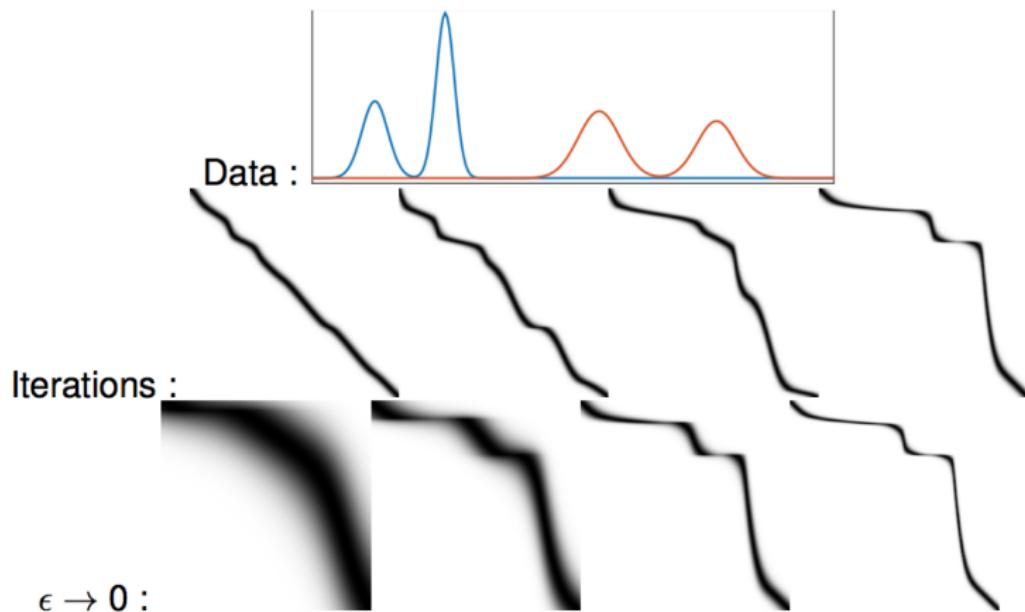
$$a_i^\varepsilon = \frac{\mu_i}{\sum_j \bar{\gamma}_{ij}^\varepsilon b_j^\varepsilon} \text{ and } b_j^\varepsilon = \frac{\nu_j}{\sum_i \bar{\gamma}_{ij}^\varepsilon a_i^\varepsilon}.$$

- IPFP is the relaxation :

$$a_i^{\varepsilon,k+\frac{1}{2}} = \frac{\mu_i}{\sum_j \bar{\gamma}_{ij}^\varepsilon b_j^{\varepsilon,k}} \quad b_j^{\varepsilon,k+1} = \frac{\nu_j}{\sum_i \bar{\gamma}_{ij}^\varepsilon a_i^{\varepsilon,k+\frac{1}{2}}}.$$

# 1-D IPFP/Sinkhorn

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- ▶ It. are contractions in the Hilbert metric  
$$d_H(p, q) = \log\left(\frac{\max_i\left(\frac{p_i}{q_i}\right)}{\min_i\left(\frac{p_i}{q_i}\right)}\right).$$
- ▶ Convergence with  $\varepsilon$  (Cominetti San Martin (94), Carlier et al (15)).
- ▶ On a cartesian grid and  $d \geq 2$  ( $x_i = \{x_{i_1}^1, x_{i_2}^2\}$ )

$$\bar{\gamma}_{ij}^\varepsilon = e^{-\frac{\|x_i - x_j\|^2}{\varepsilon}} = e^{-\frac{\|x_{i_1}^1 - x_{j_1}^1\|^2}{\varepsilon}} e^{-\frac{\|x_{i_2}^2 - x_{j_2}^2\|^2}{\varepsilon}}$$
 is separable

Store  $(\sqrt{N} \times \sqrt{N})$  matrices. One Iteration costs  $O(N^{1.5})$ .

- ▶ # iterations increase with  $\frac{1}{\varepsilon}$ . Stability problems can be fixed.
- ▶ Many Generalizations including MMOT check B. et al (2015) Chizat et al (2017).

- $M$  scalings :

$$Q_{i_1, i_1, \dots, i_M}^{\star, \varepsilon} = a_{i_1}^1 a_{i_2}^2 \dots a_{i_M}^M e^{-\frac{c(x_1, \dots, x_M)}{\varepsilon}}$$

$$c(x_1, \dots, x_M) = \frac{\|x_{i_2} - x_{i_1}\|^2 + \|x_{i_3} - x_{i_2}\|^2 + \dots + \|x_{i_M} - x_{i_{M-1}}\|^2}{dt}$$

- IPFP algebra amounts to

$$a_{i_m}^{m,(k)} = \frac{\mu_{i_m}}{\sum_{i_1, \dots, i_{m-1}, \cancel{i_m}, i_{m+1}, \dots, i_M} \{.\}}$$

$$\{.\} \stackrel{\text{def.}}{=} a_{i_1}^{1,(k)} \dots a_{i_{m-1}}^{m-1,(k)} \cancel{a_{i_m}^m} a_{i_{m+1}}^{m+1,(k-1)} \dots a_{i_M}^{M,(k-1)} \overline{Q}_{i_1, \dots, i_M}$$

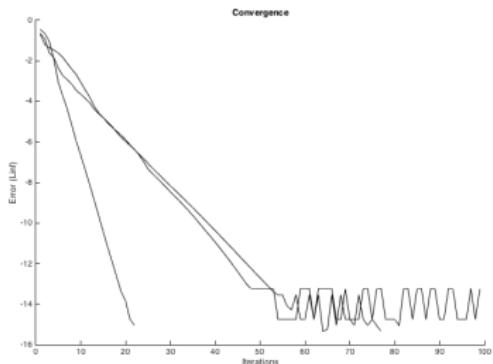
- Cost again **separable** (also along dimensions)

$$\overline{Q}_{i_1, \dots, i_M} = \prod_{m=1}^{M-1} \xi_{i_m i_{m+1}} \quad \xi_{i,j} = e^{-\frac{\|x_i - x_j\|^2}{dt \varepsilon}}$$

- Store ( $M \sqrt{N} \times \sqrt{N}$ ) matrices - one iteration costs  $O(M N^{1.5})$

# $W_2$ geodesic between characteristic functions

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- ▶ Primal :

$$\min_{\gamma} F_0(P^{D_0} \# \gamma) + F_1(P^{D_1} \# \gamma) + \varepsilon \text{KL}_{D_0 \times D_1}(\gamma | \bar{\gamma})$$

$F_i$  l.s.c. convex and proper ...

- ▶ Dual :

$$\max_{(u,v)} -F_0^*(-u) - F_1^*(-v) - \varepsilon \langle e^{\frac{1}{\varepsilon}(u \oplus v)} - 1, \bar{\gamma} \rangle_{D_0 \times D_1}$$

- ▶ Pbm is well posed and the solution is again a scaling :

$$\gamma^*(x_0, x_1) = e^{\frac{u^*(x_0)}{\varepsilon}} \bar{\gamma}(x_0, x_1) e^{\frac{v^*(x_1)}{\varepsilon}} \quad \bar{\gamma} = e^{-\frac{\|x_1 - x_0\|^2}{\varepsilon}}$$

- ▶ Scaling Algorithm:

$$u^{k+\frac{1}{2}} = \arg \max_u (-F_0^*(-u) - \varepsilon \langle e^{\frac{1}{\varepsilon}u} - 1, \Gamma e^{\frac{1}{\varepsilon}v^k} \rangle_{D_0})$$

$$v^{k+1} = \arg \max_v (-F_1^*(-v) - \varepsilon \langle e^{\frac{1}{\varepsilon}v} - 1, \Gamma^T e^{\frac{1}{\varepsilon}u^{k+\frac{1}{2}}} \rangle_{D_1})$$

$$\langle e^{\frac{1}{\varepsilon}(u \oplus v)} - 1, \bar{\gamma} \rangle_{D_0 \times D_1} = \langle e^{\frac{1}{\varepsilon}u} - 1, \Gamma e^{\frac{1}{\varepsilon}v} \rangle_{D_0} = \langle e^{\frac{1}{\varepsilon}v} - 1, \Gamma^T e^{\frac{1}{\varepsilon}u} \rangle_{D_1}$$

- Ex 1 - Sinkhorn/IPFP :  $F(\rho) = 0$  if  $\rho = \rho_0$  and  $+\infty$  else .
- Ex 2 - Unbalanced OT :  $F(\rho) = \lambda KL_D(\rho|\rho_0)$  .

The scaling Alg. :  $u_i^{\varepsilon, k+\frac{1}{2}} = \left( \frac{\mu_i}{\sum_j \bar{\gamma}_{ij}^\varepsilon b_j^{\varepsilon, k}} \right)^{\frac{\lambda}{\lambda+\varepsilon}}$

- Ex 3 - JKO Gradient Flows :  $F_0(\rho) = 0$  if  $\rho = \rho_0$  and  $+\infty$  else and  $F_1(\rho) = dt E(\rho)$   
an internal energy (simplest is  $E = V \rho$ ).

Check Carlier et al (15) for the analysis of the impact of the entropic regularization on the JKO GF.

- Congestion :  $F(\rho) = 0$  if  $\rho \leq \alpha$  and  $+\infty$  else .

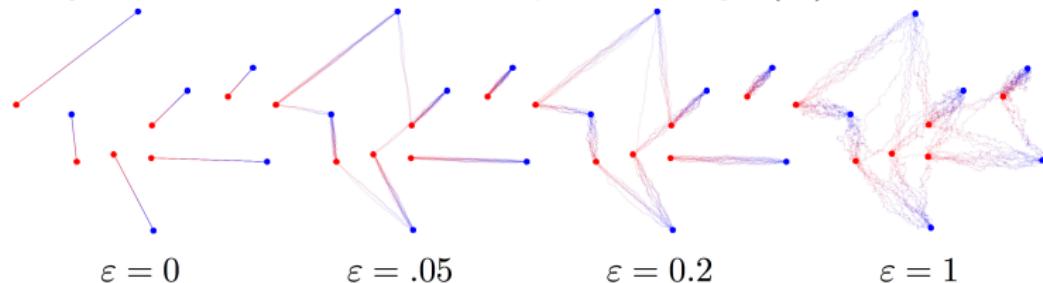
The scaling Alg. :  $a_i^{\varepsilon, k+\frac{1}{2}} = \frac{\min\{\alpha_i, \sum_j \bar{\gamma}_{ij}^\varepsilon b_j^{\varepsilon, k}\}}{\sum_j \bar{\gamma}_{ij}^\varepsilon b_j^{\varepsilon, k}}$

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# Schrödinger bridge

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Long list of refs, check Léonard survey, see also Leger (17)



## ► Schrödinger problem

$$\inf_{(\rho, v) \text{ satisfies (FP)}} \int_0^1 \int_D \frac{1}{2} \rho(t, x) \|v(t, x)\|^2 dx dt$$

$$(FP) \quad \partial_t \rho + \operatorname{div}(\rho v) = \varepsilon \Delta \rho, \quad \rho(i, \cdot) = \rho_i(\cdot)$$

## ► A variational stochastic "planning" MFG :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla \psi) = \varepsilon \Delta \rho, & \rho(i, \cdot) = \rho_i(\cdot) \\ \partial_t \psi + \frac{1}{2} \|\nabla \psi\|^2 = -\varepsilon \Delta \psi \end{cases}$$

- ▶ Hopf-Cole Change of variable :

$$\boxed{\eta = e^{\frac{1}{2\varepsilon}\psi} \quad \eta^* = \rho e^{-\frac{1}{2\varepsilon}\psi}}$$

gives  $\begin{cases} \partial_t \eta + \varepsilon \Delta \eta = 0 \\ \partial_t \eta^* - \varepsilon \Delta \eta^* = 0 \end{cases}$

Guéant (11?)

- ▶ Solve Schrödinger System : Find  $(a, b)$  s.t.

$$\begin{cases} \rho_0(x_0) = a(x_0) \int \Gamma(x_0, x_1) b(x_1) dx_1 & (t=0) \\ \rho_1(x_1) = b(x_1) \int \Gamma(x_0, x_1) a(x_0) dx_0 & (t=1) \end{cases}$$

$$\boxed{\Gamma(x_0, x_1) = \frac{1}{(4\pi\varepsilon)^{\frac{d}{2}}} e^{-\frac{1}{4\varepsilon}\|x_1 - x_0\|^2}}$$

- ▶ Back to IPFP/Sinkhorn ...

## Increasing $\varepsilon$

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See B. Carlier Santambrogio survey (16)

► PVMFG quadratic Hamiltonian

$$\inf_{(\rho, v) \text{ satisfies } (CE)} \int_0^1 \int_D \frac{1}{2} \rho(t, x) \|v(t, x)\|^2 + G(t, x, \rho(t, x)) dx dt$$

$$(CE) \quad \partial_t \rho + \operatorname{div}(\rho v) = 0, \quad \partial_\nu v = 0 \text{ on } \partial D, \quad \rho(i, \cdot) = \rho_i(\cdot)$$

► Lagrangian Relaxation

$$\inf_{\{Q \in \mathcal{P}(\Omega(D)), (e_i)_\# Q = \rho_i, i=0,1\}} \int_{\Omega(D)} K(\omega) dQ(\omega) + \int_0^1 \mathcal{G}(t, (e_t)_\# Q) dt$$

$$K(\omega) = \int_0^1 \|\dot{\omega}(t)\|^2 dt$$

$$\mathcal{G}(t, \rho) = \int_D G(t, x, \rho(t, x)) dx \text{ if } \rho \ll \mathcal{L}^d \text{ and } +\infty \text{ else.}$$

► The minimizer is an equilibrium :  $J_{h^\star}(Q) \geq J_{h^\star}(Q^\star)$  forall adm.  $Q$

$$J_{h^\star}(Q) = \int_{\Omega(D)} K(\omega) dQ(\omega) + \int_0^1 \int_D h^\star(t, x) d(e_t)_\# Q$$

$$h^\star(t, x) = G'(t, x, \rho^\star(t, x)) \quad \rho^\star(t, \cdot) = (e_t)_\# Q^\star$$

Carlier et al (WIP)

$$\inf_{\{Q \in \dots\}} \int_{\Omega(D)} K(\omega) dQ(\omega) + \int_0^1 \mathcal{G}(t, (e_t)_\# Q) dt + \varepsilon \mathcal{H}(Q | \mathcal{L}^d)$$

$$\mathcal{H}(Q|R) = \int_{\Omega(D)} \log\left(\frac{dQ}{dR}\right) dQ(\omega) \text{ if } P \ll R \text{ and } +\infty \text{ else.}$$

► Same as :

$$\inf_{(\rho, v) \text{ satisfies (FP)}} \int_0^1 \int_D \frac{1}{2} \rho(t, x) \|v(t, x)\|^2 + t \mathcal{G}(t, x, \rho(t, x)) dx dt$$

$$(FP) \quad \partial_t \rho + \operatorname{div}(\rho v) = \varepsilon' \Delta \rho, \quad \rho(i, \cdot) = \rho_i(\cdot)$$

► Same as :

$$\inf_{\{Q \in \dots\}} \varepsilon \mathcal{H}(Q | \bar{Q}) + \int_0^1 \mathcal{G}(t, (e_t)_\# Q) dt$$

$$\bar{Q}(\cdot) = \int_D \mathcal{W}_x(\cdot) dx \text{ the reference measure}$$

$\mathcal{W}_x$  : the Wiener measure induced by a Brownian motion starting at  $x$ .

- ▶ Piecewise linear time discr. ( $M$ ) + grid in space ( $N$ ):

$$Q_{i_1, i_1, \dots, i_M}^{\varepsilon, dt} \in \mathcal{M}^{M \times N}$$

- ▶ Density Margins :  $\rho_{i_m} = \sum_{i_1, \dots, i_{m-1}, \cancel{j_m}, i_{m+1}, \dots, i_M} Q_{i_1, i_1, \dots, i_M}^{\varepsilon, dt}$

- ▶ Discrete MFG :

$$\inf_{Q^{dt}} \varepsilon KL(Q^{\varepsilon, dt} | \overline{Q}^{\varepsilon, dt}) + \sum_{j \in i_1, i_1, \dots, i_M} G_j^{dt}(\rho_j)$$

$$\overline{Q}_{i_1, \dots, i_M}^{\varepsilon, dt} = \prod_{m=1}^{M-1} \xi_{i_m i_{m+1}} \quad \xi_{ij} = e^{-\frac{\|x_i - x_j\|^2}{dt \varepsilon}}$$

- ▶  $M$  scalings :  $Q_{i_1, i_1, \dots, i_M}^{\star} = a_{i_1}^1 a_{i_2}^2 \dots a_{i_M}^M \overline{Q}_{i_1, \dots, i_M}^{\varepsilon, dt}$

- ▶ Scaling Alg : Iterate (marginwise)

$$a_{i_m}^{m,k} = \arg \max_b (- (G_{i_m}^{dt})^*(b) - \varepsilon \sum_{i_m} b_{i_m} \sum_{i_1, \dots, i_{m-1}, \cancel{j_m}, i_{m+1}, \dots, i_M} \{.\})$$

$$\{.\} \stackrel{\text{def.}}{=} a_{i_1}^{1,(k)} \dots a_{i_{m-1}}^{m-1,(k)} \cancel{a_{i_m}^m} a_{i_{m+1}}^{m+1,(k-1)} \dots a_{i_M}^{M,(k-1)} \overline{Q}_{i_1, \dots, i_M}^{\varepsilon, dt}$$

## Tests with moving obstacles (strong congestion)<sup>36</sup> / 36

$G(t, x, \rho(t, x)) = 0$  if  $\rho(t, x) = 0$  for  $x \in OBS(t)$  and  $+\infty$  else .