Numerical methods for Mean Field Games: additional material

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Outline

A first kind of convergence result: convergence to classical solutions

Assumptions

- $\nu > 0$
- $u|_{t=T} = u_T$, $m|_{t=0} = m_0$, u_T and m_0 are smooth
- $$0 < \underline{m}_0 \leq m_0(x) \leq \overline{m}_0$$

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- $$0 < \underline{\mathrm{m}}_0 \leq m_0(x) \leq \overline{m}_0$$
- The Hamiltonian is of the form
$$H(x, \nabla u) = \mathcal{H}(x) + |\nabla u|^\beta$$
where $\beta > 1$ and $\mathcal{H}$ is a smooth function. For the discrete Hamiltonian: upwind scheme.

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where $\beta > 1$ and $\mathcal{H}$ is a smooth function. For the discrete Hamiltonian: upwind scheme.
- Local coupling: the cost term is
$$F[m](x) = f(m(x)) \quad f : \mathbb{R}_+ \to \mathbb{R}, \quad \mathcal{C}^1$$
- There exist constants $c_1 > 0$, $\gamma > 1$ and $c_2 \ge 0$ s.t.
$$mf(m) \ge c_1 |f(m)|^\gamma - c_2 \qquad \forall m$$
- Strong monotonicity: there exist positive constants c_3, η_1 and $\eta_2 < 1$ s.t.
$$f'(m) \ge c_3 \min(m^{\eta_1}, m^{-\eta_2}) \qquad \forall m$$

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Theorem

Assume that the MFG system of pdes has a unique classical solution (u, m).
Let u_h (resp. m_h) be the piecewise trilinear function in $\mathcal{C}([0,T] \times \Omega)$ obtained by interpolating the values u_i^n (resp m_i^n) of the solution of the discrete MFG system at the nodes of the space-time grid.

$$\lim_{h,\Delta t \to 0} \left(\|u - u_h\|_{L^\beta(0,T;W^{1,\beta}(\Omega))} + \|m - m_h\|_{L^{2-\eta_2}((0,T)\times\Omega)} \right) = 0$$

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Main steps in the proof

1. Obtain energy estimates on the solution of the discrete problem, in particular on $\|f(m_h)\|_{L^\gamma((0,T)\times\Omega)}$

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1. Obtain energy estimates on the solution of the discrete problem, in particular on $\|f(m_h)\|_{L^\gamma((0,T)\times\Omega)}$
2. Plug the solution of the system of pdes into the numerical scheme, take advantage of the consistency and stability of the scheme and prove that $\|\nabla u-\nabla u_h\|_{L^\beta((0,T)\times\Omega)}$ and $\|m-m_h\|_{L^{2-\eta_2}((0,T)\times\Omega)}$ converge to 0

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3. To get the full convergence for u, one has to pass to the limit in the Bellman equation. To pass to the limit in the term $f(m_h)$, use the equiintegrability of $f(m_h)$.

A second kind of convergence result: convergence to weak solutions

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = f(m) & \text{in } [0,T) \times \Omega \\ \dfrac{\partial m}{\partial t} - \nu \Delta m - \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, Du) \right) = 0 & \text{in } (0,T] \times \Omega \\ u|_{t=T}(x) = u_T(x) \\ m|_{t=0}(x) = m_0(x) \end{cases} \qquad \text{(MFG)}$$

Assumptions

- discrete Hamiltonian g: consistency, monotonicity, convexity
- *growth conditions*: there exist positive constants c_1, c_2, c_3, c_4 such that

$$\begin{aligned} g_q(x,q) \cdot q - g(x,q) &\geq c_1 |g_q(x,q)|^2 - c_2, \\ |g_q(x,q)| &\leq c_3 |q| + c_4. \end{aligned}$$

- f is continuous and bounded from below
- u_T continuous, m_0 bounded

A second kind of convergence result: convergence to weak solutions

Theorem

(stated in the case $d = 2$)
Let $u_{h,\Delta t}$, $m_{h,\Delta t}$ be the piecewise constant functions which take the values u_i^{n+1} and m_i^n, respectively, in $(t_n, t_{n+1}) \times (ih - h/2, ih + h/2)$.

There exist functions $\tilde{u}$, $\tilde{m}$ such that

1. *after the extraction of a subsequence, $u_{h,\Delta t} \to \tilde{u}$ and $m_{h,\Delta t} \to \tilde{m}$ in $L^\beta(Q)$ for all $\beta \in [1, 2)$*
2. *$\tilde{u}$ and $\tilde{m}$ belong to $L^\alpha(0, T; W^{1,\alpha}(\Omega))$ for any $\alpha \in [1, \frac{4}{3})$*
3. *$(\tilde{u}, \tilde{m})$ is a weak solution to the system (MFG) in the following sense:*
   1. $$H(\cdot, D\tilde{u}) \in L^1(Q), \qquad \tilde{m} f(\tilde{m}) \in L^1(Q),$$
      $$\tilde{m}\Big(H_p(\cdot, D\tilde{u}) \cdot D\tilde{u} - H(\cdot, D\tilde{u})\Big) \in L^1(Q)$$
   2. *$(\tilde{u}, \tilde{m})$ satisfies the system (MFG) in the sense of distributions*
   3. *$\tilde{u}, \tilde{m} \in C^0([0, T]; L^1(\Omega))$ and $\tilde{u}|_{t=T} = u_T$, $\tilde{m}|_{t=0} = m_0$.*

Outline

An optimal control problem driven by a PDE

Assumptions

- $\Phi : L^2(Q) \to \mathbb{R}$, $\mathcal{C}^1$, strictly convex. Set $f[m] = \nabla\Phi[m]$.
- $\Psi : L^2(\Omega) \to \mathbb{R}$, $\mathcal{C}^1$, strictly convex. Set $g[m] = \nabla\Psi[m]$.
- Running cost: $L : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $\mathcal{C}^1$ convex and coercive.
- Hamiltonian $H(x,p) = \sup_{\gamma\in\mathbb{R}^d} \{-p\cdot\gamma - L(x,\gamma)\}$, $\mathcal{C}^1$, coercive.

$$L(x,\gamma) = \sup_{p\in\mathbb{R}^d} \{-p\cdot\gamma - H(x,p)\}.$$

Optimization problem: Minimize on (m,γ), $m \in L^2(Q)$, $\gamma \in \{L^2(Q)\}^d$:

$$J(m,\gamma) = \Phi(m) + \int_0^T \int_\Omega \Big(m(t,x)L(x,\gamma(t,x))\Big)dxdt + \Psi(m(T,\cdot))$$

subject to
$$\begin{cases} \dfrac{\partial m}{\partial t} - \nu\Delta m + \mathrm{div}(m\gamma) = 0, & \text{in } (0,T)\times\Omega, \\ m(0,x) = m_0(x) & \text{in } \Omega. \end{cases}$$

An optimal control problem driven by a PDE

The optimization problem is actually the minimization of a convex functional with linear constraints:

Set

$$\widetilde{L}(x,m,z) = \begin{cases} mL(x, \frac{z}{m}) & \text{if } m > 0, \\ 0 & \text{if } m = 0 \text{ and } z = 0, \\ +\infty & \text{if } m = 0 \text{ and } z \neq 0. \end{cases}$$

$(m,z) \mapsto \widetilde{L}(x,m,z)$ is convex and LSC.

The optimization problem can be written:

$$\inf_{m \in L^2(Q), z \in \{L^1(Q)\}^d} \Phi[m] + \int_0^T \int_\Omega \Big(\widetilde{L}(x, m(t,x), z(t,x)) \Big) dx dt + \Psi(m(T,\cdot))$$

subject to

$$\begin{cases} \frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}(z) &= 0, \quad \text{in } (0,T) \times \Omega, \\ m(T,x) &= m_0(x) \quad \text{in } \Omega, \\ m &\geq 0. \end{cases}$$

Optimality conditions (1/2)

$$\delta\gamma \mapsto \delta m : \qquad \left\{ \begin{array}{rcl} \partial_t \delta m - \nu\Delta\delta m + \mathrm{div}(\delta m\, \gamma) &=& -\mathrm{div}(m\, \delta\gamma), \quad \text{in } (0,T]\times\Omega, \\ \delta m(0,x) &=& 0 \quad \text{in } \Omega, \end{array}\right.$$

$$\begin{aligned}\delta J(m,\gamma) = &\int_0^T \int_\Omega \delta m(t,x)\Big(L(x,\gamma(t,x)) + f[m](t,x))\Big) \\ &+ \int_0^T \int_\Omega \delta\gamma(t,x) m(t,x) \frac{\partial L}{\partial\gamma}(x,\gamma(t,x)) + \int_\Omega \delta m(T,x) g[m(T,\cdot)](x)dx.\end{aligned}$$

Adjoint problem

$$\left\{ \begin{array}{ll} -\dfrac{\partial u}{\partial t} - \nu\Delta u - \gamma\cdot\nabla u = L(x,\gamma) + f[m](t,x) & \text{in } [0,T)\times\Omega \\ u(t=T) = g[m|_{t=T}] & \end{array}\right.$$

$$\begin{aligned}\delta J(m,\gamma) &= \int_0^T \int_\Omega u(t,x)\left(\partial_t\delta m - \nu\Delta\delta m + \mathrm{div}(\delta m\,\gamma)\right) \\ &\quad + \int_0^T \int_\Omega m(t,x)\delta\gamma(t,x) \frac{\partial L}{\partial\gamma}(x,\gamma(t,x)) \\ &= \int_0^T \int_\Omega m(t,x)\left(\frac{\partial L}{\partial\gamma}(x,\gamma(t,x)) - \nabla u(t,x)\right)\delta\gamma(t,x)\end{aligned}$$

Optimality conditions (2/2)

If $m^* > 0$, then

$$\frac{\partial L}{\partial \gamma}(x, \gamma^*(t,x)) + \nabla u(t,x) = 0.$$

Therefore, $\gamma^*(t,x)$ achieves

$$-Du(t,x)\cdot\gamma^*(t,x) - L(x,\gamma^*(t,x)) = \max_{\gamma\in\mathbb{R}^d}\{-Du(t,x)\cdot\gamma - L(x,\gamma)\} = H(x, Du(t,x))$$

and

$$\gamma^*(t,x) = -H_p(x, Du(t,x)).$$

We recover the MFG system of PDEs:

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \nu\Delta u + H(x,\nabla u) = f[m] & \text{in } [0,T)\times\Omega \\ \dfrac{\partial m}{\partial t} - \nu\Delta m - \operatorname{div}\left(m\dfrac{\partial H}{\partial p}(x, Du)\right) = 0 & \text{in } (0,T]\times\Omega \\ u|_{t=T}(x) = g[m|_{t=T}](x) \\ m|_{t=0}(x) = m_0(x). \end{cases}$$

Duality

The optimization problem can be written

$$\inf_{m,\gamma}\sup_{p,u}\left\{\begin{array}{l}\displaystyle\int_0^T\int_\Omega\Big(m(t,x)(-p(t,x)\gamma(t,x)-H(x,p(t,x)))\Big)dxdt\\ +\Phi(m)+\Psi(m|_{t=T})\\ \displaystyle-\int_0^T\int_\Omega u(t,x)\left(\frac{\partial m}{\partial t}-\nu\Delta m+\mathrm{div}(m\,\gamma)\right)dxdt.\end{array}\right\}$$

Fenchel-Rockafellar duality theorem + integrations by parts:

$$\inf_{u,\alpha}\Phi^*(\alpha)+\Psi^*(u|_{t=T})-\int_\Omega m_0(x)u(0,x))dx,$$

subject to

$$-\frac{\partial u}{\partial t}-\nu\Delta u+H(x,\nabla u)=\alpha.$$

with

$$\Phi^*(\alpha)=\sup_{m\geq 0}\{\int_Q m(t,x)\alpha(t,x)dxdt-\Phi(m)\},\qquad G^*(u)=\sup_{m\geq 0}\{\int_\Omega m(x)u(x)dx-\Psi(m)\}.$$

Consequence : an iterative solver for the discrete MFG system (1/3)

Set $\nu = 0$, $\Phi(m) = \int_Q F(m(t,x))dxdt$, $\Psi(m) = \int_\Omega G(m(x))dx$ for simplicity. The discrete version of the latter optimization problem is

$$\inf_u \left\{ \begin{array}{l} h\Delta t \sum_{n=0}^{M-1} \sum_{i=0}^{N-1} F^* \left(\frac{u_i^n - u_i^{n+1}}{\Delta t} + g\left(x_i, \frac{u_{i+1}^n - u_i^n}{h}, \frac{u_i^n - u_{i-1}^n}{h}\right)\right) \\ +h \sum_{i=0}^{N-1} G^*\left(u_i^M\right) - h \sum_{i=0}^{N-1} m_i^0 u_i^0 \end{array} \right\}$$

Set

$$a_i^n = \frac{u_i^n - u_i^{n+1}}{\Delta t}, \quad b_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \quad c_i^n = \frac{u_i^n - u_{i-1}^n}{h},$$

$$q_i^n = (a_i^n, b_i^n, c_i^n) \in \mathbb{R}^3, \qquad q = (q_i^n)_{0\le n<M, i\in\mathbb{R}/N\mathbb{Z}}$$

$$q = \Lambda u \qquad \Lambda: \text{ linear operator}$$

The optimization problem has the form

$$\inf_{u,q:q=\Lambda u} \left\{\Theta(q) + \chi(u)\right\} = \inf_{u,q} \sup_\sigma \left\{\Theta(q) + \chi(u) + \langle \sigma, \Lambda u - q\rangle\right\}$$

where $\sigma_i^n = (m_i^{n+1}, z_{1,i}^{n+1}, z_{2,i}^{n+1})$.

Consequence : an iterative solver for the discrete MFG system (2/3)

Setting $\mathcal{L}(u,q,\sigma) = \Theta(q) + \chi(u) + \langle \sigma, \Lambda u - q\rangle$, we get the saddle point problem:

$$\inf_{u,q} \sup_{\sigma} \mathcal{L}(u,q,\sigma).$$

Consider the augmented Lagrangian

$$\begin{aligned}\mathcal{L}_r(u,q,\sigma) &= \mathcal{L}(u,q,\sigma) + \frac{r}{2}\|\Lambda u - q\|_2^2 \\ &= \Theta(q) + \chi(u) + \langle \sigma, \Lambda u - q\rangle + \frac{r}{2}\|\Lambda u - q\|_2^2.\end{aligned}$$

It is equivalent to solving

$$\inf_{u,q} \sup_{\sigma} \mathcal{L}_r(u,q,\sigma).$$

The Alternating Direction Method of Multipliers is an iterative method

$$\left(u^{(k)}, q^{(k)}, \sigma^{(k)}\right) \to \left(u^{(k+1)}, q^{(k+1)}, \sigma^{(k+1)}\right)$$

Consequence : an iterative solver for the discrete MFG system (3/3)

Alternating Direction Method of Multipliers

Step 1: $u^{(k+1)} = \arg\min_u \mathcal{L}_r\left(u, q^{(k)}, \sigma^{(k)}\right)$, i.e.

$$0 \in \partial\chi\left(u^{(k+1)}\right) + \Lambda^T\sigma^{(k)} + r\Lambda^T\left(\Lambda u^{(k+1)} - q^{(k)}\right)$$

This is a discrete Poisson equation in the discrete time-space cylinder, + possibly nonlinear boundary conditions.

Step 2: $q^{(k+1)} = \arg\min_q \mathcal{L}_r\left(u^{(k+1)}, q, \sigma^{(k)}\right)$, i.e.

$$\sigma^{(k)} - r\left(q^{(k+1)} - \Lambda u^{(k+1)}\right) \in \partial\Theta\left(q^{(k+1)}\right)$$

can be done by a loop on the time-space grid nodes, with a low dimensional optimization problem at each node (nonlinearity).

Step 3: Update σ so that $q^{(k+1)} = \arg\min_q \mathcal{L}\left(u^{(k+1)}, q, \sigma^{(k+1)}\right)$, by

$$\sigma^{(k+1)} = \sigma^{(k)} + r\left(\Lambda u^{(k+1)} - q^{(k+1)}\right).$$

Loop on the time-space grid nodes.

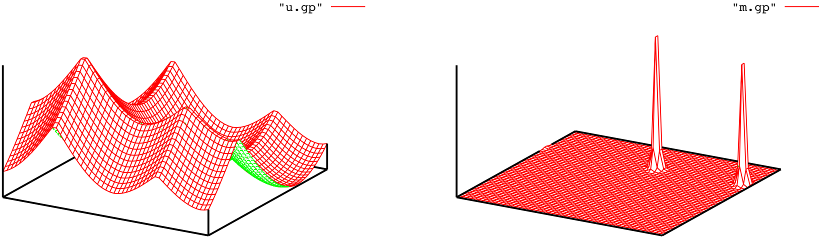
$$\sigma^{(k+1)} \in \partial\Theta\left(q^{(k+1)}\right) \quad \Rightarrow \quad m_i^{n+1} \geq 0 \quad \forall 0 \leq n < M, i.$$

Outline

Deterministic infinite horizon MFG with nonlocal coupling

$$\nu = 0.001,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$
$$F[m] = (1-\Delta)^{-1}(1-\Delta)^{-1} m$$

left: u, right m.



Outline

Main purpose

- Many models for crowd motion are inspired by statistical mechanics (socio-physics)
- microscopic models: pedestrians = particles with more or less complex interactions (e.g. B. Maury et al)
- macroscopic models similar to fluid dynamics models (e.g. Hughes et al)
- in all these models, rational anticipation is not taken into account
- MFG may lead to crowd motion models including rational anticipation
- The systems of PDEs can be simulated numerically

A Model of crowd motion with congestion

- One (possibly several) population(s) of identical agents: the pedestrians
- The impact of a single agent on the global behavior is negligible
- Rational anticipation: the global model is obtained by considering Nash equilibria with N pedestrians and passing to the limit as $N \to \infty$
- The strategy of a single pedestrian depends on some global information, for example the density $m(t,x)$ of pedestrians at space-time point (t,x)
- In congestion models, the cost of motion of a pedestrian located at (t,x) is an increasing function of the density of pedestrians at (t,x), namely $m(t,x)$
- Each pedestrian may be affected by a random idiosynchratic (or common) noise

A typical application: exit from a hall or a stadium

The cost to be minimized by a pedestrian is made of three parts

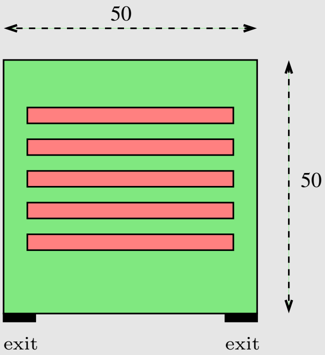
1. the exit-time. More complex models can be written for modelling e.g. panic.
2. the cost of motion, which may be quadratic w.r.t. velocity and increase in crowded regions

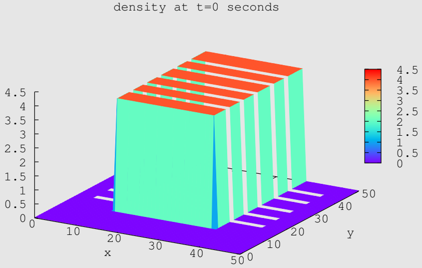
$$\begin{aligned} \text{cost of motion} &= (c+m(t,x))^\alpha V^2 \\ V &= \text{ instantaneous velocity of the agent} \\ m(t,x) &= \text{ density of the population at } (t,x) \\ \alpha &= \text{ some exponent, for example } \tfrac{3}{4} \\ c &= \text{some positive parameter} \end{aligned}$$

3. possibly, an exit cost

A typical case: exit from a hall with obstacles

The geometry and initial density





The initial density m_0 is piecewise constant and takes two values 0 and 4 people/m^2. There are 3300 people in the hall.
The horizon is $T = 40$ min. The two doors stay open from $t = 0$ to $t = T$.

$$\frac{\partial u}{\partial t} + \nu \Delta u - H(x, \nabla u, m) = 0$$
$$\frac{\partial m}{\partial t} - \nu \Delta m - \text{div}\left(m \frac{\partial H}{\partial p}(\cdot, \nabla u, m) \right) = 0$$

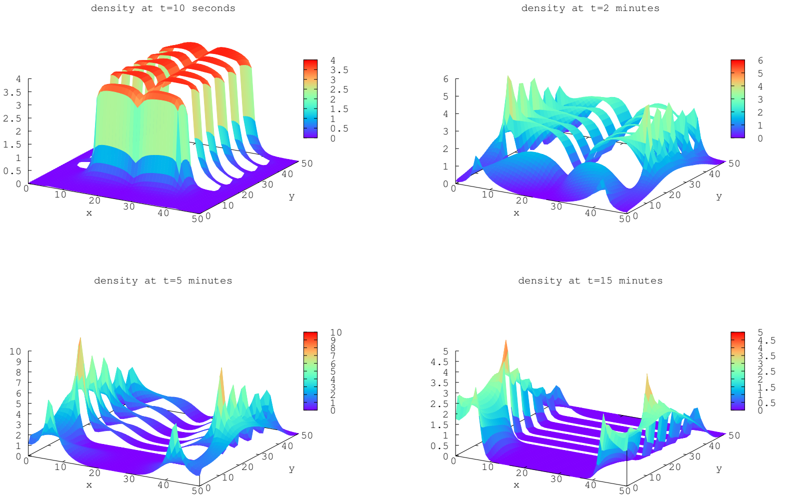
with the Hamiltonian

$$H(x, p, m) = \mathcal{H}(x, m) + \frac{|p|^2}{(c+m)^\alpha}$$

and $c \geq 0$, $0 \leq \alpha < 2$

The function $\mathcal{H}(x, m)$ may model the panic

Evolution of the distribution of pedestrians



(the scale varies w.r.t. t)

Evolution of the density of pedestrians

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Figure : The evolution of the distribution of pedestrians

Exit from a hall with a common uncertainty

(idea of J-M. Lasry)

Similar geometry. The horizon is T.

- Before $t = T/2$, the two doors are closed.

Exit from a hall with a common uncertainty

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Similar geometry. The horizon is T.

- Before $t = T/2$, the two doors are closed.
- People know that one of the two doors will be opened at $t = T/2$ and will stay open until $t = T$, but they do not know which one.

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- At $T/2$, the probability that a given door be opened is $1/2$: A common source of risk for all the agents.
- **Interest of this example: the behavior of the agents can be predicted only if rational anticipation is taken into account.**

The evolution of the distribution of pedestrians

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Outline

The system of PDEs

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \nu\Delta u_1 - H_1(t,x,m_1+m_2,\nabla u_1) &= -F_1(m_1,m_2)\\
\frac{\partial m_1}{\partial t} - \nu\Delta m_1 - \operatorname{div}\left(m_1\frac{\partial H_1}{\partial p}(t,x,m_1+m_2,\nabla u_1)\right) &= 0\\
\frac{\partial u_2}{\partial t} + \nu\Delta u_2 - H_2(t,x,m_1+m_2,\nabla u_2) &= -F_2(m_1,m_2)\\
\frac{\partial m_2}{\partial t} - \nu\Delta m_2 - \operatorname{div}\left(m_2\frac{\partial H_2}{\partial p}(t,x,m_1+m_2,\nabla u_2)\right) &= 0
\end{aligned}
$$

with on the boundary,

$$
\begin{aligned}
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} &= 0\\
\nu\frac{\partial m_i}{\partial n} + m_i\mathbf{n}\cdot\frac{\partial H_i}{\partial p}(t,x,m_1+m_2,\nabla u_i) = 0, &\quad i=1,2
\end{aligned}
$$

Two populations must cross each other

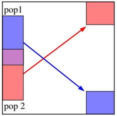
$$F_1(m_1, m_2) = 2\left(\frac{m_1}{m_1+m_2} - 0.8\right)_- + (m_1 + m_2 - 8)_+$$

$$F_2(m_1, m_2) = \left(\frac{m_2}{m_1+m_2} - 0.6\right)_- + (m_1 + m_2 - 8)_+$$

$$\Omega = (0,1)^2, \quad \nu = 0.03,$$

$$H_1(x,p) = |p|^2 - 1.4 \times 1_{\{x_1<0.7, x_2>0.2\}}$$

$$H_2(x,p) = |p|^2 - 1.4 \times 1_{\{x_1<0.7, x_2<0.8\}}$$



Evolution of the densities $\nu = 0.03$

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