Time Parallel Time Integration

Part II

Waveform Relaxation and Domain Decomposition

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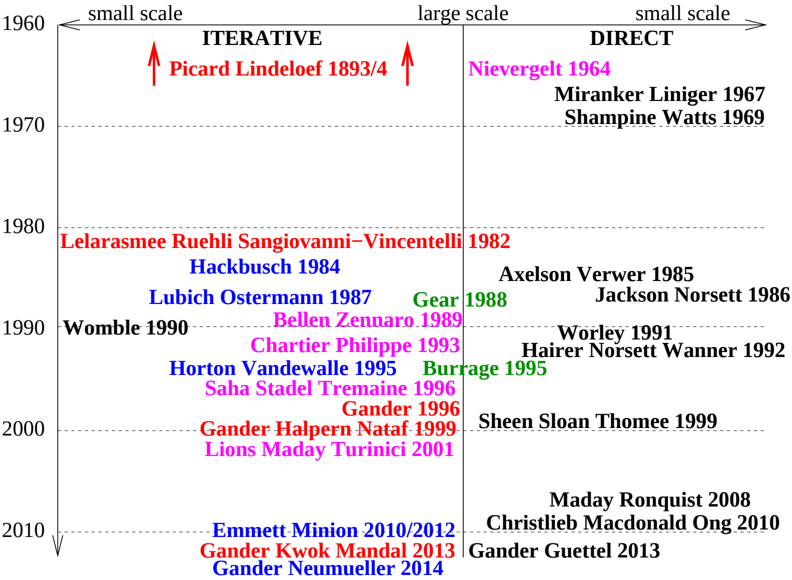
Waveform Relaxation and Domain Decomposition



Picard 1893 and Lindelöf 1894

Émile Picard (1893): Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires

$$v' = f(v) \quad \Longrightarrow \quad v^n(t) = v(0) + \int_0^t f(v^{n-1}(\tau))d\tau$$

Ernest Lindelöf (1894): Sur l'application des méthodes d'approximations successives à l'étude des intégrales réelles des équations différentielles ordinaires

Theorem (Superlinear Convergence)

On bounded time intervals $t \in [0, T]$, the iterates satisfy the superlinear error bound

$$||\mathbf{v} - \mathbf{v}^n|| \leq \frac{(CT)^n}{n!} ||\mathbf{v} - \mathbf{v}^0||,$$

where C is a positive constant.

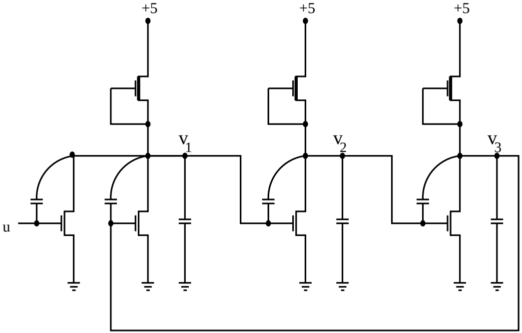
Lelarasmee, Ruehli and Sangiovanni-Vincentelli

The Waveform Relaxation Method for Time-Domain Analysis of Large Scale Integrated Circuits. IEEE Trans. on Computer-Aided Design of Int. Circ. a. Sys. 1982

> *"The spectacular growth in the scale of integrated circuits being designed in the VLSI era has generated the need for new methods of circuit simulation. "Standard" circuit simulators, such as SPICE and ASTAP, simply take too much CPU time and too much storage to analyze a VLSI circuit".*



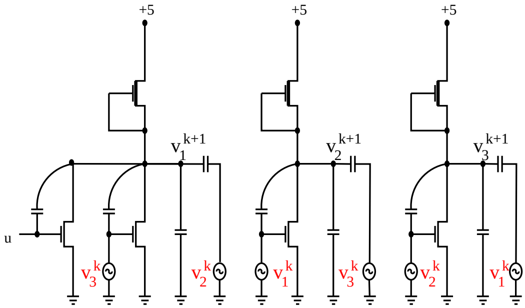
MOS ring oscillator from 1982



Using Kirchhoff's and Ohm's laws gives system of ODEs:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= f(\mathbf{v}), \qquad 0 < t < T \\ \mathbf{v}(0) &= \mathbf{g} \end{aligned}$$

Waveform Relaxation Decomposition

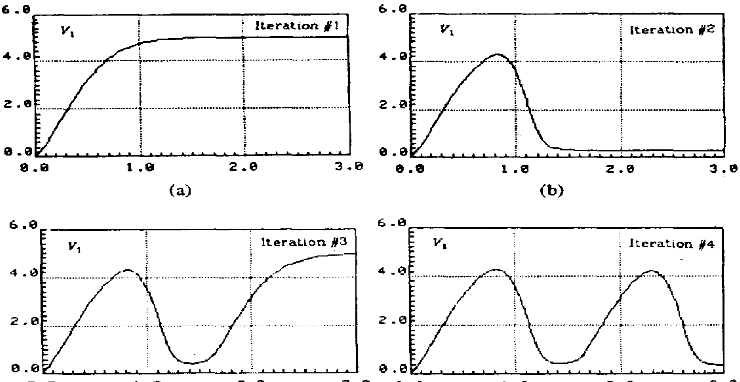


Iteration using sub-circuit solutions only:

$$\begin{aligned} \partial_t v_1^{k+1} &= f_1(v_1^{k+1}, v_2^k, v_3^k) \\ \partial_t v_2^{k+1} &= f_2(v_1^k, v_2^{k+1}, v_3^k) \\ \partial_t v_3^{k+1} &= f_3(v_1^k, v_2^k, v_3^{k+1}) \end{aligned}$$

Signals along wires are called 'waveforms', which gave the algorithm its name: **Waveform Relaxation**.

Historical Numerical Convergence Study



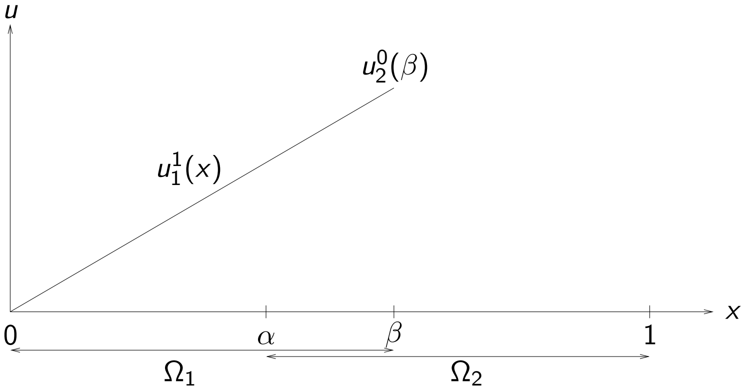
(a) (b) (c) (d)

Lelarasmee et al (1982): "Note that since the oscillator is highly non unidirectional due to the feedback from v_3 to the NOR gate, the convergence of the iterated solutions is achieved with the number of iterations being proportional to the number of oscillating cycles of interest"

Alternating Schwarz (Schwarz 1869)

For $u_{xx} = 0$ in $\Omega = (0,1)$, $u(0) = u(1) = 0$:

$$\begin{aligned} \partial_{xx} u_1^n &= 0 \quad \text{in } \Omega_1 & \partial_{xx} u_2^n &= 0 \quad \text{in } \Omega_2 \\ u_1^n(0) &= 0 & u_2^n(1) &= 0 \\ u_1^n(\beta) &= u_2^{n-1}(\beta) & u_2^n(\alpha) &= u_1^n(\alpha) \end{aligned}$$



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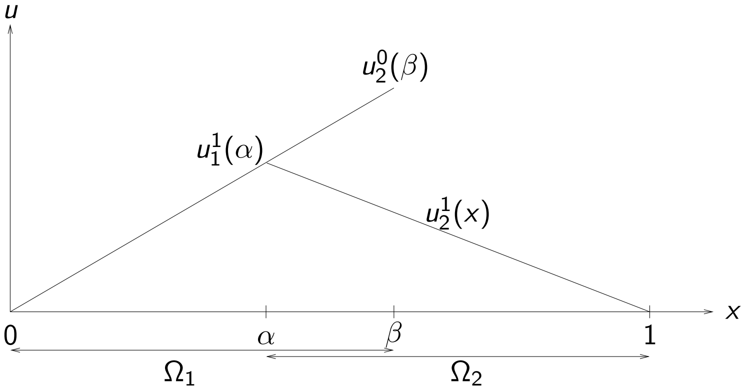

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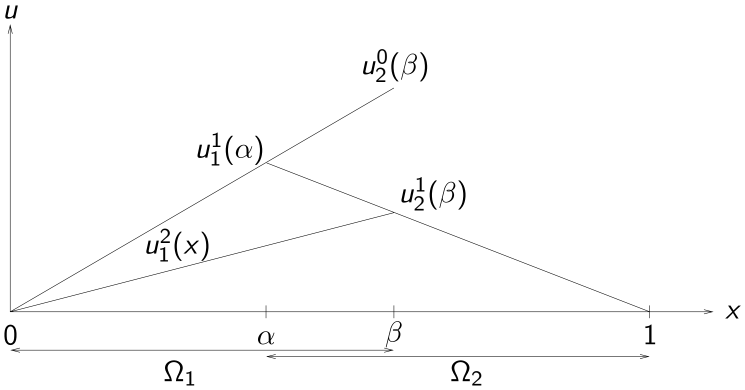

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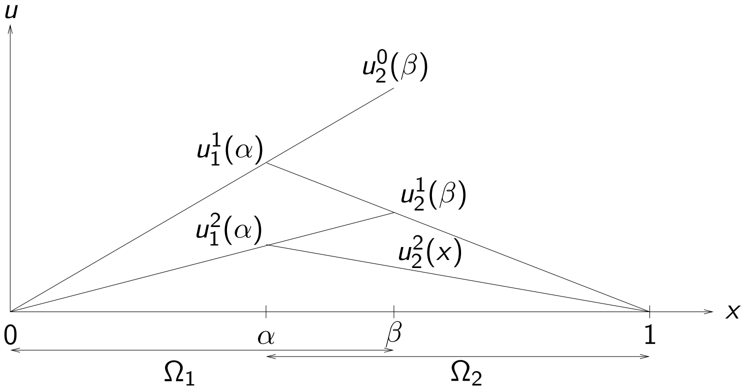

Dirichlet Neumann (Bjørstad, Widlund 1986)

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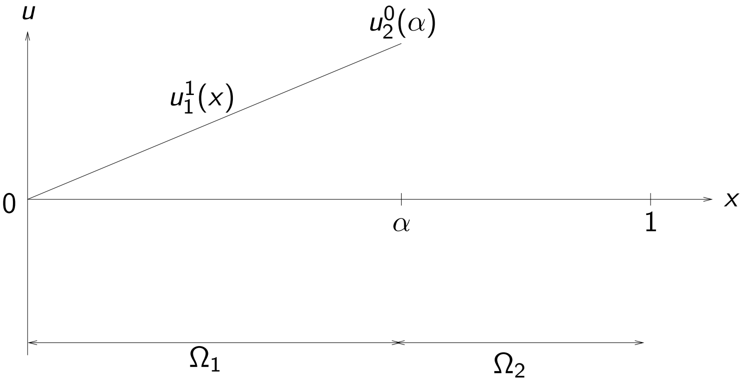

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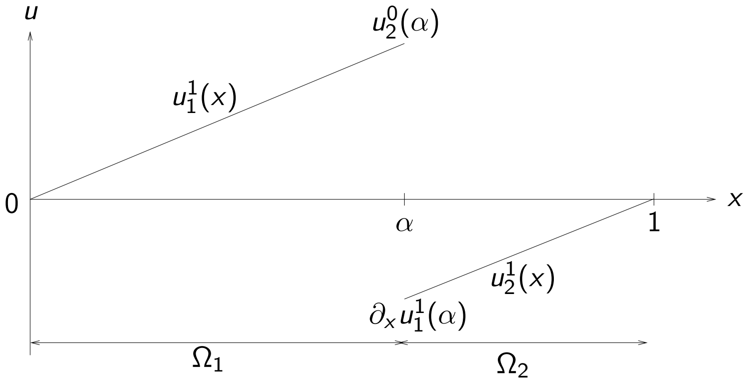

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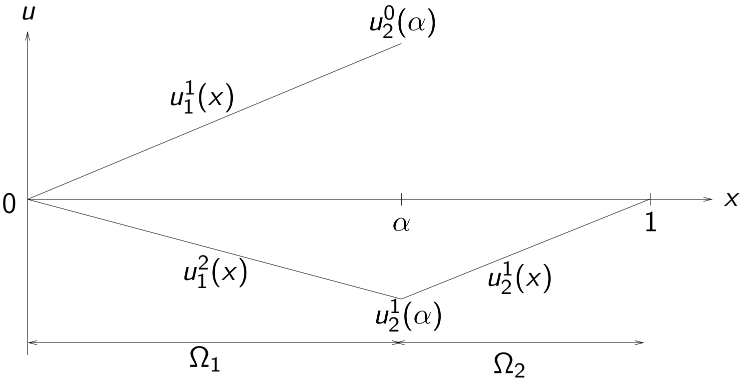

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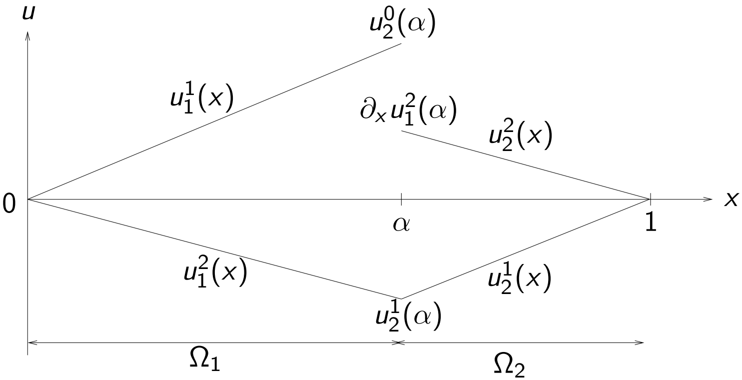

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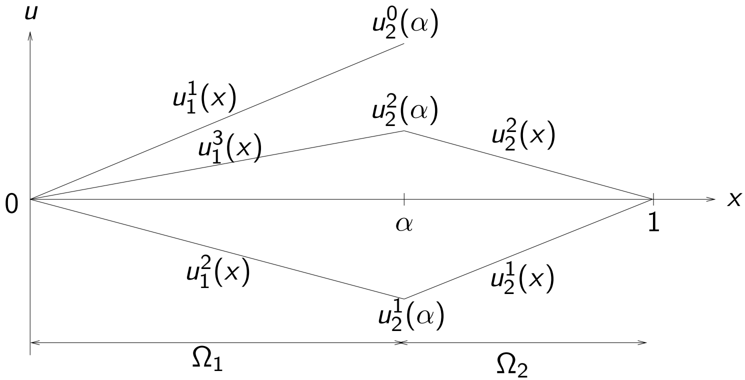

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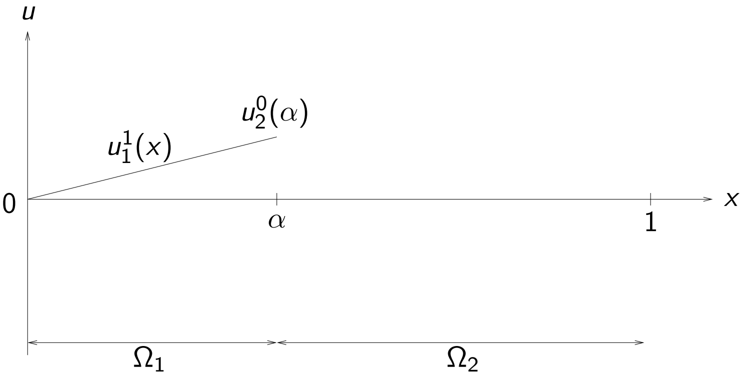

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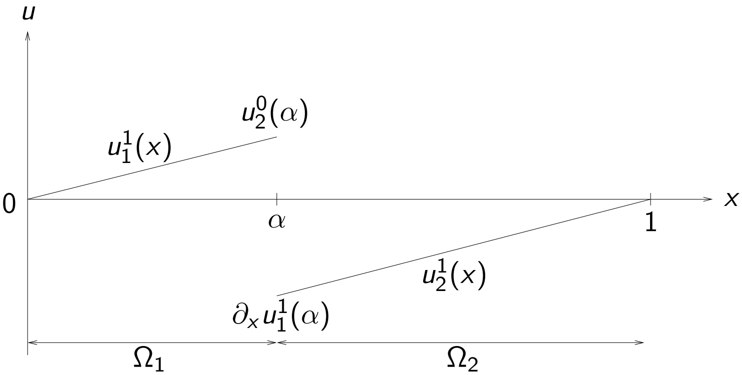

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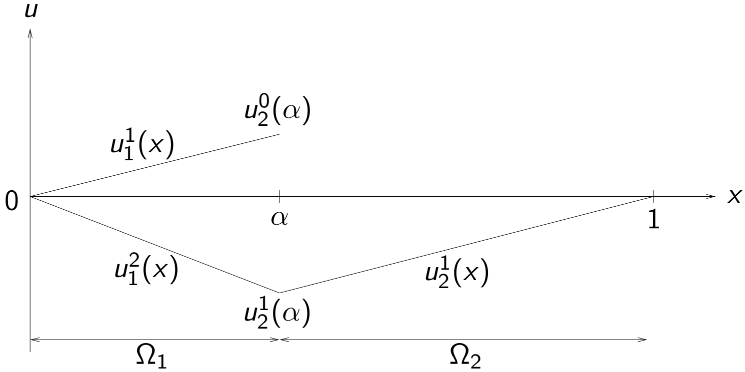

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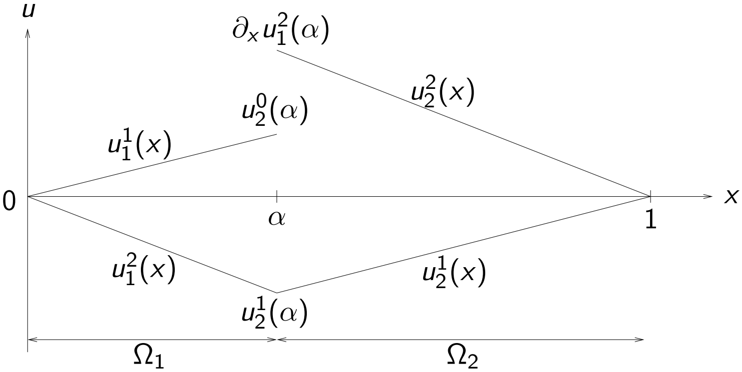

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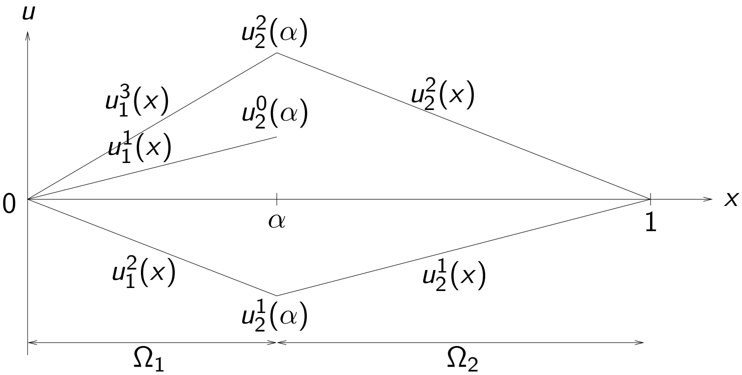

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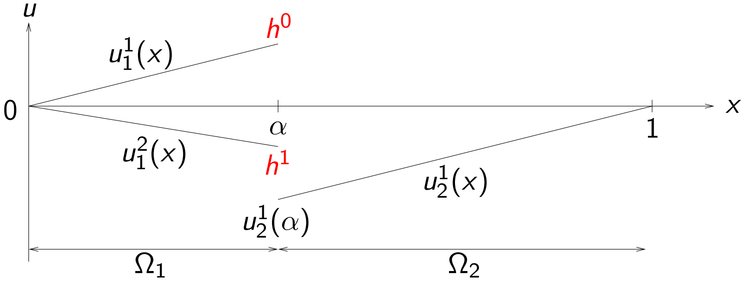
$$h^n = \theta u_2^n(\alpha) + (1-\theta) h^{n-1}$$



Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

For $u_{xx} = 0$ in $\Omega = (0,1)$, $u(0) = u(1) = 0$:

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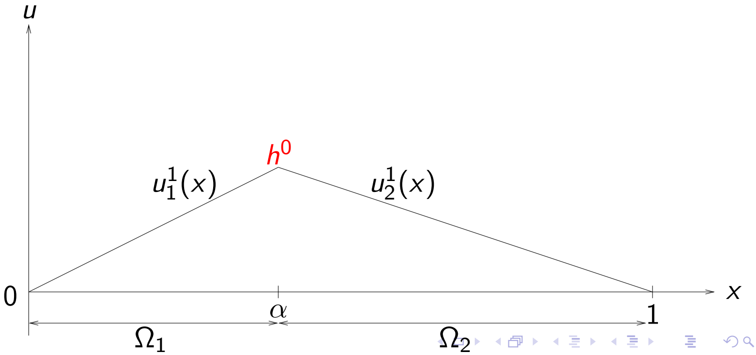
$$h^n = h^{n-1} - \theta(\psi_1(\alpha) + \psi_2(\alpha))$$



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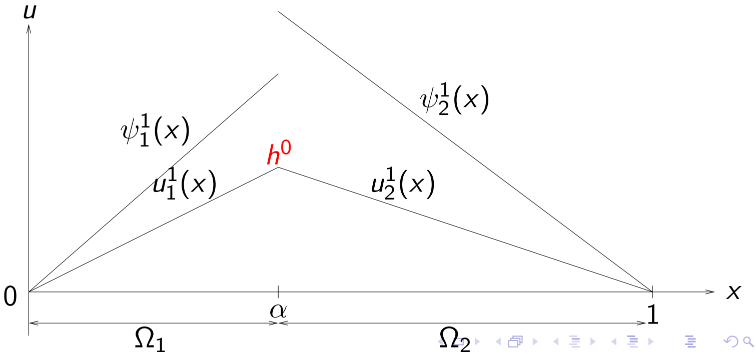

Time Parallel Methods Part II WR and DD

Martin J. Gander

Waveform Relaxation
Picard Lindelöf
Ruehli et al

Domain Decomposition
Schwarz
Bjørstad, Widlund
Bourgat, Glowinski, Tallec, Vidrascu

WR variants
Evolution Problems
Schwarz WR
Dirichlet-Neumann WR
Neumann-Neumann WR
Heat Equation

Parareal Schwarz WR

Conclusions

Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

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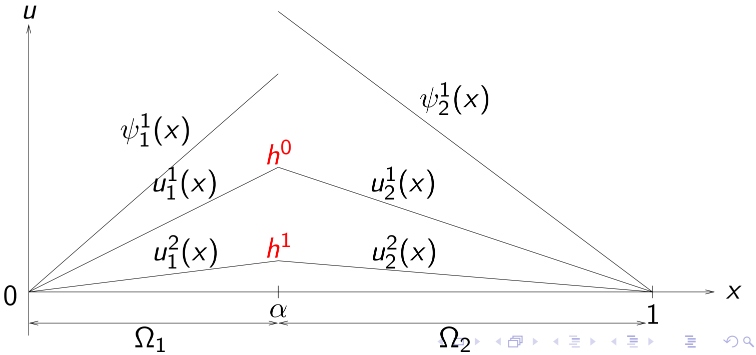


Time Dependent Problems

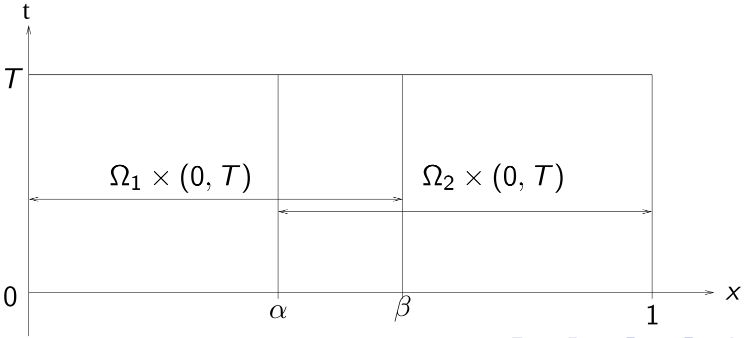
What happens if the PDE is time dependent, e.g. a heat equation

$$\partial_t u = \partial_{xx} u, \quad \text{in } \Omega$$

or a wave equation,

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad \text{in } \Omega,$$

where the domain is now in space-time ?



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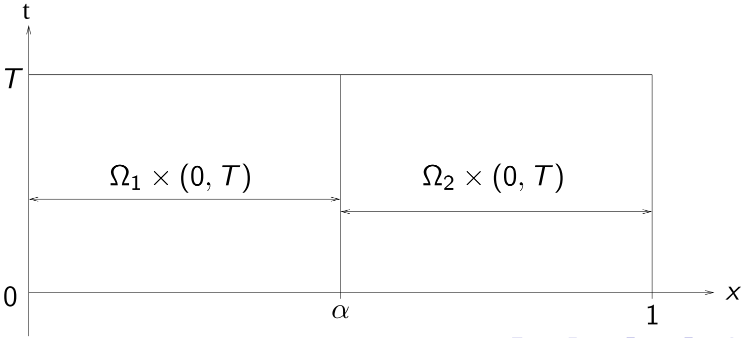


Schwarz Waveform Relaxation: wave equation

$$\begin{array}{ll} \partial_{tt} u_1^n = c^2 \partial_{xx} u_1^n \text{ in } \Omega_1 \times (0,T) & \partial_{tt} u_2^n = c^2 \partial_{xx} u_2^n \text{ in } \Omega_2 \times (0,T) \\ u_1^n(0,t) = 0 & u_2^n(1,t) = 0 \\ u_1^n(\beta,t) = u_2^{n-1}(\beta,t) & u_2^n(\alpha,t) = u_1^n(\alpha,t) \end{array}$$

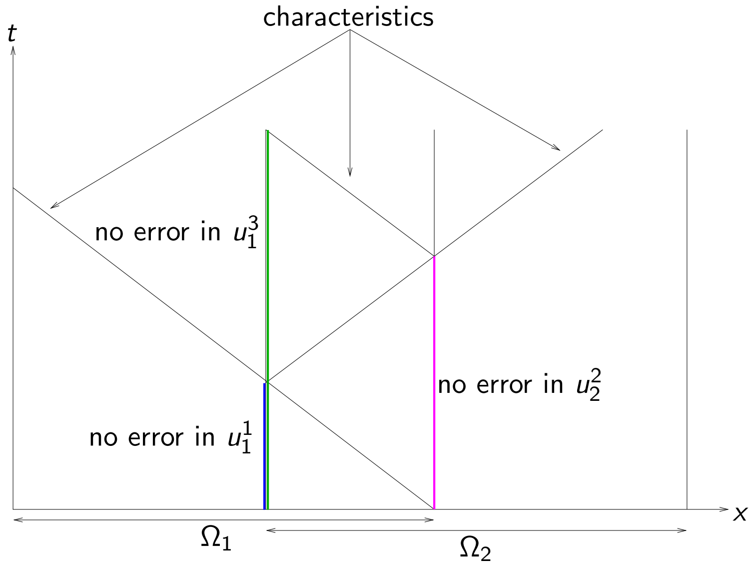
Theorem (Wave equation (G 1997))

The algorithm converges in a finite number of steps, i.e. when

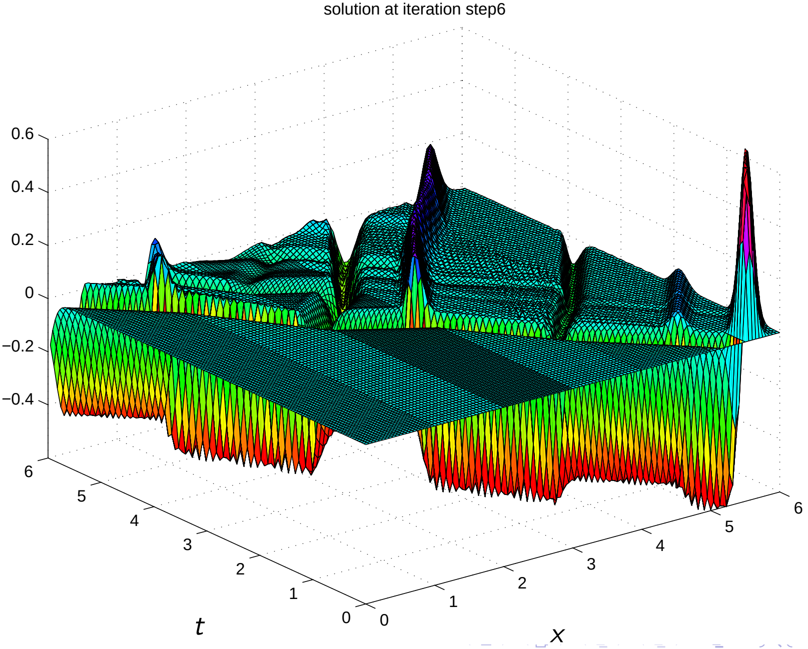
$$n \geq \frac{Tc}{\beta - \alpha}.$$

- Analogous results for many subdomains and general decompositions (G, Halpern 2004)
- Results with absorbing transmission conditions for non-overlapping decompositions (G, Halpern, Nataf 2003)

Graphical Convergence Proof in 1D



An Example with non-matching grids



Dirichlet-Neumann WR: wave equation

$$\begin{array}{ll}
\partial_{tt} u_1^n = c^2 \partial_{xx} u_1^n \text{ in } \Omega_1 \times (0,T) & \partial_{tt} u_2^n = c^2 \partial_{xx} u_2^n \text{ in } \Omega_2 \times (0,T) \\
u_1^n(0,t) = 0 & u_2^n(1,t) = 0 \\
u_1^n(\alpha,t) = {\color{red} h^{n-1}(t)} & \partial_x u_2^n(\alpha,t) = \partial_x u_1^n(\alpha,t)
\end{array}$$

$$h^n(t) = \theta u_2^n(\alpha,t) + (1-\theta) h^{n-1}(t)$$

Theorem (G, Kwok, Mandal 2014)

If $\alpha = 0.5$ and $\theta = 0.5$, DNWR converges in 2 iterations.

If $\alpha \neq 0.5$ and $\theta = 0.5$ the algorithm converges in a finite number of steps, as soon as

$$n \geq \frac{Tc}{2 \min(\alpha, 1-\alpha)}.$$

If $\alpha = 0.5$ and $\theta \in (0,1)$, $\theta \neq 0.5$, the algorithm converges linearly.

Neumann-Neumann WR: wave equation

$$\begin{aligned} \partial_{tt} u_i^n &= c^2 \partial_{xx} u_i^n \text{ in } \Omega_i \times (0,T) & \partial_{tt}\psi_i^n &= c^2 \partial_{xx}\psi_i^n \text{ in } \Omega_i \times (0,T) \\ u_1^n(0) &= 0 & \psi_1^n(0) &= 0 \\ u_2^n(1) &= 0 & \psi_2^n(1) &= 0 \\ u_i^n(\alpha,t) &= h^{n-1}(t) & \partial_{n_i}\psi_i^n(\alpha,t) &= \partial_{n_1} u_1^n(\alpha,t) + \partial_{n_2} u_2^n(\alpha,t) \end{aligned}$$

$$h^n(t) = h^{n-1}(t) - \theta(\psi_1(\alpha,t) + \psi_1(\alpha,t))$$

Theorem (G, Kwok, Mandal 2014)

If $\alpha = 0.5$ and $\theta = 0.25$, NNWR converges in 2 iterations.

If $\alpha \neq 0.5$ and $\theta = 0.25$ the algorithm converges in a finite number of steps, as soon as

$$n > \frac{Tc}{4\min(\alpha, 1-\alpha)}.$$

If $\alpha = 0.5$ and $\theta \in (0, 0.5)$, $\theta \neq 0.25$, the algorithm converges linearly.

Convergence estimates: heat equation

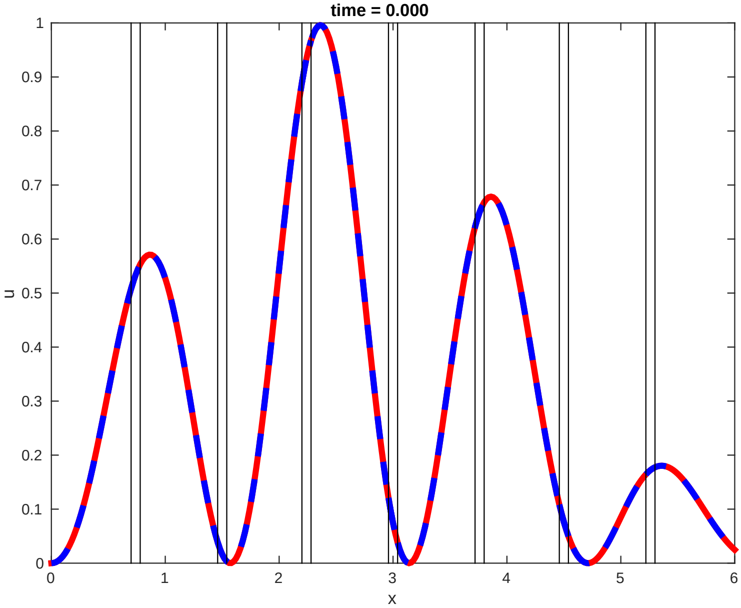
Methods	2 subdomains	N equal subdomains
SWR	$\operatorname{erfc}\left(\frac{n(\beta-\alpha)}{\sqrt{T}}\right)$	$2^n \operatorname{erfc}\left(\frac{n\delta}{2\sqrt{T}}\right)$
DNWR	$\left(\frac{1-2\alpha}{1-\alpha}\right)^n \operatorname{erfc}\left(\frac{n\alpha}{2\sqrt{T}}\right)$	$(N-2)^n \operatorname{erfc}\left(\frac{n}{2N\sqrt{T}}\right)$
NNWR	$\left(\frac{(1-2\alpha)^2}{\alpha(1-\alpha)}\right)^n \operatorname{erfc}\left(\frac{n\alpha}{\sqrt{T}}\right)$	$\left(\frac{\sqrt{6}}{1-e^{-\frac{(2n+1)}{N^2 T}}}\right)^{2n} e^{\frac{-n^2}{N^2 T}}$

Optimized Schwarz Waveform Relaxation (OSWR)

$$\begin{aligned} \partial_t u_1^n &= \partial_{xx} u_1^n \text{ in } \Omega_1 \times (0,T) & \partial_t u_2^n &= \partial_{xx} u_2^n \text{ in } \Omega_2 \times (0,T) \\ u_1^n(0,t) &= 0 & u_2^n(1,t) &= 0 \\ \mathcal{B}_1 u_1^n(\beta,t) &= \mathcal{B}_1 u_2^{n-1}(\beta,t) & \mathcal{B}_2 u_2^n(\alpha,t) &= \mathcal{B}_2 u_1^n(\alpha,t) \end{aligned}$$

- Many convergence results: heat equation, wave equation, advection reaction diffusion, Maxwell, shallow water, ...
- Recent methods like Sweeping Preconditioner and Source Transfer are based on the same approach

An Example with 8 Subdomains

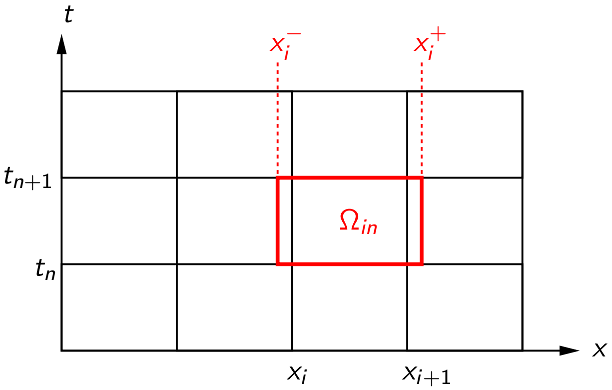


Parareal Schwarz Waveform Relaxation

G, Jiang, Li (2011), see also **Maday, Turinici (2007)**

Model problem: $\partial_t u = \partial_{xx} u \quad$ in $\Omega = (0,1) \times (0,T)$

Decomposition of the space-time domain:



$$\Omega_{in} := (x_i - \frac{L}{2}, x_{i+1} + \frac{L}{2}) \times (t_n, t_{n+1})$$

Parareal Schwarz Waveform Relaxation

Given an initial condition u_0 and boundary conditions g^- and g^+, we define $F_{in}(u_0, g^-, g^+)$ and $G_{in}(u_0, g^-, g^+)$ to be fine and coarse approximations of the solution at $t = t_{n+1}$ of

$$\begin{array}{rcll} \partial_t u &=& \partial_{xx} u, & x \in (x_i^-, x_i^+),\ t \in (t_n, t_{n+1}) \\ u(x, t_n) &=& u_0 & x \in (x_i^-, x_i^+) \\ \mathcal{B}_i^- u(x_i^-, t) &=& g^- & t \in (t_n, t_{n+1}) \\ \mathcal{B}_i^+ u(x_i^+, t) &=& g^+ & t \in (t_n, t_{n+1}) \end{array}$$

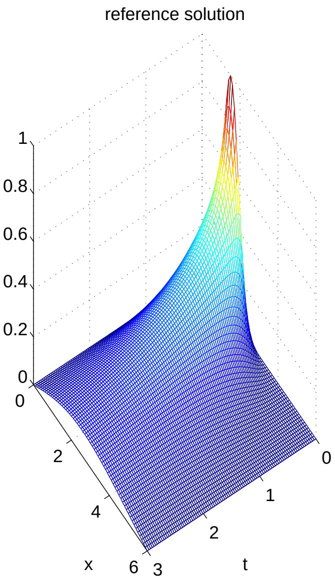
A Parareal Schwarz Waveform Relaxation Algorithm:
Given initial conditions $u_{0,in}^k(x)$ and boundary conditions $\mathcal{B}_i^- u_{i-1,n}^k(t)$ and $\mathcal{B}_i^+ u_{i+1,n}^k(t)$, we compute

1. All $u_{in}^{k+1} := F_{in}(u_{0,in}^k, \mathcal{B}_i^- u_{i-1,n}^k, \mathcal{B}_i^+ u_{i+1,n}^k)$ in parallel
2. Compute new initial conditions using

$$u_{0,i,n+1}^{k+1} = F(u_{0,in}^k, \mathcal{B}_i^- u_{i-1,n}^k, \mathcal{B}_i^+ u_{i+1,n}^k)$$

$$+G(u_{0,in}^{k+1}, \mathcal{B}_i^- u_{i-1,n}^{k+1}, \mathcal{B}_i^+ u_{i+1,n}^{k+1}) - G(u_{0,in}^k, \mathcal{B}_i^- u_{i-1,n}^k, \mathcal{B}_i^+ u_{i+1,n}^k)$$

Parareal Schwarz WR Numerical Example



Model problem:

1D Heat equation

$$\partial_t u = \partial_{xx} u$$

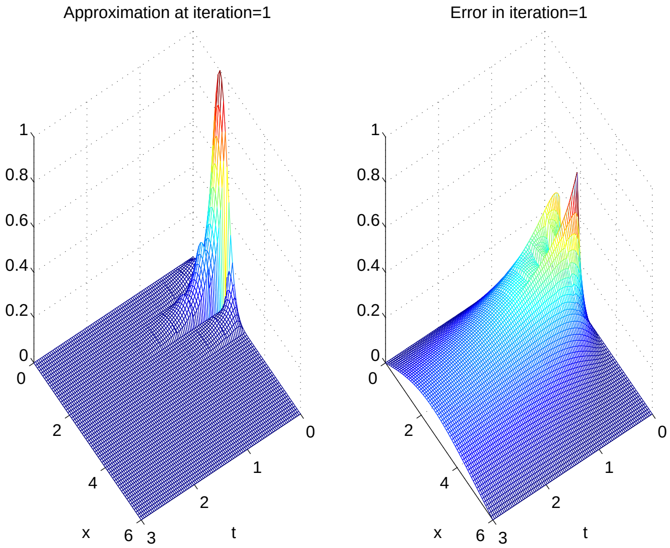
on $\Omega = (0,6)\times(0,T)$, $T = 3$

Space time decomposition into 6 spatial subdomains, and 10 time subdomains

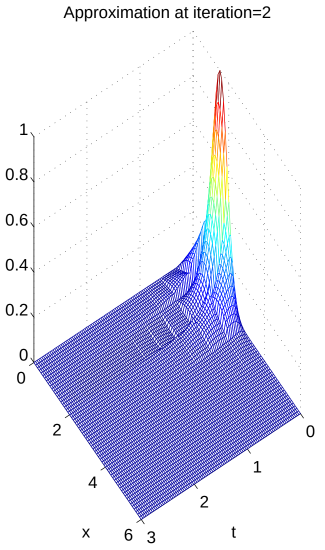
Discretization with $\Delta x = \frac{1}{10}$, $\Delta t = \frac{3}{100}$

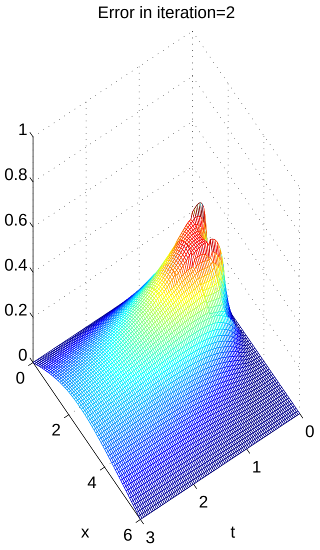
Overlap in space of $2\Delta x$

Parareal Schwarz WR: Iteration 1

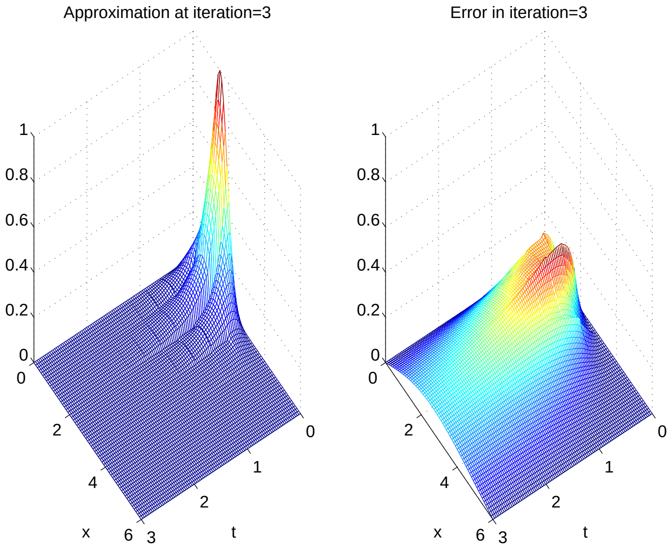


Parareal Schwarz WR: Iteration 2

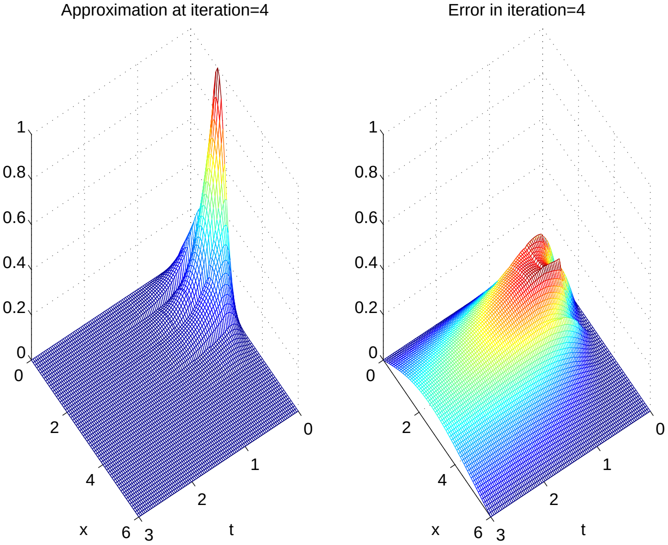




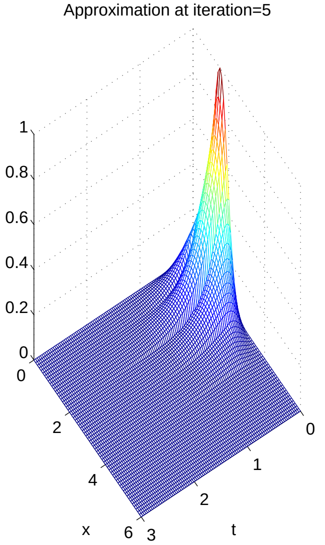
Parareal Schwarz WR: Iteration 3

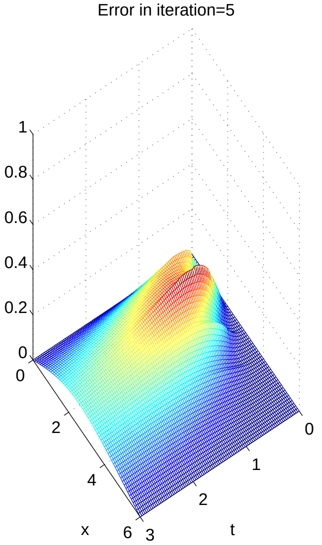


Parareal Schwarz WR: Iteration 4

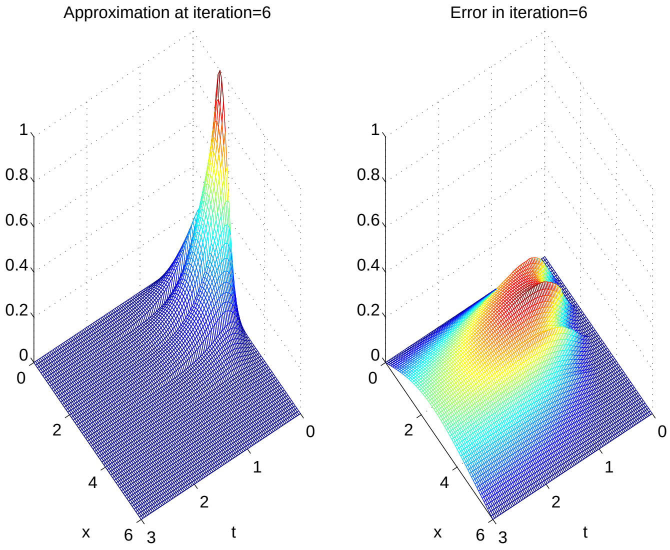


Parareal Schwarz WR: Iteration 5

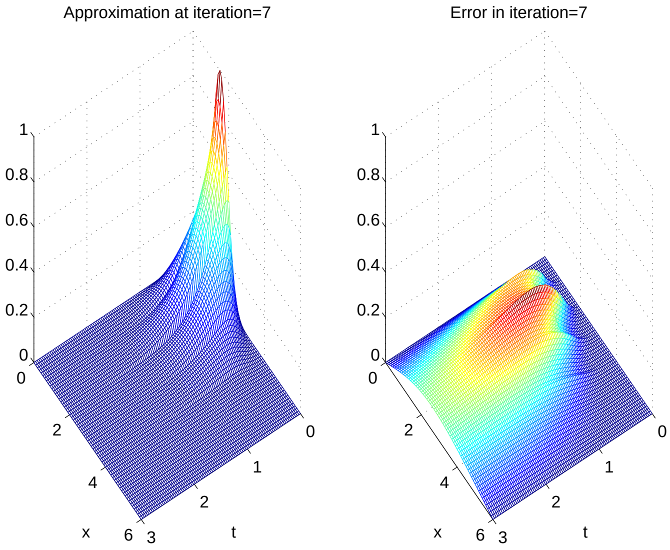




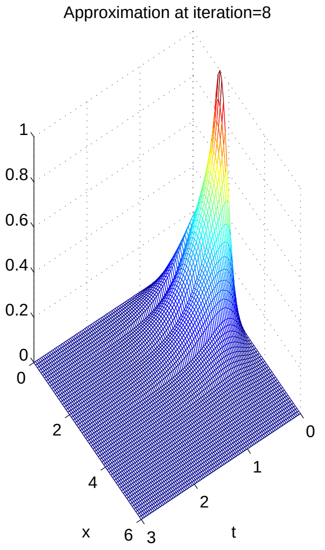
Parareal Schwarz WR: Iteration 6

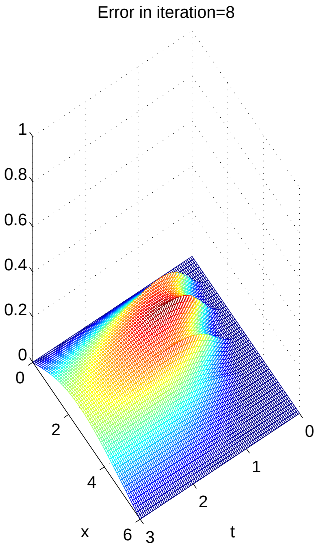


Parareal Schwarz WR: Iteration 7

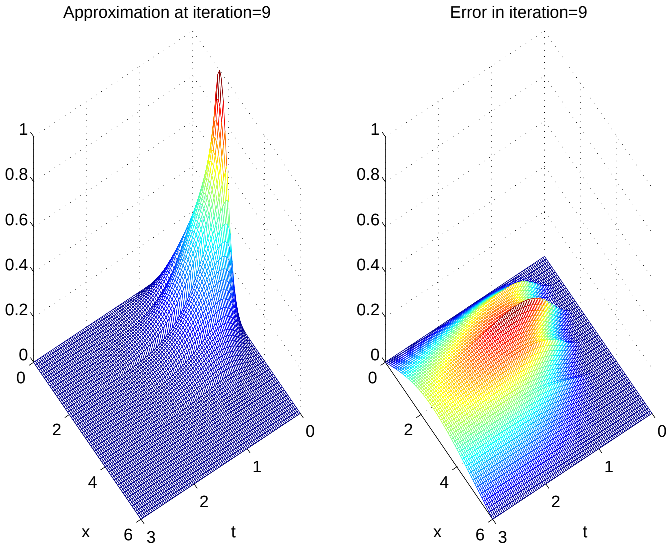


Parareal Schwarz WR: Iteration 8

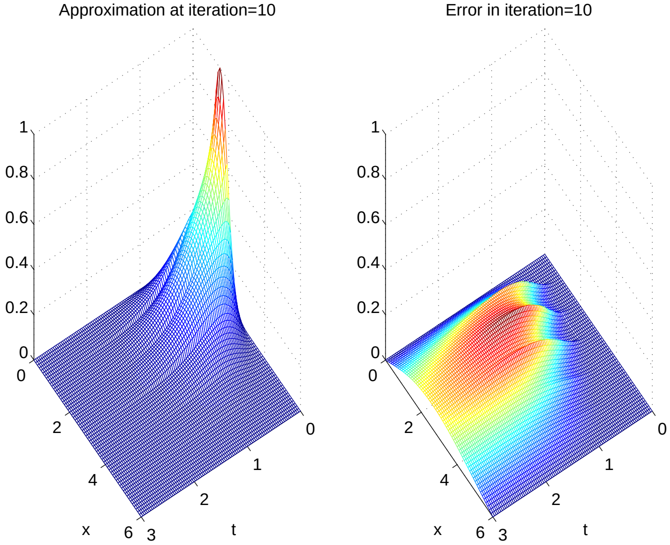




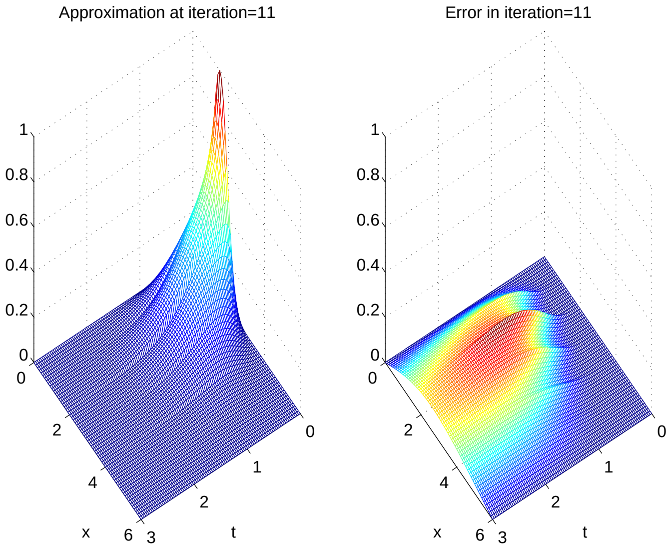
Parareal Schwarz WR: Iteration 9



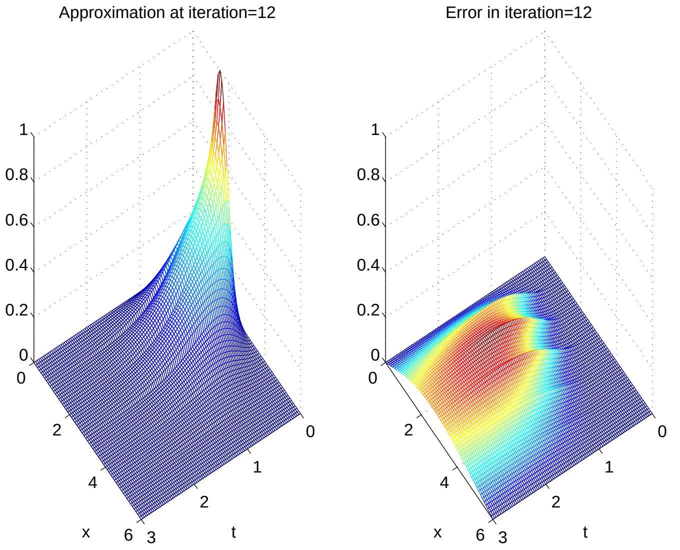
Parareal Schwarz WR: Iteration 10



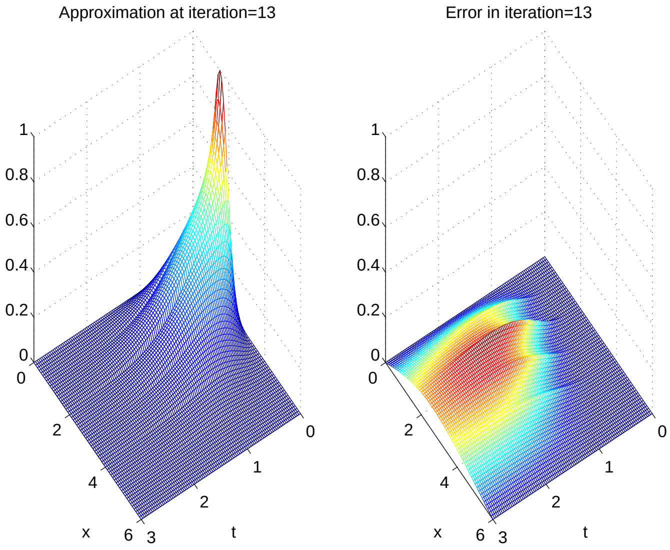
Parareal Schwarz WR: Iteration 11



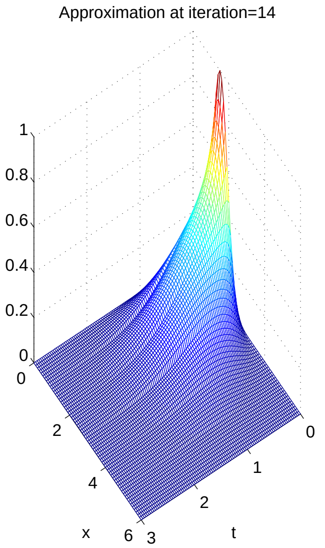
Parareal Schwarz WR: Iteration 12

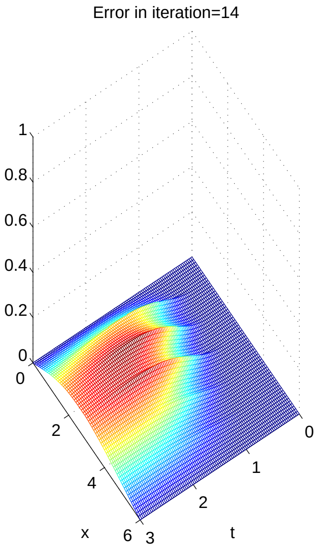


Parareal Schwarz WR: Iteration 13

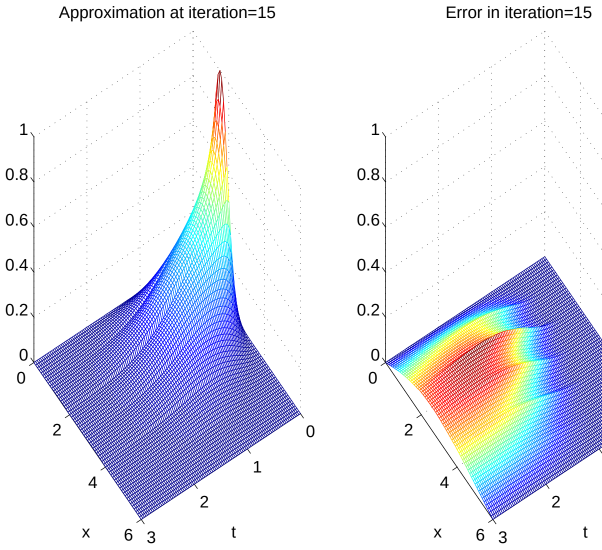


Parareal Schwarz WR: Iteration 14

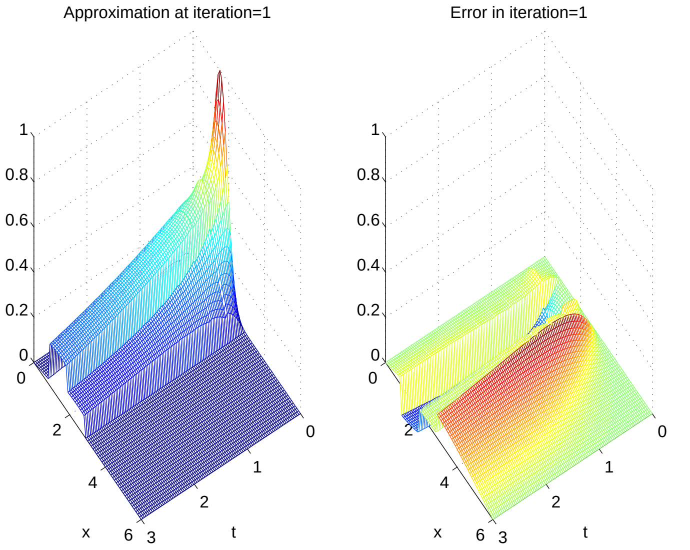




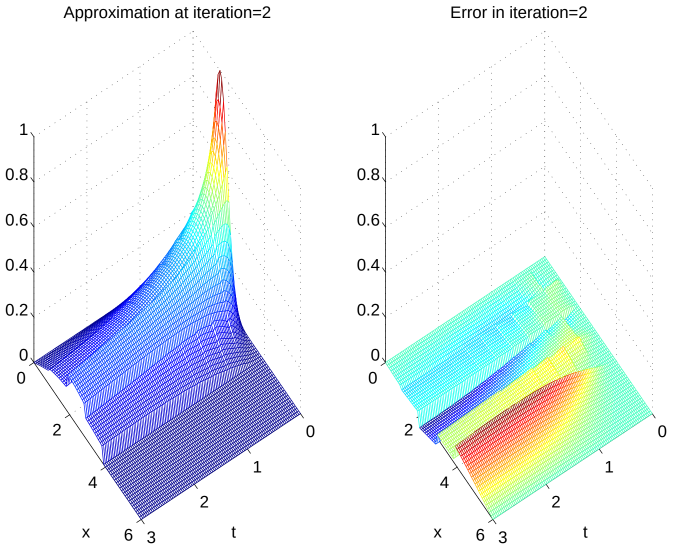
Parareal Schwarz WR: Iteration 15



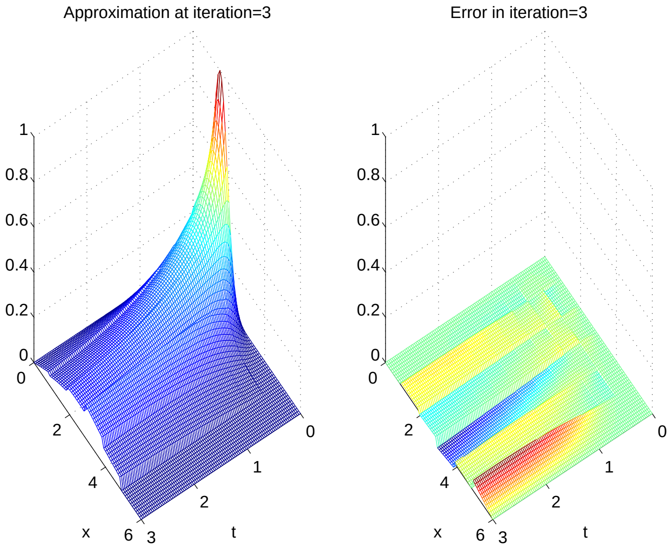
Optimized Parareal Schwarz WR: Iteration 1



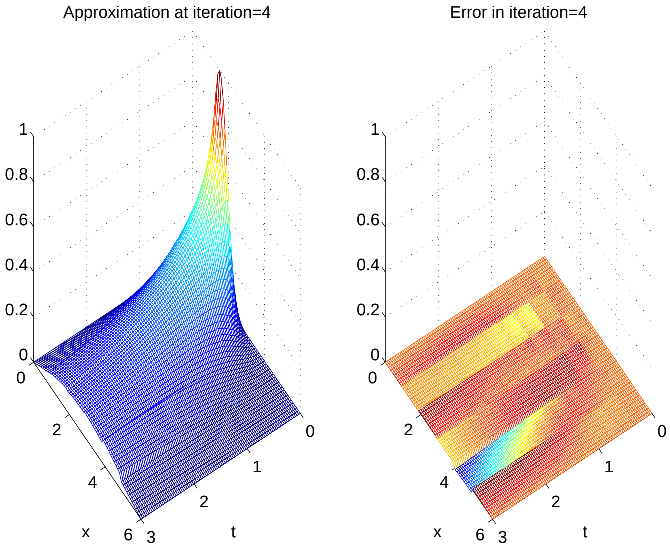
Optimized Parareal Schwarz WR: Iteration 2



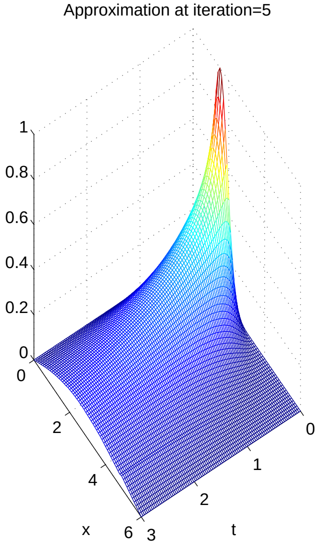
Optimized Parareal Schwarz WR: Iteration 3

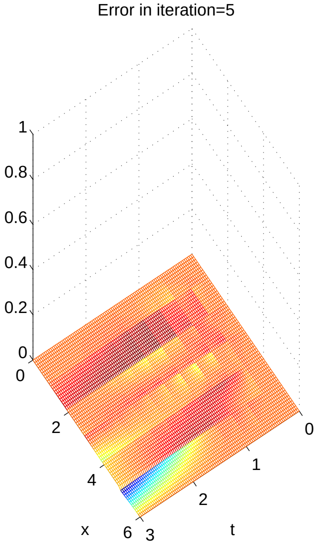


Optimized Parareal Schwarz WR: Iteration 4

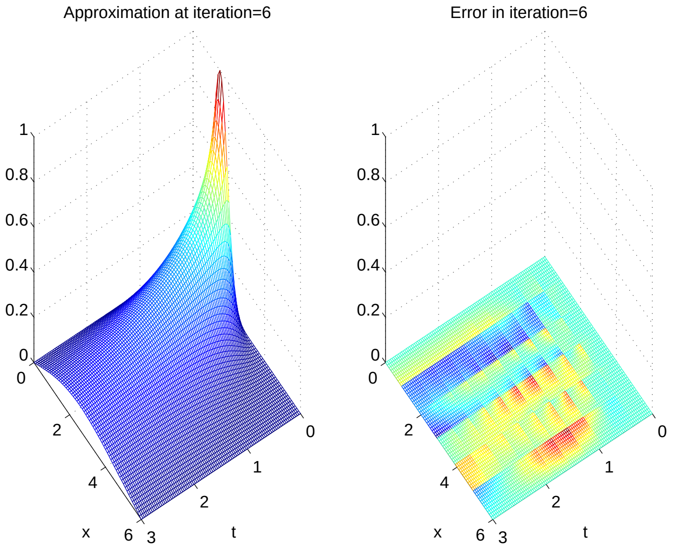


Optimized Parareal Schwarz WR: Iteration 5



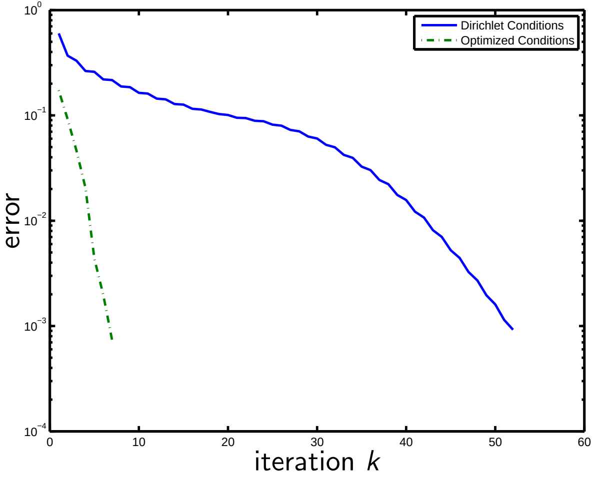


Optimized Parareal Schwarz WR: Iteration 6



Convergence Behavior of PSWR

Convergence comparison between Dirichlet and optimized transmission conditions in space:



Conclusions Part II: WR and DD

- Waveform relaxation methods for ODEs have their roots in the Picard interation (1893) and the analysis by Lindelöf (1894)
- Waveform Relaxation was invented by Lelarasmee, Ruehli and Sangiovanni-Vincentelli (1982) for VLSI simulations
- Schwarz Waveform Relaxation goes back to the PhD thesis of Gander (1996)
- Optimized Schwarz Waveform Relaxation (Gander, Halpern, Nataf 1999)
- Dirichlet-Neumann and Neumann-Neumann Waveform Relaxation (Gander, Kwok Mandal 2013, and Hoang, Jaffré, Japhet, Kern and Roberts 2013)

Preprints are available at www.unige.ch/∼gander