Time Parallel Time Integration
Part II
Waveform Relaxation and Domain Decomposition

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CEMRACS, July 2016
Waveform Relaxation and Domain Decomposition

- **Time Parallel Methods Part II**
  - WR and DD

**WR variants**
- Waveform Relaxation
- DIRECT

**Evolution Problems**
- Schwarz WR
- Neumann-Neumann WR
- Heat Equation

**Conclusions**
Picard 1893 and Lindelöf 1894

Émile Picard (1893): Sur l’application des méthodes d’approxpimations successives à l’étude de certaines équations différentielles ordinaires

\[ v' = f(v) \implies v^n(t) = v(0) + \int_0^t f(v^{n-1}(\tau)) d\tau \]

Ernest Lindelöf (1894): Sur l’application des méthodes d’approxpimations successives à l’étude des intégrales réelles des équations différentielles ordinaires

Theorem (Superlinear Convergence)

On bounded time intervals \( t \in [0, T] \), the iterates satisfy the superlinear error bound

\[ ||v - v^n|| \leq \frac{(CT)^n}{n!} ||v - v^0||, \]

where \( C \) is a positive constant.
Lelarasmee, Ruehli and Sangiovanni-Vincentelli


“The spectacular growth in the scale of integrated circuits being designed in the VLSI era has generated the need for new methods of circuit simulation. “Standard” circuit simulators, such as SPICE and ASTAP, simply take too much CPU time and too much storage to analyze a VLSI circuit”.
Using Kirchhoff’s and Ohm’s laws gives system of ODEs:

\[
\frac{\partial \mathbf{v}}{\partial t} = f(\mathbf{v}), \quad 0 < t < T
\]
\[
\mathbf{v}(0) = \mathbf{g}
\]
Iteration using sub-circuit solutions only:

\[
\begin{align*}
\partial_t v_1^{k+1} &= f_1(v_1^k, v_2^k, v_3^k) \\
\partial_t v_2^{k+1} &= f_2(v_1^k, v_2^{k+1}, v_3^k) \\
\partial_t v_3^{k+1} &= f_3(v_1^k, v_2^k, v_3^{k+1})
\end{align*}
\]

Signals along wires are called 'waveforms', which gave the algorithm its name: **Waveform Relaxation**.
Historical Numerical Convergence Study

Lelarasmee et al (1982): “Note that since the oscillator is highly non unidirectional due to the feedback from $v_3$ to the NOR gate, the convergence of the iterated solutions is achieved with the number of iterations being proportional to the number of oscillating cycles of interest”
Alternating Schwarz (Schwarz 1869)

For \( u_{xx} = 0 \) in \( \Omega = (0, 1) \), \( u(0) = u(1) = 0 \):

\[
\begin{align*}
\partial_{xx} u_1^n &= 0 \quad \text{in } \Omega_1 \\
u_1^n(0) &= 0 \\
u_1^n(\beta) &= u_2^{n-1}(\beta) \\
\partial_{xx} u_2^n &= 0 \quad \text{in } \Omega_2 \\
u_2^n(1) &= 0 \\
u_2^n(\alpha) &= u_1^n(\alpha)
\end{align*}
\]
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\]

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\partial_{xx} u_2^n & = 0 \quad \text{in } \Omega_2 \\
u_2^n(1) & = 0 \\
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\partial_{xx} u_1^n & = 0 \quad \text{in } \Omega_1 \quad \partial_{xx} u_2^n & = 0 \quad \text{in } \Omega_2 \\
u_1^n(0) & = 0 \quad \quad \quad \quad \quad \quad \quad u_2^n(1) & = 0 \\
u_1^n(\beta) & = u_2^{n-1}(\beta) \quad \quad \quad u_2^n(\alpha) = u_1^n(\alpha)
\end{align*}
\]
Dirichlet Neumann (Bjørstad, Widlund 1986)

For $u_{xx} = 0$ in $\Omega = (0, 1)$, $u(0) = u(1) = 0$:

$$
\begin{align*}
\partial_{xx} u_1^n & = 0 \quad \text{in } \Omega_1 \\
u_1^n(0) & = 0 \\
u_1^n(\alpha) & = u_2^{n-1}(\alpha)
\end{align*}
$$

$$
\begin{align*}
\partial_{xx} u_2^n & = 0 \quad \text{in } \Omega_2 \\
u_2^n(1) & = 0 \\
\partial_x u_2^n(\alpha) & = \partial_x u_1^n(\alpha)
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$$
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For \( u_{xx} = 0 \) in \( \Omega = (0,1) \), \( u(0) = u(1) = 0 \):

\[
\begin{align*}
\partial_{xx} u^n_1 &= 0 \quad \text{in } \Omega_1 \\
u^n_1(0) &= 0 \\
u^n_1(\alpha) &= u^{n-1}_2(\alpha) \\

\partial_{xx} u^n_2 &= 0 \quad \text{in } \Omega_2 \\
u^n_2(1) &= 0 \\
\partial_x u^n_2(\alpha) &= \partial_x u^n_1(\alpha)
\end{align*}
\]
Dirichlet Neumann (Bjørstad, Widlund 1986)

For \( u_{xx} = 0 \) in \( \Omega = (0, 1) \), \( u(0) = u(1) = 0 \):

\[
\begin{align*}
\Delta x u_1^n & = 0 \quad \text{in } \Omega_1 & \Delta x u_2^n & = 0 \quad \text{in } \Omega_2 \\
u_1^n(0) & = 0 & u_2^n(1) & = 0 \\
u_1^n(\alpha) & = u_2^{n-1}(\alpha) & \partial x u_2^n(\alpha) & = \partial x u_1^n(\alpha)
\end{align*}
\]
Dirichlet Neumann (Bjørstad, Widlund 1986)

For $u_{xx} = 0$ in $\Omega = (0, 1)$, $u(0) = u(1) = 0$:

\[
\begin{align*}
\partial_{xx} u_1^n &= 0 \quad \text{in } \Omega_1 \\
u_1^n(0) &= 0 \\
u_1^n(\alpha) &= \nu_2^{n-1}(\alpha)
\end{align*}
\]

\[
\begin{align*}
\partial_{xx} u_2^n &= 0 \quad \text{in } \Omega_2 \\
u_2^n(1) &= 0 \\
\partial_x u_2^n(\alpha) &= \partial_x u_1^n(\alpha)
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u_1^n(0) & = 0 \\
u_2^n(1) & = 0 \\
u_1^n(\alpha) & = u_2^{n-1}(\alpha) \\
u_2^n(\alpha) & = \partial_x u_1^n(\alpha)
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\partial_{xx} u_2^n &= 0 \quad \text{in } \Omega_2 \\
u_2^n(1) &= 0 \\
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\end{align*}
\]
Dirichlet Neumann (Bjørstad, Widlund 1986)

For $u_{xx} = 0$ in $\Omega = (0, 1)$, $u(0) = u(1) = 0$:

$$
\begin{align*}
\partial_{xx} u_1^n &= 0 & \text{in } \Omega_1 \\
u_1^n(0) &= 0 \\
u_1^n(\alpha) &= \nu_2^{n-1}(\alpha) \\
\partial_x u_2^n(\alpha) &= \partial_x u_1^n(\alpha) \\
\partial_{xx} u_2^n &= 0 & \text{in } \Omega_2 \\
u_2^n(1) &= 0
\end{align*}
$$
**Dirichlet Neumann (Bjørstad, Widlund 1986)**

For \( u_{xx} = 0 \) in \( \Omega = (0, 1) \), \( u(0) = u(1) = 0 \):

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u_1^n(0) & = 0 \\
u_1^n(\alpha) & = u_2^{n-1}(\alpha)
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\partial_{xx} u_1^n & = 0 \quad \text{in } \Omega_1 & \partial_{xx} u_2^n & = 0 \quad \text{in } \Omega_2 \\
u_1^n(0) & = 0 & u_2^n(1) & = 0 \\
u_1^n(\alpha) & = h^{n-1} & \partial_x u_2^n(\alpha) & = \partial_x u_1^n(\alpha)
\end{align*}
\]

\[h^n = \theta u_2^n(\alpha) + (1 - \theta) h^{n-1}\]
Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

For \( u_{xx} = 0 \) in \( \Omega = (0, 1) \), \( u(0) = u(1) = 0 \):

\[
\begin{align*}
\partial_{xx} u_i^n &= 0 \quad \text{in } \Omega_i & \partial_{xx} \psi_i^n &= 0 \quad \text{in } \Omega_i \\
 u_1^n(0) &= 0 & \psi_1^n(0) &= 0 \\
 u_2^n(1) &= 0 & \psi_2^n(1) &= 0 \\
 u_i^n(\alpha) &= h^{n-1} & \partial_n \psi_i^n(\alpha) &= \partial_n u_1^n(\alpha) + \partial_n u_2^n(\alpha)
\end{align*}
\]

\[
h^n = h^{n-1} - \theta (\psi_1(\alpha) + \psi_2(\alpha))
\]
Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

For \( u_{xx} = 0 \) in \( \Omega = (0, 1) \), \( u(0) = u(1) = 0 \):

\[
\begin{align*}
\partial_{xx} u^n_i & = 0 \quad \text{in } \Omega_i \quad \partial_{xx} \psi^n_i & = 0 \quad \text{in } \Omega_i \\
u^n_1(0) & = 0 \quad \psi^n_1(0) = 0 \\
u^n_2(1) & = 0 \quad \psi^n_2(1) = 0 \\
u^n_i(\alpha) & = h^{n-1} \quad \partial_n \psi^n_i(\alpha) = \partial_1 u^n_1(\alpha) + \partial_2 u^n_2(\alpha)
\end{align*}
\]

\( h^n = h^{n-1} - \theta(\psi_1(\alpha) + \psi_2(\alpha)) \)
Neumann-Neumann (Bourgat, Glowinski, Tallec, Vidrascu 1989)

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\end{align*}
\]

\[
h^n = h^{n-1} - \theta(\psi_1(\alpha) + \psi_2(\alpha))
\]
Time Dependent Problems

What happens if the PDE is time dependent, e.g. a heat equation
\[ \partial_t u = \partial_{xx} u, \quad \text{in } \Omega \]
or a wave equation,
\[ \partial_{tt} u = c^2 \partial_{xx} u, \quad \text{in } \Omega, \]
where the domain is now in space-time?

\[ t \]
\[ \Omega_1 \times (0, T) \]
\[ \Omega_2 \times (0, T) \]
\[ x \]
Time Dependent Problems

What happens if the PDE is time dependent, e.g. a heat

equation

\[ \partial_t u = \partial_{xx} u, \quad \text{in } \Omega \]

or a wave equation,

\[ \partial_{tt} u = c^2 \partial_{xx} u, \quad \text{in } \Omega, \]

where the domain is now in space-time?
Schwarz Waveform Relaxation: wave equation

\[ \partial_{tt} u_1^n = c^2 \partial_{xx} u_1^n \quad \text{in} \quad \Omega_1 \times (0, T) \quad \partial_{tt} u_2^n = c^2 \partial_{xx} u_2^n \quad \text{in} \quad \Omega_2 \times (0, T) \]

\[ u_1^n(0, t) = 0 \quad u_2^n(1, t) = 0 \]

\[ u_1^n(\beta, t) = u_2^{n-1}(\beta, t) \quad u_2^n(\alpha, t) = u_1^n(\alpha, t) \]

Theorem (Wave equation (G 1997))

*The algorithm converges in a finite number of steps, i.e.* when

\[ n \geq \frac{Tc}{\beta - \alpha}. \]

- Analogous results for many subdomains and general decompositions (G, Halpern 2004)
- Results with absorbing transmission conditions for non-overlapping decompositions (G, Halpern, Nataf 2003)
Graphical Convergence Proof in 1D

\[ \begin{align*}
\Omega_1 & \quad \text{no error in } u_1^1 \\
\Omega_2 & \quad \text{no error in } u_2^2 \\
\Omega_3 & \quad \text{no error in } u_3^3
\end{align*} \]
An Example with non-matching grids

solution at iteration step 6
Dirichlet-Neumann WR: wave equation

\[ \partial_{tt} u_1^n = c^2 \partial_{xx} u_1^n \quad \text{in} \ \Omega_1 \times (0, T) \]
\[ \partial_{tt} u_2^n = c^2 \partial_{xx} u_2^n \quad \text{in} \ \Omega_2 \times (0, T) \]
\[ u_1^n(0, t) = 0 \]
\[ u_2^n(1, t) = 0 \]
\[ u_1^n(\alpha, t) = h^{n-1}(t) \]
\[ \partial_x u_2^n(\alpha, t) = \partial_x u_1^n(\alpha, t) \]
\[ h^n(t) = \theta u_2^n(\alpha, t) + (1 - \theta) h^{n-1}(t) \]

Theorem (G, Kwok, Mandal 2014)

If \( \alpha = 0.5 \) and \( \theta = 0.5 \), DNWR converges in 2 iterations.

If \( \alpha \neq 0.5 \) and \( \theta = 0.5 \) the algorithm converges in a finite number of steps, as soon as

\[ n \geq \frac{Tc}{2 \min(\alpha, 1 - \alpha)}. \]

If \( \alpha = 0.5 \) and \( \theta \in (0, 1) \), \( \theta \neq 0.5 \), the algorithm converges linearly.
Neumann-Neumann WR: wave equation

\[ \partial_{tt} u^n_i = c^2 \partial_{xx} u^n_i \text{ in } \Omega_i \times (0, T) \]
\[ u^n_1(0) = 0 \]
\[ u^n_2(1) = 0 \]
\[ u^n_i(\alpha, t) = h^{n-1}(t) \]

\[ \partial_{tt} \psi^n_i = c^2 \partial_{xx} \psi^n_i \text{ in } \Omega_i \times (0, T) \]
\[ \psi^n_1(0) = 0 \]
\[ \psi^n_2(1) = 0 \]
\[ \partial_{n_i} \psi^n_i(\alpha,t) = \partial_{n_1} u^n_1(\alpha,t) + \partial_{n_2} u^n_2(\alpha,t) \]

\[ h^n(t) = h^{n-1}(t) - \theta(\psi_1(\alpha, t) + \psi_1(\alpha, t)) \]

Theorem (G, Kwok, Mandal 2014)

If \( \alpha = 0.5 \) and \( \theta = 0.25 \), NNWR converges in 2 iterations.

If \( \alpha \neq 0.5 \) and \( \theta = 0.25 \) the algorithm converges in a finite number of steps, as soon as

\[ n > \frac{T_c}{4 \min(\alpha, 1 - \alpha)}. \]

If \( \alpha = 0.5 \) and \( \theta \in (0, 0.5) \), \( \theta \neq 0.25 \), the algorithm converges linearly.
### Convergence estimates: heat equation

<table>
<thead>
<tr>
<th>Methods</th>
<th>2 subdomains</th>
<th>$N$ equal subdomains</th>
</tr>
</thead>
<tbody>
<tr>
<td>SWR</td>
<td>$\text{erfc} \left( \frac{n(\beta - \alpha)}{\sqrt{T}} \right)$</td>
<td>$2^n \text{erfc} \left( \frac{n\delta}{2\sqrt{T}} \right)$</td>
</tr>
<tr>
<td>DNWR</td>
<td>$\left( \frac{1 - 2\alpha}{1 - \alpha} \right)^n \text{erfc} \left( \frac{n\alpha}{2\sqrt{T}} \right)$</td>
<td>$(N - 2)^n \text{erfc} \left( \frac{n}{2N\sqrt{T}} \right)$</td>
</tr>
<tr>
<td>NNWR</td>
<td>$\left( \frac{(1 - 2\alpha)^2}{\alpha(1 - \alpha)} \right)^n \text{erfc} \left( \frac{n\alpha}{\sqrt{T}} \right)$</td>
<td>$\left( \frac{\sqrt{6}}{1 - e^{-\frac{(2n+1)N}{N^2T}}} \right)^{2n} e^{-\frac{n^2}{N^2T}}$</td>
</tr>
</tbody>
</table>

**Optimized Schwarz Waveform Relaxation (OSWR)**

\[
\partial_t u^n_1 = \partial_{xx} u^n_1 \text{ in } \Omega_1 \times (0, T) \quad \quad \partial_t u^n_2 = \partial_{xx} u^n_2 \text{ in } \Omega_2 \times (0, T)
\]

\[
\begin{align*}
    u^n_1(0, t) &= 0 \\
    u^n_2(1, t) &= 0 \\
    B_1 u^n_1(\beta, t) &= B_1 u^{n-1}_2(\beta, t) \\
    B_2 u^n_2(\alpha, t) &= B_2 u^n_1(\alpha, t)
\end{align*}
\]

- Many convergence results: heat equation, wave equation, advection reaction diffusion, Maxwell, shallow water, ...  
- Recent methods like Sweeping Preconditioner and Source Transfer are based on the same approach.
An Example with 8 Subdomains
Parareal Schwarz Waveform Relaxation

G, Jiang, Li (2011), see also Maday, Turinici (2007)

Model problem: \( \partial_t u = \partial_{xx} u \) in \( \Omega = (0, 1) \times (0, T) \)

Decomposition of the space-time domain:

\[
\Omega_{in} := \left( x_i - \frac{L}{2}, x_i + \frac{L}{2} \right) \times (t_n, t_{n+1})
\]
Parareal Schwarz Waveform Relaxation

Given an initial condition $u_0$ and boundary conditions $g^-$ and $g^+$, we define $F_{in}(u_0, g^-, g^+)$ and $G_{in}(u_0, g^-, g^+)$ to be fine and coarse approximations of the solution at $t = t_{n+1}$ of

$$
\partial_t u = \partial_{xx} u, \quad x \in (x_i^-, x_i^+), \quad t \in (t_n, t_{n+1})
$$

$$
u(x, t_n) = u_0, \quad x \in (x_i^-, x_i^+)$$

$$
\mathcal{B}_i^- u(x_i^-, t) = g^- \quad t \in (t_n, t_{n+1})
$$

$$
\mathcal{B}_i^+ u(x_i^+, t) = g^+ \quad t \in (t_n, t_{n+1})
$$

A Parareal Schwarz Waveform Relaxation Algorithm:

Given initial conditions $u_{0, in}(x)$ and boundary conditions $\mathcal{B}_i^- u_{i-1, n}(t)$ and $\mathcal{B}_i^+ u_{i+1, n}(t)$, we compute

1. All $u_{in}^{k+1} := F_{in}(u_{0, in}^k, \mathcal{B}_i^- u_{i-1, n}^k, \mathcal{B}_i^+ u_{i+1, n}^k)$ in parallel
2. Compute new initial conditions using

$$
u_{0, i, n+1}^{k+1} = F(u_{0, in}^k, \mathcal{B}_i^- u_{i-1, n}^k, \mathcal{B}_i^+ u_{i+1, n}^k)
$$

$$
+ G(u_{0, in}^{k+1}, \mathcal{B}_i^- u_{i-1, n}^{k+1}, \mathcal{B}_i^+ u_{i+1, n}^{k+1}) - G(u_{0, in}^k, \mathcal{B}_i^- u_{i-1, n}^k, \mathcal{B}_i^+ u_{i+1, n}^k)
$$
Parareal Schwarz WR Numerical Example

Model problem:

1D Heat equation

\[ \partial_t u = \partial_{xx} u \]

on \( \Omega = (0, 6) \times (0, T), \ T = 3 \)

Space-time decomposition into 6 spatial subdomains, and 10 time subdomains

Discretization with \( \Delta x = \frac{1}{10} \), \( \Delta t = \frac{3}{100} \)

Overlap in space of \( 2\Delta x \)
Parareal Schwarz WR: Iteration 1

Approximation at iteration=1

Error in iteration=1
Parareal Schwarz WR: Iteration 2

Approximation at iteration=2

Error in iteration=2

Parareal Schwarz WR
Conclusions
Parareal Schwarz WR: Iteration 3

Approximation at iteration=3

Error in iteration=3
Parareal Schwarz WR: Iteration 4

Approximation at iteration=4

Error in iteration=4
Parareal Schwarz WR: Iteration 5

Approximation at iteration=5

Error in iteration=5
Parareal Schwarz WR: Iteration 6

Approximation at iteration=6

Error in iteration=6
Parareal Schwarz WR: Iteration 7

Approximation at iteration=7

Error in iteration=7
Parareal Schwarz WR: Iteration 8

Approximation at iteration=8

Error in iteration=8

Conclusions
Parareal Schwarz WR: Iteration 9

Approximation at iteration=9

Error in iteration=9
Parareal Schwarz WR: Iteration 10

Approximation at iteration=10

Error in iteration=10

Approximation at iteration=10

Error in iteration=10
Parareal Schwarz WR: Iteration 11

Approximation at iteration=11

Error in iteration=11
Parareal Schwarz WR: Iteration 12

Approximation at iteration=12

Error in iteration=12
Parareal Schwarz WR: Iteration 13

Approximation at iteration=13

Error in iteration=13
Parareal Schwarz WR: Iteration 14

Approximation at iteration=14

Error in iteration=14
Parareal Schwarz WR: Iteration 15

Approximation at iteration=15

Error in iteration=15
Optimized Parareal Schwarz WR: Iteration 1

Approximation at iteration=1

Error in iteration=1
Optimized Parareal Schwarz WR: Iteration 2

Approximation at iteration=2

Error in iteration=2
Optimized Parareal Schwarz WR: Iteration 3

Approximation at iteration=3

Error in iteration=3

Conclusions
Optimized Parareal Schwarz WR: Iteration 4

Approximation at iteration=4

Error in iteration=4

Parareal Schwarz WR

Conclusions
Optimized Parareal Schwarz WR: Iteration 5

Approximation at iteration=5

Error in iteration=5
Optimized Parareal Schwarz WR: Iteration 6

Approximation at iteration=6

Error in iteration=6
Convergence Behavior of PSWR

Convergence comparison between Dirichlet and optimized transmission conditions in space:

![Graph showing convergence behavior of PSWR with error on the y-axis and iteration k on the x-axis. The graph compares Dirichlet Conditions (solid blue line) and Optimized Conditions (dashed green line).]
Conclusions Part II: WR and DD

- Waveform relaxation methods for ODEs have their roots in the Picard iteration (1893) and the analysis by Lindelöf (1894)

- Waveform Relaxation was invented by Lelarasmee, Ruehli and Sangiovanni-Vincentelli (1982) for VLSI simulations

- Schwarz Waveform Relaxation goes back to the PhD thesis of Gander (1996)

- Optimized Schwarz Waveform Relaxation (Gander, Halpern, Nataf 1999)

- Dirichlet-Neumann and Neumann-Neumann Waveform Relaxation (Gander, Kwok Mandal 2013, and Hoang, Jaffré, Japhet, Kern and Roberts 2013)

Preprints are available at www.unige.ch/~gander