REDUCED BASIS METHOD
approximation of PDE’s, interpolation, a posteriori estimates

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This talk deals with linear constructive approximation methods specifically tailored to approximate the functions in $F$ (or in a set close to $F$ see PBDW).

For this, we consider approximations on “appropriate” $n$-dimensional spaces $X_n$ of relatively small dimension under the hypothesis that the Kolmogorov $n$-width of $F$ is small.
The Kolmogorov $n$-width of $F$ in $\mathcal{X}$, is defined as

$$d_n(F, \mathcal{X}) = \inf_{X_n \subseteq \mathcal{X}} \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_{\mathcal{X}},$$

And the assumption is that $d_n(F, \mathcal{X})$ is small.
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$$d_n(F, \mathcal{X}) = \inf_{X_n \subset \mathcal{X}} \sup_{\dim(X_n) \leq n} \inf_{u \in F} \inf_{v \in X_n} \|u - v\|_{\mathcal{X}} ,$$

And the assumption is that $d_n(F, \mathcal{X})$ is small.

In order to satisfy the constraint on the computing time, we consider continuous linear approximations $\mathcal{J}_n : \mathcal{X} \to X_n$ on “appropriate” $n$-dimensional spaces $X_n$ of relatively small dimension. A first bound on the approximation error is then

$$\sup_{u \in F} \|u - \mathcal{J}_n[u]\|_{\mathcal{X}} \leq (1 + \Lambda_n) \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_{\mathcal{X}} ,$$

where

$$\Lambda_n = \sup_{\varphi \in \mathcal{X}} \frac{\|\mathcal{J}_n[\varphi]\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{X}}}$$

is the norm of the operator $\mathcal{J}_n : \mathcal{X} \to X_n$ and is known as the Lebesgue constant.
The Kolmogorov $n$-width of $F$ in $X$, is defined as

$$d_n(F, X) = \inf_{X_n \subseteq X} \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_X,$$

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$$\sup_{u \in F} \|u - J_n[u]\|_X \leq (1 + \Lambda_n) \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_X,$$  \hspace{1cm} (1)

where

$$\Lambda_n = \sup_{\varphi \in X} \frac{\|J_n[\varphi]\|_X}{\|\varphi\|_X}$$  \hspace{1cm} (2)

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EIM/GEIM

In 2004 with M. Barrault, N. C. Nguyen, and A. T. Patera, we proposed an generic approach: the ‘Empirical Interpolation’ Method: Application to Efficient Reduced-Basis Discretization Of Partial Differential Equations, that has proven successful.

This approach allows to determine an “empirical” optimal set of interpolation points and/or set of interpolating functions.

In 2013, with Olga Mula, we have generalized it (GEIM) to include more general output from the functions we want to interpolate: not only pointwise values but also some moments.
The first generating function is \( \varphi_1 = \arg \max_{\varphi \in F} \| \varphi(\cdot) \|_{L^\infty(\Omega)} \), the associated interpolation point satisfies

\[
x_1 = \arg \max_{x \in \Omega} |\varphi_1(x)|,
\]

we then set \( q_1 = \varphi_1(\cdot)/\varphi_1(x_1) \) and \( B_{11}^1 = 1 \).

Then the construction proceeds by induction: assume the nested sets of interpolation points \( \Xi_{M-1} = \{ x_1, \ldots, x_{M-1} \} \), \( M \leq M_{\text{max}} \), and the associated nested sets of basis functions \( \{ q_1, \ldots, q_{M-1} \} \) are given\(^1\). We first solve the interpolation problem: Find

\[
\mathcal{I}_{M-1}[\varphi(\cdot)] = \sum_{j=1}^{M-1} \alpha_{M-1,j}[\varphi] q_j,
\]

such that

\[
\mathcal{I}_{M-1}[\varphi(\cdot)](x_i) = \varphi(x_i), \quad i = 1, \ldots, M - 1,
\]

that allows to define the \( \alpha_{M-1,j}[\varphi], 1 \leq j \leq M - 1 \), as it can be proven indeed that the \( (M-1) \times (M-1) \) matrix of running entry \( q_j(x_i) \) is invertible, actually it is lower triangular with unity diagonal.

\(^1\) where \( M_{\text{max}} \leq M \) is some given upper bound fixed \textit{a priori}
We then set
\[
\forall \varphi \in F, \quad \varepsilon_{M-1}(\varphi) = \|\varphi - \mathcal{I}_{M-1}[\varphi]\|_{L^\infty(\Omega)},
\]
and define
\[
\varphi_M = \arg \max_{\varphi \in F} \varepsilon_{M-1}(\varphi),
\]
and
\[
x_M = \arg \max_{x \in \Omega} |\varphi_M(x) - \mathcal{J}_{M-1}[\varphi_M](x)|,
\]
we finally set \(r_M(x) = \varphi_M(x) - \mathcal{J}_{M-1}[\varphi_M](x)\), \(q_M = r_M/r_M(x_M)\) and
\[
B_{ij}^M = q_j(x_i), 1 \leq i, j \leq M.
\]

The Lagrangian functions — that can be used to build the interpolation operator \(\mathcal{I}_M\) in
\[
X_M = \text{span}\{\varphi_i, 1 \leq i \leq M\} = \text{span}\{q_i, 1 \leq i \leq M\}
\]
over the set of points \(\Xi_M = \{x_i, 1 \leq i \leq M\}\) — verify for any given \(M\),
\[
\mathcal{I}_M[u(\cdot)] = \sum_{i=1}^{M} u(x_i)h_i^M(\cdot)
\]
where
\[
h_i^M(\cdot) = \sum_{j=1}^{M} q_j(\cdot)[B^M]_{ji}^{-1}
\]
(note indeed that \(h_i^M(x_j) = \delta_{ij}\)).
For GEIM, we assume now that we do not have access to the values of $\varphi \in F$ at points in $\Omega$ easily, but, on the contrary, that we have a dictionary of linear forms $\sigma \in \Sigma$ — assumed to be continuous in some sense, e.g. in $L^2(\Omega)$ with norm 1 — the application of which over each $\varphi \in F$ is easy. Our extension consists in defining $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_M$ and a family of associated linear forms $\sigma_1, \sigma_2, \ldots, \sigma_M$ such that the following generalized interpolation process (our GEIM) is well defined:

$$J_M[\varphi] = \sum_{j=1}^{M} \beta_j \tilde{\varphi}_j, \text{ such that } \forall i = 1, \ldots, M, \quad \sigma_i(J_M[\varphi]) = \sigma_i(\varphi) \quad (1)$$

Note that the GEIM reduces to the EIM when the dictionary is composed of dirac masses, defined in the dual space of $C^0(\Omega)$.
Note that — obviously — everything has to be implemented on a computer and thus discretized!!
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Note also that — for some reasons — you may want to use your proper set of basis/interpolating function in your preferred space $X_N$ that may come from intuition, previous knowledge, or POD/SVD.
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**Remark 2.** If for some reasons, a set of functions $u_i \in \mathcal{U}$, $i \in \mathbb{N}$ were to be given, all linearly independent, then the procedure of finding the interpolation points through the process $\forall i, 1 \leq i \leq M - 1$, $u(x_i) = \sum_{j=1}^{M-1} \alpha_{i,j}[u]u_j(x_i)$ and set $x_M = \arg \max_{x \in \Omega} |u_M(x) - \sum_{j=1}^{M-1} \alpha_{i,j}[u_M]u_j(x)|$ is also well defined and leads to a set of interpolation points that have similar properties as above. The rational for the greedy approach is that it allows us to get a better sense of the interpolation properties since $\forall u$,

$$
\|u(\cdot) - \mathcal{I}_M[u(\cdot)]\|_{L^\infty(\Omega)} \leq \|u_{M+1}(\cdot) - \mathcal{I}_M[u_{M+1}(\cdot)]\|_{L^\infty(\Omega)} = \varepsilon_M(x_{M+1})
$$

(13)

and this last quantity is one of the outputs of the construction process.
Note that — obviously — everything has to be implemented on a computer and thus discretized!!

Note also that — for some reasons — you may want to use your proper set of basis/interpolating function in your preferred space $X_N$ that may come from intuition, previous knowledge, or POD/SVD.

**Remark 2.** If for some reasons, a set of functions $u_i \in U$, $i \in \mathbb{N}$ were to be given, all linearly independent, then the procedure of finding the interpolation points through the process $\forall i, 1 \leq i \leq M - 1, \ u(x_i) = \sum_{j=1}^{M-1} \alpha_{i,j}[u]u_j(x_i)$ and set $x_M = \text{arg max}_{x \in \Omega} \|u_M(x) - \sum_{j=1}^{M-1} \alpha_{i,j}[u_M]u_j(x)\|$ is also well defined and leads to a set of interpolation points that have similar properties as above. The rational for the greedy approach is that it allows us to get a better sense of the interpolation properties since $\forall u$,

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A general multipurpose interpolation procedure: the magic points
Yvon Maday - Ngoc Cuong Nguyen - Anthony T. Patera - George S. H. Pau

The important thing is to measure to which extent the EIM/GEIM misses the optimality of the best approximation in the best optimal discrete space suggested by the definition of the Kolmogorov $n$-width.
Formula

\[
\sup_{u \in F} \| u - \mathcal{J}_n[u] \|_X \leq (1 + \Lambda_n) \sup_{u \in F} \inf_{v \in X_n} \| u - v \|_X,
\]

suggests that \( \Lambda_n \) plays an important role in the result and it is therefore important to discuss its behavior as \( n \) increases. First of all, \( \Lambda_n \) depends both on the choices of the interpolating functions and interpolation points.

We have proven (YM-Mula-Patera-Yano) that \( \Lambda_n = 1/\beta_n \), where

\[
\beta_n = \inf_{\varphi \in X_n} \sup_{\sigma \in \text{Span}\{\sigma_0, \ldots, \sigma_{n-1}\}} \frac{\langle \varphi, \sigma \rangle_{X,x} \cdot \langle x, x' \rangle}{\| \varphi \|_X \cdot \| \sigma \|_X}. 
\]
Formula

\[ \sup_{u \in F} \| u - \mathcal{J}_n[u] \| \leq (1 + \Lambda_n) \sup_{u \in F} \inf_{v \in X_n} \| u - v \| \chi, \]

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GEIM interpreted as an oblic projection …
Figure 3: (a) Variation of Lebesgue constant, $\Lambda_M$ with $n$ where $M = \frac{1}{2}(n + 1)(n + 2)$, and (b) distribution of magic points, for $\Omega_{\text{hex}}$.

Figure 4: Results for a “lunar croissant” domain $\Omega_{\text{cro}}$: (a) variation of the Lebesgue constant $\Lambda_n$ with $n$, and (b) distribution of magic points for $n = 12$. 
We are now working on the justification of this nice behavior.
The next point is to justify the choice of the interpolating functions. It can be proven that

**Lemma**

For any $n \geq 1$, the $n$th interpolating function $\varphi_n$ verifies

$$
\|\varphi_n - P_n(\varphi_n)\|_x \geq \frac{\eta}{1 + \Lambda_n} \max_{f \in F} \|f - P_n(f)\|_x.
$$

Which allows to use the frame “weak greedy” of the papers by

- P. Binev, A. Cohen, W. Dahmen, R.A. DeVore, G. Petrova, and P. Wojtaszczyk,

- and R. A. DeVore, G. Petrova, and P. Wojtaszczyk,

to analyse the convergence properties of our algorithm

Work done with O. Mula and G. Turinici (SINUM (2016))
In a nutshell, in the case where we have a Hilbert framework, our result states that

**Theorem**

If $(\Lambda_n)_{n=1}^\infty$ is a monotonically increasing sequence then

i) if $d_n \leq C_0 n^{-\alpha}$ for any $n \geq 1$, then $\tau_n \leq C_0 \tilde{\beta}_n n^{-\alpha}$, with

$$\tilde{\beta}_n := 2^{3\alpha+1} \Lambda_n^2, \text{ if } n \geq 2.$$  

ii) if $d_n \leq C_0 e^{-c_1 n^\alpha}$ for $n \geq 1$ and $C_0 \geq 1$, then $\tau_n \leq C_0 \tilde{\beta}_n e^{-c_2 n^{-\alpha}}$, with

$$\tilde{\beta}_n := \sqrt{2} \Lambda_n, \text{ if } n \geq 2.$$  

Work done with O. Mula and G. Turinici (SINUM (2016))
Another application: fluid flow

\[ \Omega = [0; 1] \times [0; 1] \subset \mathbb{R}^2. \]

Find the solution \((u_\mu, p_\mu) \in \left( H^1(\Omega) \right)^2 \times L^2_0(\Omega)\) of:

\[
\begin{align*}
-\Delta u_\mu + \text{grad}(p_\mu) &= f_\mu, \quad \text{a.e. in } \Omega \\
\text{div}(u_\mu) &= 0, \quad \text{a.e. in } \Omega \\
u_\mu &= \begin{pmatrix} x(1-x) \\ 0 \end{pmatrix}, \quad \text{a.e. on } \Gamma_1 \\
u_\mu &= 0, \quad \text{a.e. on } \partial \Omega \setminus \Gamma_1
\end{align*}
\]

\[ f_\mu = \begin{pmatrix}
100 \sin(\mu_1 \pi y) \\
-100 \sin(\mu_2 \pi \frac{1-x}{2})
\end{pmatrix} \]
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\[
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u_\mu = \begin{pmatrix} x(1-x) \\ 0 \end{pmatrix}, & \text{a.e. on } \Gamma_1 \\
u_\mu = 0, & \text{a.e. on } \partial \Omega \setminus \Gamma_1
\end{cases}
\]

\[ f_\mu = \begin{pmatrix} 100 \sin(\mu_1 \Pi y) \\ -100 \sin(\mu_2 \Pi \frac{1-x}{2}) \end{pmatrix} \]
Application of EIM for Reduced Basis Methods for non affine and nonlinear PDE:

see


or the two recent books

Certified Reduced Basis Methods for Parametrized Partial Differential Equations Authors: Hesthaven, Jan S, Rozza, Gianluigi, Stamm, Benjamin

Reduced Basis Methods for Partial Differential Equations An Introduction Authors: Quarteroni, Alfio, Manzoni, Andrea, Negri, Federico
More about (G)EIM
In a recent paper, with J.P. Argaud, B. Bouriquet, H. Gong and O. Mula, we have introduced the GEIM to monitor nuclear reactor.

The challenge here is that we do not have access inside the core of the reactor and thus have only the possibility to place the captors inside the surrounding region.
The two group diffusion equation in matrix notation reads

\[
A(\mu)\varphi = \frac{1}{k_{\text{eff}}} F(\mu)\varphi
\]

Where \( \mu \) is the parameters set, e.g. \( D, \Sigma, \nu\Sigma_f \). \( A \) and \( F \) are 2×2 matrix and \( \varphi \) is a 2-element column vector:

\[
A(\mu) = \begin{pmatrix}
-\nabla \cdot D^1 \nabla + (\Sigma_a^1 + \Sigma_s^{1\rightarrow 2}) & 0 \\
-\Sigma_s^{1\rightarrow 2} & -\nabla \cdot D^2 \nabla + \Sigma_a^2
\end{pmatrix}
\]

\[
F(\mu) = \begin{pmatrix}
\chi_1 \nu \Sigma_f^1 & \chi_1 \nu \Sigma_f^2 \\
\chi_2 \nu \Sigma_f^1 & \chi_2 \nu \Sigma_f^2
\end{pmatrix}
\]

\[
\varphi = \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}
\]

Where \( D^i, i = 1, 2 \) is called the diffusion coefficient of each group; \( \Sigma_a^i, i = 1, 2 \) is the absorption cross section of each group; \( \varphi_i, i = 1, 2 \) is the neutron flux of each group; \( \Sigma_s^{1\rightarrow 2} \) is called the removal cross section from group 1 to group 2; \( \nu \Sigma_f^i, i = 1, 2 \) is the fission source term of each group; \( \chi_i, i = 1, 2 \) is called the fission spectrum of each group; finally \( k_{\text{eff}} \) is the effective multiplication factor, also the eigenvalue of equation.
In 1D, this looks like

Figure 1: Flux and power distribution in the Core for the benchmark problem
In order you are convinced that the Kolmogorov dimension is small
First classical EIM

EIM, Magic-Point choosed in [0,30]cm, Flux, Group 2

Dimension M

x (in cm)
Figure 6: The relative error of different functions in $L^2$ and semi-$H^1$ norm versus the EIM dimension (Noise Free). The function used by EIM is the flux of group 2, the magic points are chosen from the whole area, $[0, 30]$ cm. The interpolation for flux of group 1 and the power use the same coefficients solved by EIM for the flux of group 2.
Then with a restriction on the position of the points...
Similar results in two dimensions

The first twenty interpolation points distribution (constrained in fuel region)
Similar results in two dimensions
Similar results in two dimensions
What about noisy data

We clearly see the effect of the Lebesgue constant
What about noisy data:
how to minimize the effect of the Lebesgue constant
Application of GEIM in data assimilation and monitoring
Application of GEIM in data assimilation and monitoring
Application of GEIM in data assimilation and monitoring
Incorporating the model error:
Parametrized-Background Data-Weak (PBDW) formulation
with A.T. Patera, J. D. Penn and M. Yano

The PBDW formulation integrates a parametrized mathematical model and \( M \) experimental observations associated with the configuration \( \mathcal{C} \) to estimate the true field \( u^{true}[\mathcal{C}] \) as well as any desired output \( l^{out}(u^{true}[\mathcal{C}]) \in \mathcal{C} \) for given output functional \( l^{out} \).

We first introduce a sequence of background spaces that reflect our (prior) best knowledge,
\[
\mathcal{Z}_1 \subset \cdots \subset \mathcal{Z}_{N_{max}} \subset \mathcal{U};
\]
here the second ellipsis indicates that we may consider the sequence of length \( N_{max} \) as resulting from a truncation of an infinite sequence. Our goal is to choose the background spaces such that
\[
\lim_{N \to \infty} \inf_{w \in \mathcal{Z}_N} \|u^{true}[\mathcal{C}] - w\| \leq \epsilon_{\mathcal{Z}} \quad \forall \mathcal{C} \in \mathcal{S},
\]
In words, we choose the background spaces such that the most dominant physics that we anticipate to encounter for various system configurations is well represented for a relatively small \( N \).
Incorporating the model error: Parametrized-Background Data-Weak (PBDW) formulation with A.T. Patera, J. D. Penn and M. Yano

We now characterize our data acquisition procedure. Given a system in configuration $\mathcal{C} \in \mathcal{S}$, we assume our observed data $y^{\text{obs}}[\mathcal{C}] \in \mathbb{C}^M$ is of the form,

$$\forall m = 1, \ldots, M, \quad y_m^{\text{obs}}[\mathcal{C}] = \ell_m^o(u^{\text{true}}[\mathcal{C}]) + e_m;$$

here $y_m^{\text{obs}}[\mathcal{C}]$ is the value of the $m$-th observation, $\ell_m^o \in \mathcal{U}'$ is the linear (and not antilinear) functional of the functional depends on the specific transducer used to acquire data.

Concerning the form of the noises $(e_m)_m$, we make the following three assumptions:

(A1) zero mean: $E[e_m] = 0, m = 1, \ldots, M$;
(A2) homoscedastic: $E[e_m^2] = \sigma^2, m = 1, \ldots, M$;
(A3) uncorrelated: $E[e_me_n] = 0, m \neq n$. 

We first associate with each observation functional $\ell^o_m \in \mathcal{U}'$ an observable function,

$$\forall m = 1, \ldots, M, \quad q_m = R_{\mathcal{U}} \ell^o_m,$$

the Riesz representation of the functional [1]. We then introduce hierarchical observable spaces,

$$\forall M = 1, \ldots, M_{\text{max}}, \ldots, \quad \mathcal{U}_M = \text{span}\{q_m\}_{m=1}^M;$$

We may now state the PBDW estimation statement: given a physical system in configuration $\mathcal{C} \in \mathcal{S}$, find $(u^*_{N,M}[\mathcal{C}] \in \mathcal{U}, z^*_{N,M}[\mathcal{C}] \in \mathcal{Z}_N, \eta^*_{N,M}[\mathcal{C}] \in \mathcal{U})$ such that

$$(u^*_{N,M}[\mathcal{C}], z^*_{N,M}[\mathcal{C}], \eta^*_{N,M}[\mathcal{C}]) = \arg \inf_{u_{N,M} \in \mathcal{U}, z_{N,M} \in \mathcal{Z}_N, \eta_{N,M} \in \mathcal{U}} \|\eta_{N,M}\|^2$$

subject to

$$(u_{N,M}, v) = (\eta_{N,M}, v) + (z_{N,M}, v) \quad \forall v \in \mathcal{U},$$

$$(u_{N,M}, \phi) = (u^\text{obs}_M[\mathcal{C}], \phi) \quad \forall \phi \in \mathcal{U}_M.$$ 

We may readily derive the associated (reduced) Euler-Lagrange equations as a saddle problem [16]: given a physical system in configuration $\mathcal{C} \in \mathcal{S}$, find $(\eta^*_{N,M}[\mathcal{C}] \in \mathcal{U}_M, z^*_{N,M}[\mathcal{C}] \in \mathcal{Z}_N)$ such that

$$(\eta^*_{N,M}[\mathcal{C}], q) + (z^*_{N,M}[\mathcal{C}], q) = (u^\text{obs}_M[\mathcal{C}], q) \quad \forall q \in \mathcal{U}_M,$$

$$(\eta^*_{N,M}[\mathcal{C}], p) = 0 \quad \forall p \in \mathcal{Z}_N,$$

and set

$$u^*_{N,M}[\mathcal{C}] = \eta^*_{N,M}[\mathcal{C}] + z^*_{N,M}[\mathcal{C}].$$
Algebraic Form: Offline-Online Computational Procedure

\[
\begin{pmatrix}
A & B \\
B^H & 0
\end{pmatrix}
\begin{pmatrix}
\eta^*[C] \\
z^*[C]
\end{pmatrix} = \begin{pmatrix}
y^{\text{obs}}[C] \\
0
\end{pmatrix},
\]

where

\[
A \equiv Q^\dagger UQ = LQ \in \mathbb{C}^{M \times M}
\]

\[
B \equiv Q^\dagger UZ = LZ \in \mathbb{C}^{M \times N},
\]
Analysis

Lemma 1. The expectation of the norm of the state error may be decomposed into deterministic and stochastic components and is bounded by

\[ E[\|u^{\text{true}}[C] - u^*_N, M[C]\|] \leq \|u^{\text{true}}[C] - u_{N,M}^{\text{nf}}[C]\| + E[\|u_{N,M}^{\text{nf}}[C] - u^*_N, M[C]\|]; \]

here \(u^{\text{true}}[C]\) is the true deterministic state, \(u^*_N, M[C]\) is the PBDW estimate given by (3), \(u_{N,M}^{\text{nf}}[C]\) is the noise-free estimate given by (6), and \(E\) refers to expectation.

Proposition 2. The deterministic component of the error is bounded by

\[ \|u^{\text{true}}[C] - u_{N,M}^{\text{nf}}[C]\| \leq \left(1 + \frac{1}{\beta_{N,M}}\right) \inf_{q \in U_N \cap Z_N^{\perp}} \|\Pi_{Z_N^{\perp}} u^{\text{true}}[C] - q\|, \]

where \(\beta_{N,M}\) is the inf-sup constant given by

\[ \beta_{N,M} \equiv \inf_{w \in Z_N} \sup_{v \in U_M} \frac{(w, v)}{\|w\| \|v\|}; \]

here \(u^{\text{true}}[C]\) is the true deterministic state, and \(u_{N,M}^{\text{nf}}[C]\) is the noise-free estimate given by (6).
**Analysis**

**Lemma 1.** The expectation of the norm of the state error may be decomposed into deterministic and stochastic components and is bounded by

$$ E[\| u^{\text{true}}[C] - u^*_{N,M}[C] \|] \leq \| u^{\text{true}}[C] - u^{\text{nf}}_{N,M}[C] \| + E[\| u^{\text{nf}}_{N,M}[C] - u^*_{N,M}[C] \|]; $$

here $u^{\text{true}}[C]$ is the true deterministic state, $u^*_{N,M}[C]$ is the PBDW estimate given by (3), $u^{\text{nf}}_{N,M}[C]$ is the noise-free estimate given by (6), and $E$ refers to expectation.

**Proposition 3.** Suppose the observation error $e$ satisfies the assumptions (A1), (A2), and (A3). Then, the mean of the stochastic error is zero:

$$ E[ u^{\text{nf}}_{N,M}[C] - u^*_{N,M}[C] ] = 0; $$

(9)

here $u^{\text{nf}}_{N,M}[C]$ is the noise-free estimate given by (6), and $u^*_{N,M}[C]$ is the PBDW estimate given by (3). Moreover, the variance of the stochastic error is bounded by

$$ \sqrt{E[\| u^{\text{nf}}_{N,M}[C] - u^*_{N,M}[C] \|^2]} \leq \sigma \left( 1 + \frac{2}{\beta^2_{N,M}} \right) \sqrt{\text{trace}(A^{-1})}, $$

(10)

where $A \equiv Q^\dagger UQ \in \mathbb{C}^{M \times M}$, $\beta_{N,M}$ is the inf-sup constant defined in (8), and $\sigma^2$ is the variance of the measurement noise.
Experimental settings

- Physical system:
  - Speaker diaphragm center: $x_1^{\text{dim}} = (0.89, 6.35)$ cm
  - $r_{\text{mic}}^{\text{dim}} = 0.064$ cm
  - Acrylic walls: $t_{\text{wall}}^{\text{dim}} = 1.5$ mm
  - $H^{\text{dim}} = 12.7$ cm
  - $W^{\text{dim}} = 17.8$ cm
  - 5.6 mm gap

- Computational domain:
  - $\Gamma_{\text{rad}}$
  - $\Gamma_{\text{spk}}$
  - $\Omega$
  - $\Omega^{\text{bk}}$
Figure 4. Behavior of the stability constant $\beta_{N,M}$ and the observation-conditioning metric $\text{trace}(A^{-1})$. 
FIGURE 6. Frequency response (amplitude and phase) at the assessment center (a) $\xi_j^c = (2.67, 2.67, 4.50)$ and (b) $\xi_j^c = (9.33, 2.67, 4.50)$. 

---

**Explanation:**

The figure illustrates the frequency response of two different scenarios, each with distinct parameters $\xi_j^c$.

(a) For the set $(2.67, 2.67, 4.50)$:
- The amplitude and phase responses are shown as a function of $\tilde{k}$.
- The plots compare the prior response $P_{N=7, M=12}^\text{prior}(j; \tilde{k})$ with the true response $P_{N=7, M=12}^\text{true}(j; \tilde{k})$.

(b) For the set $(9.33, 2.67, 4.50)$:
- Similar comparison as in (a) but with different parameter values.

The plots provide a visual comparison of the model predictions against the true responses, highlighting the accuracy and discrepancies at various frequencies.
We now devise a strategy to systematically incorporate the unmodeled physics identified by the update space $\mathcal{U}_M$ to augment the background space $\mathcal{Z}_N$ for subsequent data assimilation. The goal is to reduce the number of observations for future configurations. We consider the following algorithm:

1. Find the configuration that maximizes the relative error (indicator):

$$\tilde{k}^* = \arg \sup_{k \in [0.3, 0.7]} \frac{E_{\text{avg}}[C_k](u_{N=N_{\text{max}}, M=12})}{E_{\text{avg}}[C_k](u_{N=0, M=0})}.$$  

2. Compute the update state associated with the configuration $C_{\tilde{k}^*}$,

$$\eta^*_{N=N_{\text{max}}=8, M=M_{\text{max}}=48}[C_{\tilde{k}^*}].$$

3. Construct the “augmented” best-knowledge space

$$\mathcal{Z}_{N_{\text{max}}+1}^{\text{aug}} \equiv \text{span}\{\mathcal{Z}_{N_{\text{max}}}, \eta^*_{N=N_{\text{max}}=8, M=M_{\text{max}}=48}[C_{\tilde{k}^*}]\};$$

note that $\eta^*_{N_{\text{max}}, M_{\text{max}}}[C_{\tilde{k}^*}] \in \mathcal{Z}_{N_{\text{max}}} \cap \mathcal{U}_{M_{\text{max}}}$ and hence $\eta^*_{N_{\text{max}}, M_{\text{max}}}[C_{\tilde{k}^*}]$ is orthogonal to $\mathcal{Z}_{N_{\text{max}}}$. 


Figure 12. The relative error in the PBDW estimate for the original background space $\mathcal{Z}_{N=8}$ and the augmented background space $\mathcal{Z}_{N=9}^{\text{aug}}$ both using $M = 12$ observations.
Conclusion
• EIM and GEIM have been used now extensively in order to implement easily the reduced basis method
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- From its initial “magic point” statement to “empirical interpolation” we have been now able (thanks to the work initiated by deVore on greedy algorithms) to state it on more firm ground
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- The pure interpolation process can be enriched by more data as in the PBDW method with A.T. Patera, J. Penn and M. Yano.
The PBDW formulation is endowed with the following characteristics:

- **Weak formulation.**
- **Actionable a priori theory.** The weak formulation facilitates the construction of a priori error estimates
- **Background space** that best reflects our (prior) best knowledge of the phenomenon under consideration
- **Design of quasi-optimal set of observations from a library of experimentally realizable observations** in order to maximize the stability of the data assimilation.
- **Correction of unmodeled physics with uncertainty.**
- **Online computational cost is O(M).** We may realize real-time state estimation
- **Simple non-intrusive implementation and generality.** The mathematical model appears only in the offline stage.

- Recently a work of P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk has analyzed this approach in terms of optimal recovery and our algorithm is optimal in this frame. They have extended it to the Multi-Space Case.
Reduced Basis method for a bifurcation study of a thermal convection problem

Henar Herrero, Yvon Maday, Francisco Pla
A sketch of the domain and the physical situation is shown here. The domain is a rectangle of depth $d$ and width $L$. The domain contains a fluid that is heated from below, so that on the bottom plate a temperature $T_0$ is imposed and on the upper plate the temperature is

$$T_1 = T_0 - \Delta T = T_0 - \beta d$$

where $\beta$ is the vertical temperature gradient.
Formulation of the problem

The equations governing the system are the incompressible Navier-Stokes equations with the Boussinesq approximation coupled with a heat equation.
Bifurcation diagram
Stationary equations : Legendre collocation method

The Navier Stokes system with proper boundary conditions has a simple conductive solution $u^c = 0$, $\theta^c = (1-z)/2$, $P^c = R(z-z^2/2)/2$. The stationary problem with the change of variables $\theta' = \theta - \theta^c$, $P' = P - P^c$ and dropping the primes to simplify notation, is the following,

$$
\nabla \cdot u = 0, \quad \text{in } \Omega,
$$

$$
R \theta e_z - \nabla P + \Delta u = 0, \quad \text{in } \Omega,
$$

$$
u \cdot \nabla \theta - u_z = \Delta \theta, \quad \text{in } \Omega.
$$

with the boundary conditions,

$$
u = 0, \theta = 0 \text{ on } z = 0; \quad \theta = \partial_z u_x = u_z = 0 \text{ on } z = 1,$$

$$
\partial_x \theta = \partial_x u_z = u_x = 0, \quad \text{on } x = 0, \text{ and on } x = \Gamma.
$$

The “standard” numerical method used here to solve it provided for different values of the Rayleigh number $R$ is a Legendre spectral collocation method. The fields are expanded into Legendre polynomials, $U = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} L_i(x)L_j(z)$, where $L_i$ is the Legendre polynomial of degree $i$. 
Greedy procedure

We choose a value of the Rayleigh number that we name $R_1$, with its corresponding solution $\Phi(R_1)$, i.e. in this work it is the smallest value of $R$ in the interval we consider. We normalize this stationary solution according to the $L^2$ scalar product:

$$
\Psi_1 = \left( \psi_1^u = \frac{u_1}{\|u_1\|_{L^2}}, \psi_1^\theta = \frac{\theta_1}{\|\theta_1\|_{L^2}}, \psi_1^P = \frac{P_1}{\|P_1\|_{L^2}} \right),
$$

then we consider a first space $X_1 = \text{span}\{\psi_1^u\} \times \text{span}\{\psi_1^\theta\} \times \text{span}\{\psi_1^P\}$.

In this calculations we have considered expansions of order $n=35$ in the $x$-direction and $m=13$ in the $z$-direction.

We then choose $R_2$ where the Galerkin error when approximated on

$$
X_1 = \text{span}\{\psi_1^u\} \times \text{span}\{\psi_1^\theta\} \times \text{span}\{\psi_1^P\}
$$

and the corresponding stationary solution is $\Phi(R_2)$. We orthonormalize both functions by Gram-Schmidt procedure in order to obtain a new $\Psi_2$ and we consider the second space $X_2 = \text{span}\{\psi_1^u, \psi_2^u\} \times \text{span}\{\psi_1^\theta, \psi_2^\theta\} \times \text{span}\{\psi_1^P, \psi_2^P\}$.

An so on, until we reach a value $j = N < \text{card}(\Xi_{\text{trial}})$ for which the stopping criterium $\epsilon_i^{(N)} \leq 10^{-7}$, $i = 1, 2$ is satisfied.
Greedy procedure

Therefore, we obtain the reduced basis \( \{\Psi_1, \Psi_2, \ldots, \Psi_N\} \) and a corresponding discrete space \( X_N \equiv X_N^u \times X_N^\theta \times X_N^P \). For each branch we have constructed the reduced basis with solutions on that branch. We have constructed a reduced basis for each stable branch, i.e. for \( \Phi_1 \) and \( \gamma \Phi_1 \). The branch for \( \Phi_2 \) needs two reduced basis, one for the unstable part and another for the stable part. The same situation for \( \Phi_3 \). The totally unstable branches need more care, and the interval has been divided in three parts, so that a reduced basis has been calculated in each part, i.e. [1, 539; 1, 600], [1, 600; 2, 000] and [2, 000; 3, 000]. Table 1 shows the number of snapshots used to calculate the reduced basis and the number of elements of the reduced basis in each branch of solutions.

Table 1: Number of snapshots in the trial set for the different reduced basis (RB) in each branch of solutions.

<table>
<thead>
<tr>
<th></th>
<th>( \Phi_1 )</th>
<th>( \Phi_2 )</th>
<th>( \Phi_2 ) (two sets)</th>
<th>( \Phi_4 )</th>
<th>( \Phi_4 ) (three sets)</th>
</tr>
</thead>
<tbody>
<tr>
<td># ( \Xi_{trial} )</td>
<td>23</td>
<td>22</td>
<td>29, 19</td>
<td>19</td>
<td>22, 26, 51</td>
</tr>
<tr>
<td># RB</td>
<td>9</td>
<td>8</td>
<td>6, 6</td>
<td>10</td>
<td>6, 7, 7</td>
</tr>
</tbody>
</table>
Table 2: $\epsilon_1^{(j)}, \epsilon_2^{(j)}, j = 1, ..., N$ and the respective Rayleigh number $R$ in which the maximum takes place for different dimensions $j$ of the reduced basis space. $R$ is in the interval [1, 102; 3, 000] on the stable branch of solutions $\Phi_1$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\epsilon_1^{(j)}$</th>
<th>$\epsilon_2^{(j)}$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.809</td>
<td>0.236</td>
<td>1,102</td>
</tr>
<tr>
<td>2</td>
<td>0.085</td>
<td>0.009</td>
<td>3,000</td>
</tr>
<tr>
<td>3</td>
<td>0.008</td>
<td>0.001</td>
<td>1,500</td>
</tr>
<tr>
<td>4</td>
<td>0.003</td>
<td>4.7 \cdot 10^{-4}</td>
<td>2,200</td>
</tr>
<tr>
<td>5</td>
<td>3.3 \cdot 10^{-4}</td>
<td>6.7 \cdot 10^{-5}</td>
<td>1,110</td>
</tr>
<tr>
<td>6</td>
<td>1.1 \cdot 10^{-4}</td>
<td>1.7 \cdot 10^{-5}</td>
<td>2,700</td>
</tr>
<tr>
<td>7</td>
<td>1.7 \cdot 10^{-5}</td>
<td>2.7 \cdot 10^{-6}</td>
<td>1,900</td>
</tr>
<tr>
<td>8</td>
<td>2.8 \cdot 10^{-6}</td>
<td>4.1 \cdot 10^{-7}</td>
<td>1,800</td>
</tr>
<tr>
<td>9</td>
<td>1.8 \cdot 10^{-7}</td>
<td>1.7 \cdot 10^{-8}</td>
<td>1,300</td>
</tr>
</tbody>
</table>
Greedy procedure

Table 3: $\epsilon_1^{(j)}, \epsilon_2^{(j)}, j = 1, \ldots, N$ and the respective Rayleigh number $R$ in which the maximum takes place for different dimensions $j$ of the reduced basis space. $R$ is in the interval $[1, 253; 3, 000]$ on the branch of solutions $\Phi_2$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\epsilon_1^{(j)}$</th>
<th>$\epsilon_2^{(j)}$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.748</td>
<td>0.588</td>
<td>1,253</td>
</tr>
<tr>
<td>2</td>
<td>0.056</td>
<td>0.015</td>
<td>3,000</td>
</tr>
<tr>
<td>3</td>
<td>0.007</td>
<td>0.001</td>
<td>1,600</td>
</tr>
<tr>
<td>4</td>
<td>0.002</td>
<td>2.5 \cdot 10^{-4}</td>
<td>2,200</td>
</tr>
<tr>
<td>5</td>
<td>1.1 \cdot 10^{-4}</td>
<td>3.5 \cdot 10^{-5}</td>
<td>1,300</td>
</tr>
<tr>
<td>6</td>
<td>2.6 \cdot 10^{-5}</td>
<td>7.4 \cdot 10^{-6}</td>
<td>2,700</td>
</tr>
<tr>
<td>7</td>
<td>4.6 \cdot 10^{-6}</td>
<td>7.5 \cdot 10^{-7}</td>
<td>1,400</td>
</tr>
<tr>
<td>8</td>
<td>7.0 \cdot 10^{-7}</td>
<td>9.6 \cdot 10^{-8}</td>
<td>1,260</td>
</tr>
</tbody>
</table>
Galerkin procedure for each branch

Figure 6: Norm of the difference between the stationary solution obtained with Legendre collocation and with a post-processed reduced basis method based on Legendre collocation for the stable branch $\Phi_1$ in the interval of $R \in [1,101; 3,000]$. 

Galerkin procedure for each branch

Figure 9: Norm of the difference between the stationary solution obtained with Legendre collocation and with a post-processed reduced basis method based on Legendre collocation for the first part of the unstable branch of $\Phi_4$ in the interval of $R [1,539; 1,600]$. 
Galerkin procedure for each branch

The reduced basis method is supported by standard discretizations. The off-line work for the calculation of the solutions to construct the reduced basis needs these standard methods. But, once the work of the standard method is done, the use of the reduced basis has several advantages.

The size of the matrices after the discretization is very small. For a single value of the Rayleigh number $R$ the size of the matrices that appear after the discretizations are 2,016 in Legendre collocation with expansions of order $13 \times 35$, whereas in the case of the reduced basis with 8 elements the size of matrices are 16. A factor of 126 in the size of the matrices for each value of $R$.

The behavior of the Newton method for the nonlinearity is improved with respect to standard methods. In the Legendre collocation method, for instance, we obtain the first solution in the branch in the interval $[1, 101; 3, 000]$ near $R = 1, 101$. To obtain the solution at $R = 3, 000$ we need to calculate the solution at $R = 1, 102$, take this solution as initial guess for $R = 1, 110$ and calculate the solutions increasing the value of $R$ in steps of 10 till $R = 3, 000$. Sometimes the steps of increase on $R$ can be larger. So, it is not possible to jump from $R = 1, 101$ till $R = 3, 000$ with Legendre collocation.

In the reduced basis this is not the case, the solution can be directly calculated for any value of $R$. The reason for this behavior must be that nothing drive the solutions to be attracted by a different branch since there is not unexpected elements in the basis set.

This is reflected in the computational cost in time, it is 122 s for Legendre collocation and it is 6 s for reduced basis. Therefore the reduction is of a factor of 20 in time.
A new concept of reduced basis approximation for convection dominated problems

Nicolas Cagniart, Yvon Maday, Benjamin Stamm
And if the Kolmogorov n-width is not small?

Try to better look at the set of solutions
A case where the dimension is not small

1D viscous Burger equation

\[ u_t + \nu uu_x - \epsilon u_{xx} = 0 \]

Snapshots of the solution to the unsteady viscous burger equation with

\[ u_0 = \sin(x), \nu = 4, \epsilon = 0.04 \]
A case where the dimension is not small

1D viscous Burger equation

\[ u_t + \nu u u_x - \epsilon u_{xx} = 0 \]

Snapshots of the solution to the unsteady viscous burger equation with

\[ u_0 = \lambda + \sin(x), \nu = 4, \epsilon = 0.04 \]
Nevertheless ... all solutions look alike

what we have done above is to “center” the solutions

these are the $u(x - \gamma_n, t_n)$
and the dimension diminishes largely!!

Eigenvalues of the POD decomposition of the original set of snapshots (in red) and of the centered set of snapshots (in green)
and the dimension diminishes largely!!

Reconstruction of a snapshot (blue) using 3 POD modes. Left figure is in the centered case. Right figure is the uncentered case.
The discrete scheme we want to mimic

\[ u^{n+1}(\cdot, \mu) = u^n(\cdot, \mu) - dt\nu u^n(\cdot, \mu)u^n_x(\cdot, \mu) + dt\epsilon u^n_{xx}(\cdot, \mu) \]

leads to

At each time step, we are looking for a set of reduced coordinates \( \{\alpha_i^n\} \) and a translation parameter \( \gamma^n \) such that our ”true” solution \( u^N(\cdot, \mu, t^n) \) is well approximated by:

\[ u_N(\cdot, \mu, t^n) = \sum_i \alpha_i^n(\mu)\Phi_i(\cdot - \gamma^n) \quad (1) \]

We replace the search of the real \( \gamma \) and \( \alpha \)s as in the usual galerkin method. That is, we are trying to minimize the residual.

\[ \min_{\gamma^{n+1}, \Omega} \min_{\alpha^{n+1}} \left\| \sum_i \alpha_i^{n+1}\Phi_i(\cdot - \gamma^{n+1}) - u^n - dt\nu u^n u^n_x + dt\epsilon u^n_{xx} \right\|_2 \quad (2) \]

Complexity .. how to compute and solve ??
We have assumed periodic boundary conditions (for the sake of simplicity) thus we need to compute online the following terms:

\[
\begin{align*}
\forall \Delta \gamma, \forall i, j, \quad & \int_{\Omega} \Phi_i (\cdot - \Delta \gamma) \Phi_j (\cdot) \\
\forall \Delta \gamma, \forall i, j, p \quad & \int_{\Omega} \Phi_i (\cdot - \Delta \gamma) \Phi_j (\cdot) (\Phi_p) x (\cdot) \\
\forall \Delta \gamma, \forall i, j, \quad & \int_{\Omega} (\Phi_i) x (\cdot - \Delta \gamma) (\Phi_j) x (\cdot)
\end{align*}
\]

for unknown value (to be optimized) of $\Delta \gamma$

For a sufficiently small time step, we expect $\Delta \gamma$ to be of order $\delta t * c$ where $c$ is some local characteristic velocity. We have chosen the following method:

- precompute the scalar products for a predefined small set of discrete values of $\Delta \gamma$ in $[-\delta t * c_{max}, \delta t * c_{max}]$, where $c_{max}$ is the maximum expected shock speed during the simulation.

- using the regularity of the scalar product, we use spline interpolation to get and approximate value for any $\Delta \gamma$ in $[-\delta t * c_{max}, \delta t * c_{max}]$
The RB discrete scheme thus iterates between the evaluation of the proper best translation $\gamma^{n+1}$ and the proper definition of the coefficient $\alpha_{i}^{n+1}$

$$\sum_{i} \alpha_{i}^{n+1} \Phi_{i}(\cdot - \gamma^{n+1})$$

Mean reconstruction error w.r.t number of POD basis used
extensions .. what needs to be done

- non periodic (superposition of basic space and convection space)
- higher dimensions (POD representation of the “translations”) see ¹)
- better fitting (add the derivatives of the POD functions)
- better fitting (replace least square with $L^1$, see ²)


²) Technical paper: Roxana Crisovan, Rémi Abgrall, David Amsallem
Robust Model Reduction by L1-norm Minimization and Approximation via Dictionaries:
Application to Linear and Nonlinear Hyperbolic Problems