REDUCED BASIS METHOD

approximation of PDE's, interpolation, a posteriori estimates

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This talk deals with linear constructive approximation methods specifically tailored to approximate the functions in F (or in a set close to F see PBDW).

For this, we consider approximations on "appropriate" n-dimensional spaces X_n of relatively small dimension under the hypothesis that the Kolmogorov n-width of F is small.

The Kolmogorov n-width of F in $\mathcal{X}$, is defined as

$$d_n(F, \mathcal{X}) = \inf_{\substack{X_n \subset \mathcal{X} \\ \dim(X_n) \leq n}} \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_{\mathcal{X}},$$

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In order to satisfy the constraint on the computing time, we consider continuous linear approximations $\mathcal{J}_n : \mathcal{X} \to X_n$ on "appropriate" n-dimensional spaces X_n of relatively small dimension. A first bound on the approximation error is then

$$\sup_{u \in F} \|u - \mathcal{J}_n[u]\|_{\mathcal{X}} \leq (1 + \Lambda_n) \sup_{u \in F} \inf_{v \in X_n} \|u - v\|_{\mathcal{X}}, \tag{1}$$

where

$$\Lambda_n = \sup_{\varphi \in \mathcal{X}} \frac{\|\mathcal{J}_n[\varphi]\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{X}}} \tag{2}$$

is the norm of the operator $\mathcal{J}_n : \mathcal{X} \to X_n$ and is known as the Lebesgue constant.

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What is the proper definition of X_n

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EIM/GEIM

In 2004 with M. Barrault, N. C. Nguyen, and A. T. Patera, we proposed an generic approach : the 'Empirical Interpolation' Method: Application to Efficient Reduced-Basis Discretization Of Partial Differential Equations, that has proven successful

This approach allows to determine an "empirical" optimal set of interpolation points and/or set of interpolating functions.

In 2013, with Olga Mula, we have generalized it (GEIM) to include more general output from the functions we want to interpolate : not only pointwize values but also some moments.

The first generating function is $\varphi_1 = \arg\max_{\varphi \in F} \|\varphi(\,\cdot\,)\|_{L^\infty(\Omega)}$, the associated interpolation point satisfies

$$x_1 = \arg\max_{x \in \overline{\Omega}} |\varphi_1(x)|,$$

we then set $q_1 = \varphi_1(\cdot)/\varphi_1(x_1)$ and $B^1_{11} = 1$.

Then the construction proceeds by induction : assume the nested sets of interpolation points $\Xi_{M-1} = \{x_1, \ldots, x_{M-1}\}, M \leq M_{\max}$, and the associated nested sets of basis functions $\{q_1, \ldots, q_{M-1}\}$ are given[1]. We first solve the interpolation problem : Find

$$\mathcal{I}_{M-1}[\varphi(\cdot)] = \sum_{j=1}^{M-1} \alpha_{M-1,j}[\varphi] q_j \ ,$$

such that

$$\mathcal{I}_{M-1}[\varphi(\cdot)](x_i) = \varphi(x_i), \quad i = 1, \ldots, M-1 \ ,$$

that allows to define the $\alpha_{M-1,j}[\varphi], 1 \leq j \leq M-1$, as it can be proven indeed that the $(M-1) \times (M-1)$ matrix of running entry $q_j(x_i)$ is invertible, actually it is lower triangular with unity diagonal.

1) where $M_{\max} \leq \mathcal{M}$ is some given upper bound fixed *a priori*

We then set

$$\forall \varphi \in F, \quad \varepsilon_{M-1}(\varphi) = \|\varphi - \mathcal{I}_{M-1}[\varphi]\|_{L^\infty(\Omega)} \ ,$$

and define

$$\varphi_M = \arg\max_{\varphi \in F} \varepsilon_{M-1}(\varphi) \ ,$$

and

$$x_M = \arg\max_{x \in \overline{\Omega}} |\varphi_M(x) - \mathcal{J}_{M-1}[\varphi_M](x)| \ ,$$

we finally set $r_M(x) = \varphi_M(x) - \mathcal{J}_{M-1}[\varphi_M(x)]$, $q_M = r_M / r_M(x_M)$ and

$$B^M_{ij} = q_j(x_i), 1 \leq i, j \leq M$$

The Lagrangian functions — that can be used to build the interpolation operator $\mathcal{I}_M$ in

$$X_M = \text{span}\ \{\varphi_i, 1 \leq i \leq M\} = \text{span}\ \{q_i, 1 \leq i \leq M\}$$

over the set of points $\Xi_M = \{x_i, 1 \leq i \leq M\}$ — verify for any given M,

$$\mathcal{I}_M[u(\,\cdot\,)] = \sum_{i=1}^{M} u(x_i) h_i^M(\,\cdot\,)$$

where

$$h_i^M(\,\cdot\,) = \sum_{j=1}^{M} q_j(\,\cdot\,)[B^M]^{-1}_{ji}$$

(note indeed that $h_i^M(x_j) = \delta_{ij}$).

For GEIM, we assume now that we do not have access to the values of $\varphi \in F$ at points in Ω easily, but, on the contrary, that we have a dictionary of linear forms $\sigma \in \Sigma$ — assumed to be continuous in some sense, e.g. in $L^2(\Omega)$ with norm 1 — the application of which over each $\varphi \in F$ is easy. Our extension consists in defining $\tilde{\varphi}_1$, $\tilde{\varphi}_2$,..., $\tilde{\varphi}_M$ and a family of associated linear forms σ_1, σ_2,..., σ_M such that the following generalized interpolation process (our GEIM) is well defined :

$$\mathcal{J}_M[\varphi] = \sum_{j=1}^{M} \beta_j \tilde{\varphi}_j, \textit{ such that} \quad \forall i = 1, \ldots, M, \quad \sigma_i(\mathcal{J}_M[\varphi]) = \sigma_i(\varphi) \tag{1}$$

Note that the GEIM reduces to the EIM when the dictionary is composed of dirac masses, defined in the dual space of $\mathcal{C}^0(\Omega)$.

Note that — obviously — everything has to be implemented on a computer and thus discretized !!

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Note also that — for some reasons — you may want to use your proper set of basis/interpolating function in your preferredspace X_N that may come from intuition, previous knowledge, or POD/SVD.

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Remark 2. *If for some reasons, a set of functions $u_i \in \mathcal{U}$, $i \in \mathbf{N}$ were to be given, all linearly independent, then the procedure of finding the interpolation points through the process $\forall i, 1 \leq i \leq M-1, \quad u(x_i) = \sum_{j=1}^{M-1} \alpha_{i,j}[u] u_j(x_i)$ and set $x_M = \arg\max_{x \in \overline{\Omega}} |u_M(x) - \sum_{j=1}^{M-1} \alpha_{i,j}[u_M] u_j(x)|$ is also well defined and leads to a set of interpolation points that have similar properties as above. The rational for the greedy approach is that it allows us to get a better sense of the interpolation properties since $\forall u$,*

$$\|u(\cdot) - \mathcal{I}_M[u(.)]\|_{L^\infty(\Omega)} \leq \|u_{M+1}(\cdot) - \mathcal{I}_M[u_{M+1}(\cdot)]\|_{L^\infty(\Omega)} = \varepsilon_M(x_{M+1}) \tag{13}$$

and this last quantity is one of the outputs of the construction process.

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Journal: Communications on Pure and Applied Analysis -

A general multipurpose interpolation procedure: the magic points
Yvon Maday - Ngoc Cuong Nguyen - Anthony T. Patera - George S. H. Pau

Pages: 383 - 404, **Volume 8**, **Issue 1**, January 2009 doi:**10.3934/cpaa.2009.8.383**

The important thing is to measure to which extent the EIM/GEIM misses the optimality of the best approximation in the best optimal discrete space suggested by the definition of the Kolmogorov n-width.

Formula

$$\sup_{u\in F}\|u-\mathcal{J}_n[u]\|_{\mathcal{X}} \leq (1+\Lambda_n)\sup_{u\in F}\inf_{v\in X_n}\|u-v\|_{\mathcal{X}},$$

suggests that Λ_n plays an important role in the result and it is therefore important to discuss its behavior as n increases. First of all, Λ_n depends both on the choices of the interpolating functions and interpolation points.

We have proven (YM-Mula-Patera-Yano) that $\Lambda_n = 1/\beta_n$, where

$$\beta_n = \inf_{\varphi\in X_n}\sup_{\sigma\in \mathrm{Span}\{\sigma_0,\ldots,\sigma_{n-1}\}}\frac{\langle\varphi,\sigma\rangle_{\mathcal{X},\mathcal{X}'}}{\|\varphi\|_{\mathcal{X}}\|\sigma\|_{\mathcal{X}'}}.$$

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GEIM interpreted as an oblic projection …

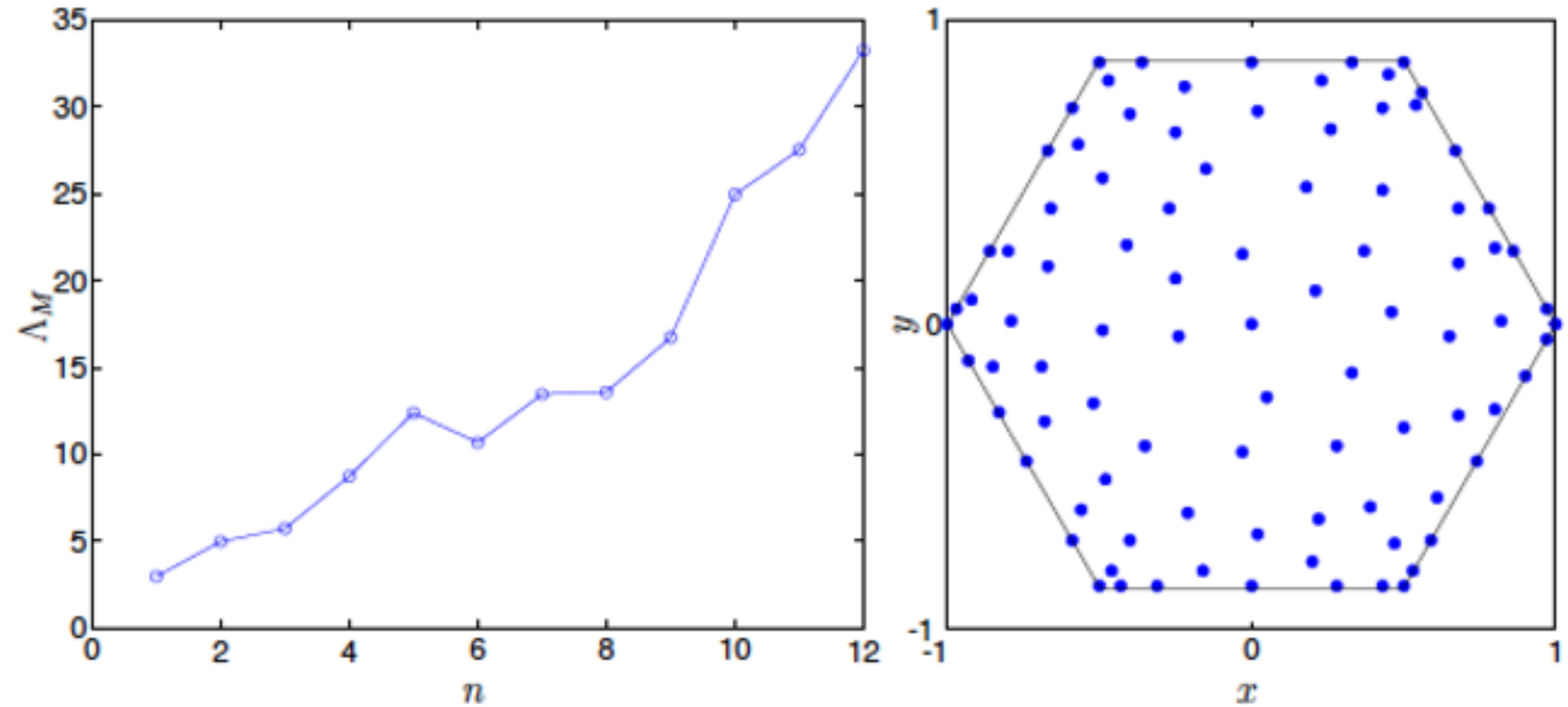


Figure 3: (a) Variation of Lebesgue constant, Λ_M with n where $M = \frac{1}{2}(n+1)(n+2)$, and (b) distribution of magic points, for Ω_{hex}.

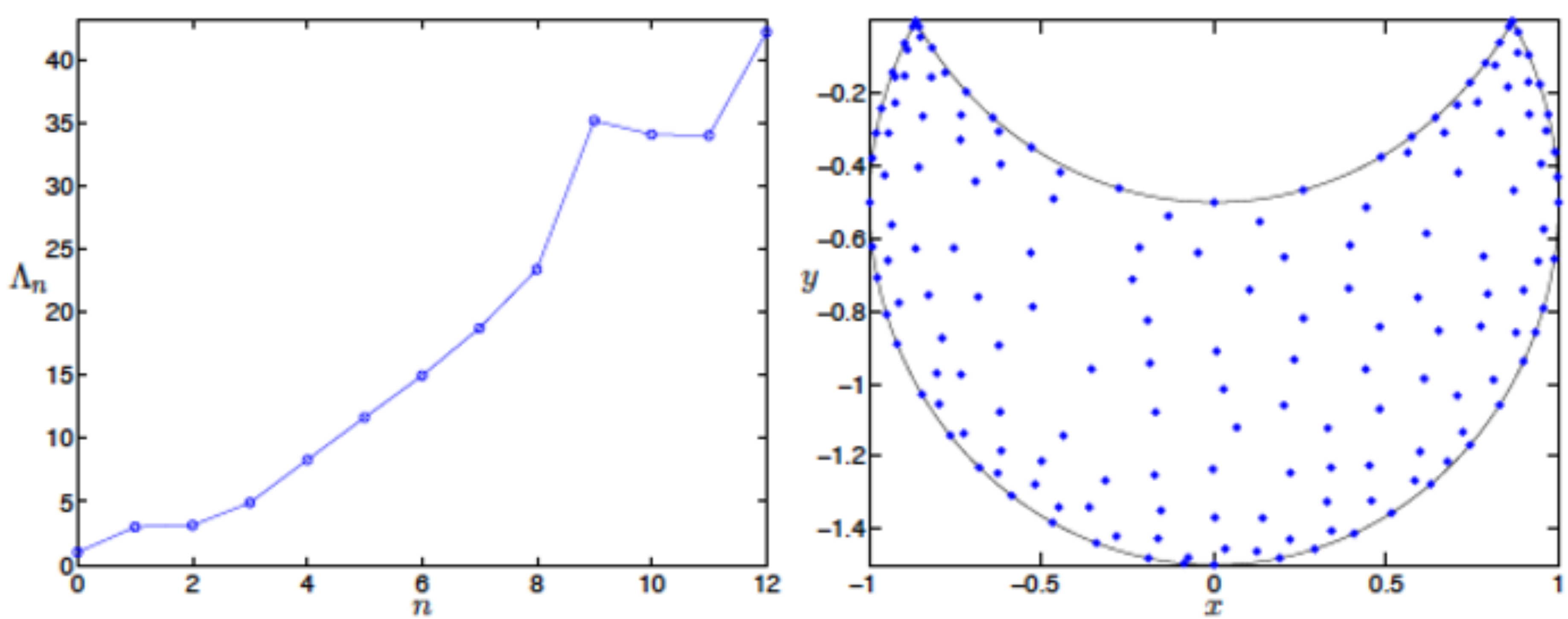
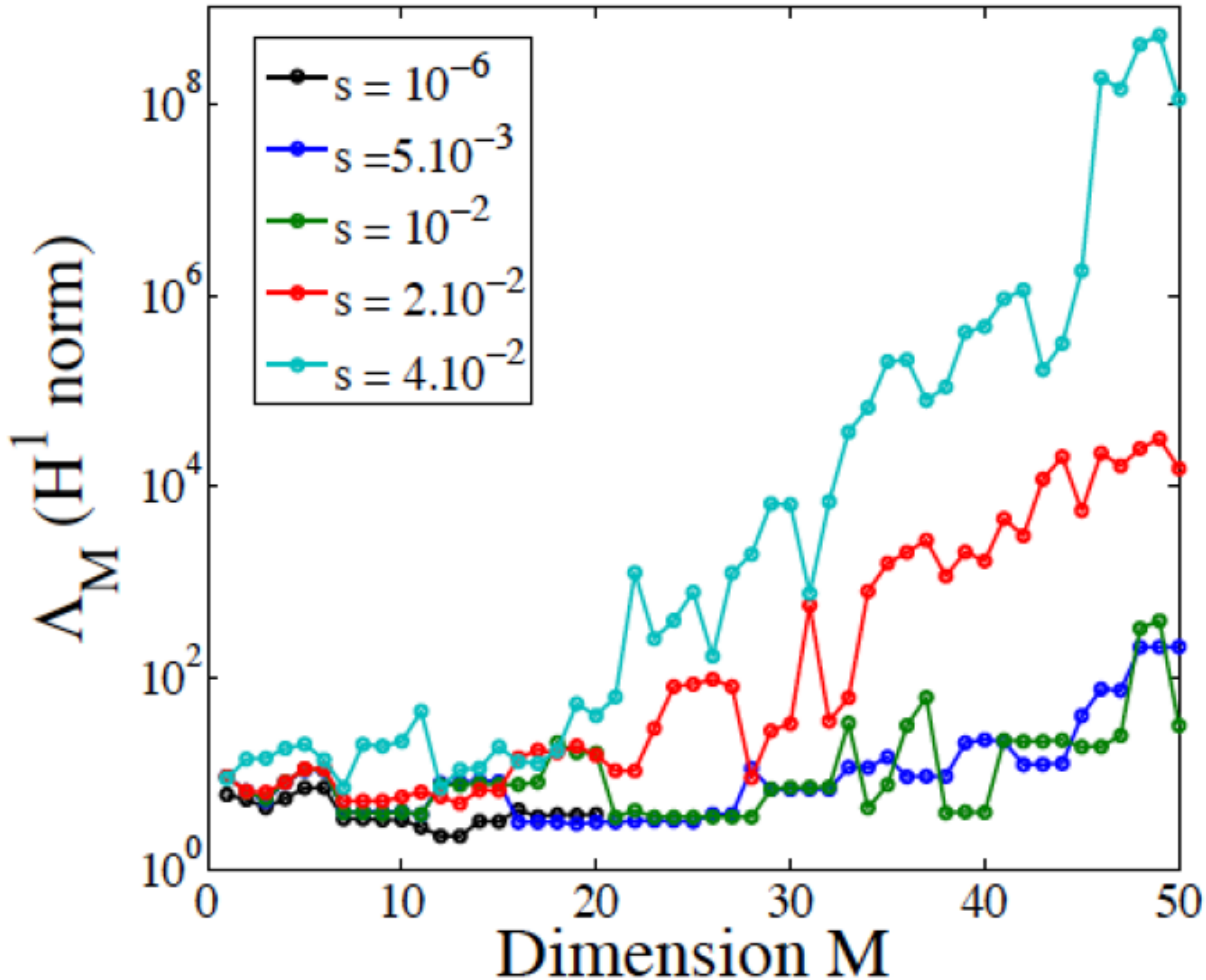
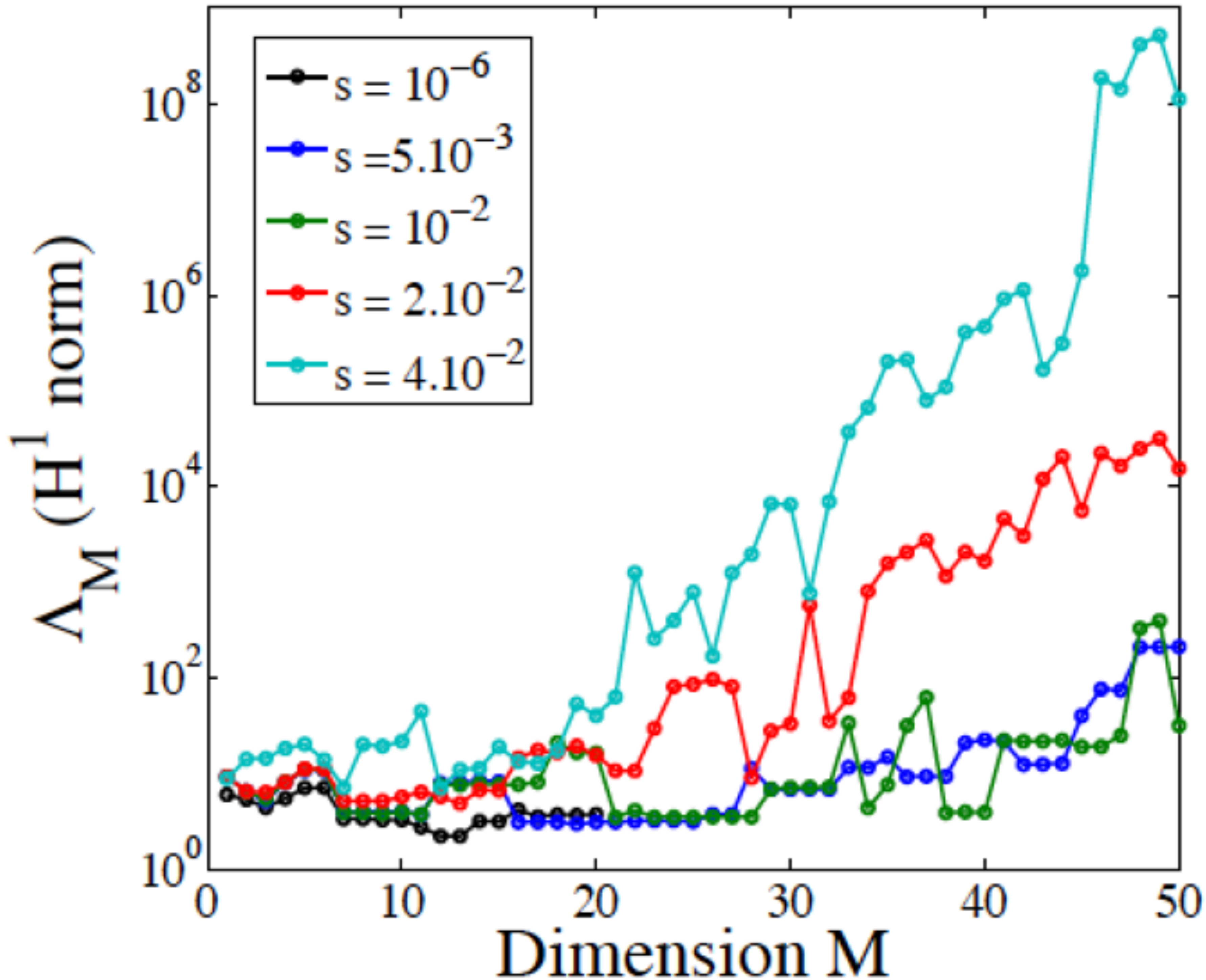


Figure 4: Results for a "lunar croissant" domain Ω_{cro}: (a) variation of the Lebesgue constant Λ_n with n, and (b) distribution of magic points for $n = 12$.



s = 10^{-6}
s =5.10^{-3}
s = 10^{-2}
s = 2.10^{-2}
s = 4.10^{-2}
10^8
10^6
10^4
10^2
10^0
0
10
20
30
40
50
Λ_M (H^1 norm)
Dimension M



We are now working on the justification of this nice behavior.

The next point is to justify the choice of the interpolating functions. It can be proven that

Lemma

For any $n \geq 1$, the nth interpolating function φ_n verifies

$$\|\varphi_n - P_n(\varphi_n)\|_{\mathcal{X}} \geq \frac{\eta}{1+\Lambda_n} \max_{f \in F} \|f - P_n(f)\|_{\mathcal{X}}.$$

Which allows to use the frame "weak greedy" of the papers by

- P. Binev, A. Cohen, W. Dahmen, R.A. DeVore, G. Petrova, and P. Wojtaszczyk,
- and R. A. DeVore, G. Petrova, and P. Wojtaszczyk,

to analyse the convergence properties of our algorithm

Work done with O. Mula and G. Turinici (SINUM (2016)

In a nutshell, in the case where we have a Hilbert framework, our result states that

Theorem
If $(\Lambda_n)_{n=1}^{\infty}$ is a monotonically increasing sequence then

i) if $d_n \leq C_0 n^{-\alpha}$ for any $n \geq 1$, then $\tau_n \leq C_0 \tilde{\beta}_n n^{-\alpha}$, with

$$\tilde{\beta}_n := 2^{3\alpha+1}\Lambda_n^2, \quad \text{if } n \geq 2.$$

ii) if $d_n \leq C_0 e^{-c_1 n^{\alpha}}$ for $n \geq 1$ and $C_0 \geq 1$, then $\tau_n \leq C_0 \tilde{\beta}_n e^{-c_2 n^{-\alpha}}$, with

$$\tilde{\beta}_n := \sqrt{2}\Lambda_n, \quad \text{if } n \geq 2.$$

Work done with O. Mula and G. Turinici (SINUM (2016)

Another application : fluid flow

$$\Omega = [0;1] \times [0;1] \subset \mathbb{R}^2.$$

$$\begin{cases} \text{Find the solution } (\boldsymbol{u}_\mu, p_\mu) \in \left(H^1(\Omega)\right)^2 \times L_0^2(\Omega) \text{ of :} \\ -\Delta \boldsymbol{u}_\mu + \mathbf{grad}(p_\mu) = \mathbf{f}_\mu, \quad \text{a.e. in } \Omega \\ \operatorname{div}(\mathbf{u}_\mu) = 0, \quad \text{a.e. in } \Omega \\ \mathbf{u}_\mu = \begin{pmatrix} x(1-x) \\ 0 \end{pmatrix}, \quad \text{a.e. on } \Gamma_1 \\ \mathbf{u}_\mu = \mathbf{0}, \quad \text{a.e. on } \partial\Omega \setminus \Gamma_1 \end{cases}$$

$$\mathbf{f}_\mu = \begin{pmatrix} 100 \sin(\mu_1 \Pi y) \\ -100 \sin\left(\mu_2 \Pi \frac{1-x}{2}\right) \end{pmatrix}$$

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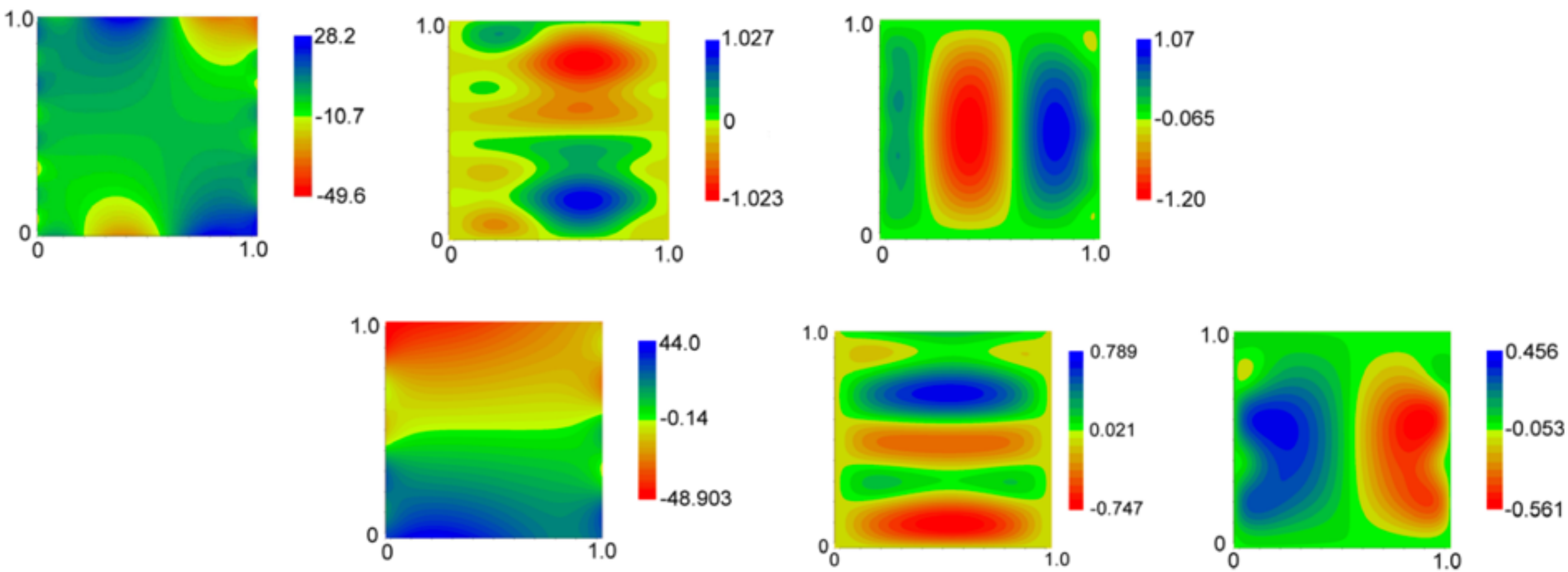
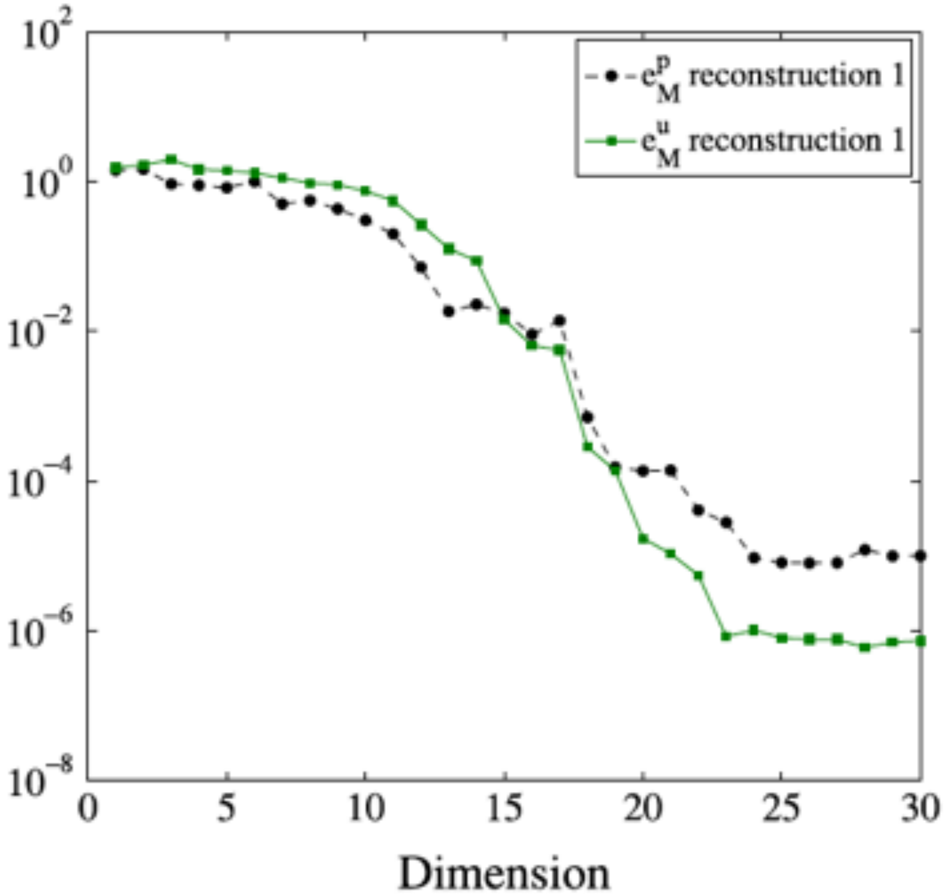
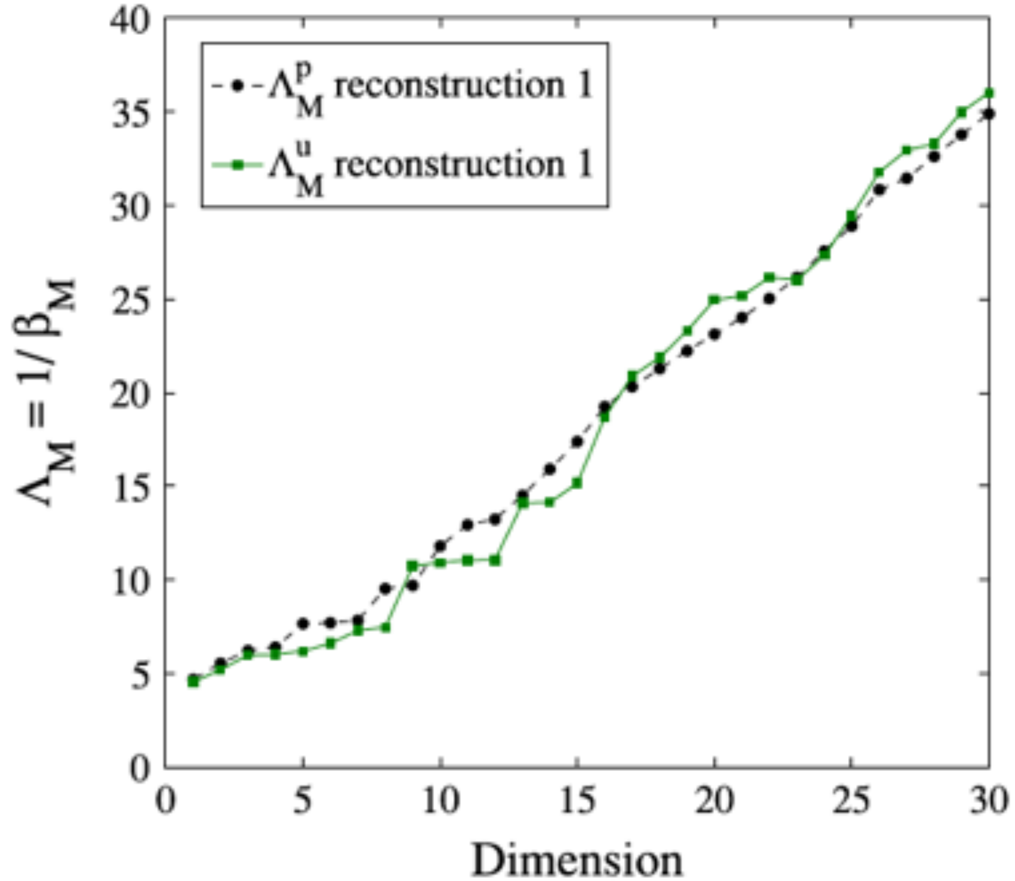
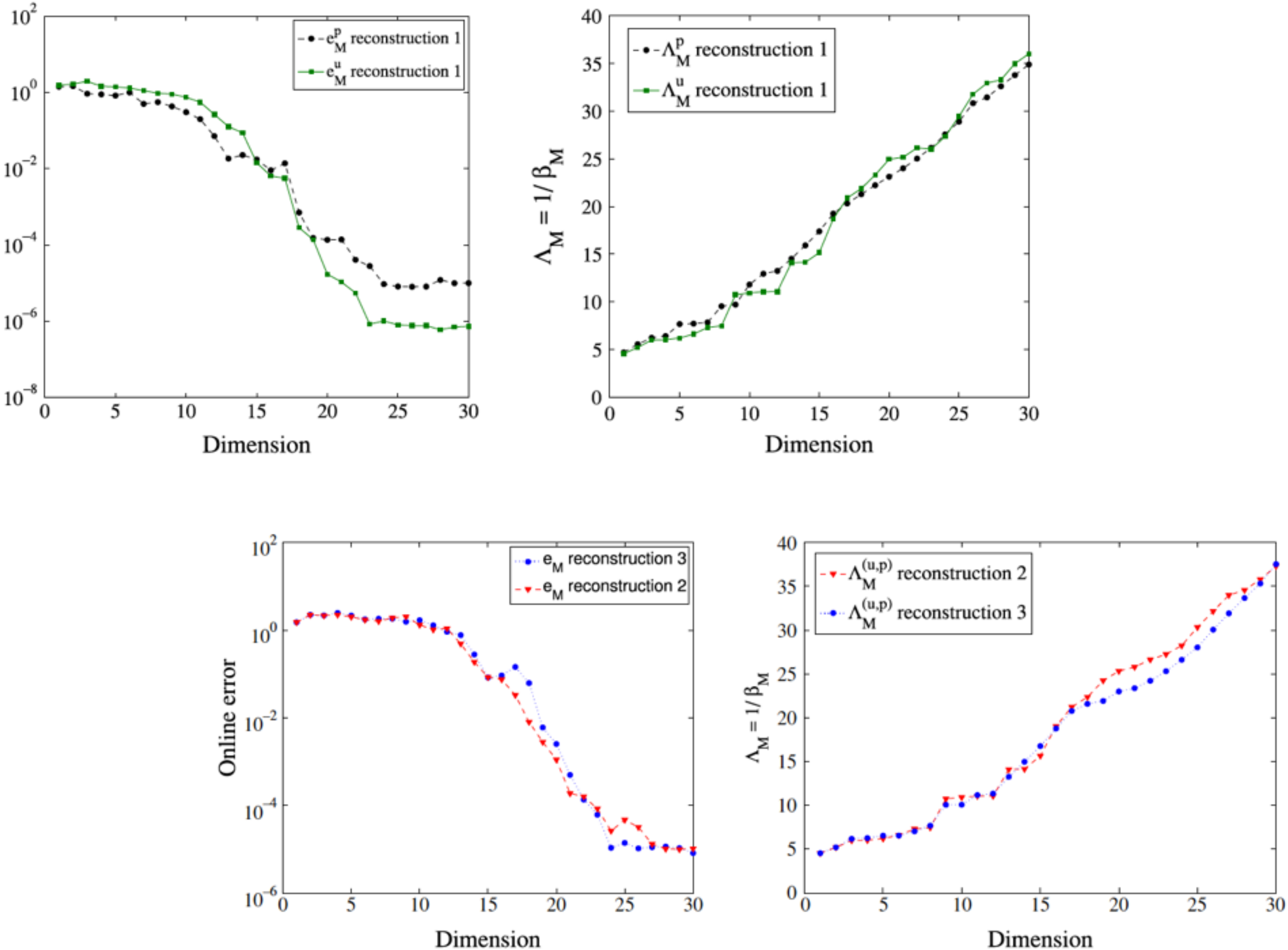




e^p_M reconstruction 1
e^u_M reconstruction 1
Dimension



Λ^p_M reconstruction 1
Λ^u_M reconstruction 1
$\Lambda_M = 1/\beta_M$
Dimension



e_M^p reconstruction 1
e_M^u reconstruction 1
Dimension
Λ_M^p reconstruction 1
Λ_M^u reconstruction 1
$\Lambda_M = 1/\beta_M$
Dimension
e_M reconstruction 3
e_M reconstruction 2
Online error
Dimension
$\Lambda_M^{(u,p)}$ reconstruction 2
$\Lambda_M^{(u,p)}$ reconstruction 3
$\Lambda_M = 1/\beta_M$
Dimension

Application of EIM for Reduced Basis Methods for non affine and nonlinear PDE:

see

Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations MA Grepl, Y Maday, NC Nguyen, AT Patera ESAIM (2007)

or the two recent books

Certified Reduced Basis Methods for Parametrized Partial Differential Equations Authors: Hesthaven, Jan S, Rozza, Gianluigi, Stamm, Benjamin

Reduced Basis Methods for Partial Differential Equations An Introduction Authors: Quarteroni, Alfio, Manzoni, Andrea, Negri, Federico

More about (G)EIM

In a recent paper, with J.P. Argaud, B. Bouriquet, H. Gong and O. Mula, we have introduced the GEIM to monitor nuclear reactor.

The challenge here is that we do not have access inside the core of the reactor and thus have only the possibility to place the captors inside the surrounding region.

The two group diffusion equation in matrix notation reads

$$\boldsymbol{A}(\boldsymbol{\mu})\boldsymbol{\varphi} = \frac{1}{k_{eff}}\boldsymbol{F}(\boldsymbol{\mu})\boldsymbol{\varphi}$$

Where $\boldsymbol{\mu}$ is the parameters set, e.g. D, Σ, $\nu\Sigma_f$. $\boldsymbol{A}$ and $\boldsymbol{F}$ are 2×2 matrix and $\boldsymbol{\varphi}$ is a 2-element column vector:

$$\boldsymbol{A}(\boldsymbol{\mu}) = \begin{pmatrix} -\nabla\cdot D^1\nabla + (\Sigma_a^1 + \Sigma_s^{1\to 2}) & 0 \\ -\Sigma_s^{1\to 2} & -\nabla\cdot D^2\nabla + \Sigma_a^2 \end{pmatrix}$$

$$\boldsymbol{F}(\boldsymbol{\mu}) = \begin{pmatrix} \chi_1\nu\Sigma_f^1 & \chi_1\nu\Sigma_f^2 \\ \chi_2\nu\Sigma_f^1 & \chi_2\nu\Sigma_f^2 \end{pmatrix}$$

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

Where $D^i,\ i = 1,2$ is called the *diffusion coefficient* of each group; $\Sigma_a^i,\ i = 1,2$ is the absorption cross section of each group; $\varphi_i,\ i = 1,2$ is the neutron flux of each group; $\Sigma_s^{1\to 2}$ is called the removal cross section from group 1 to group 2; $\nu\Sigma_f^1,\ i = 1,2$ is the fission source term of each group; $\chi_i,\ i = 1,2$ is called the fission spectrum of each group; finally k_{eff} is the effective multiplication factor, also the *eigenvalue* of equation.

In 1D, this looks like

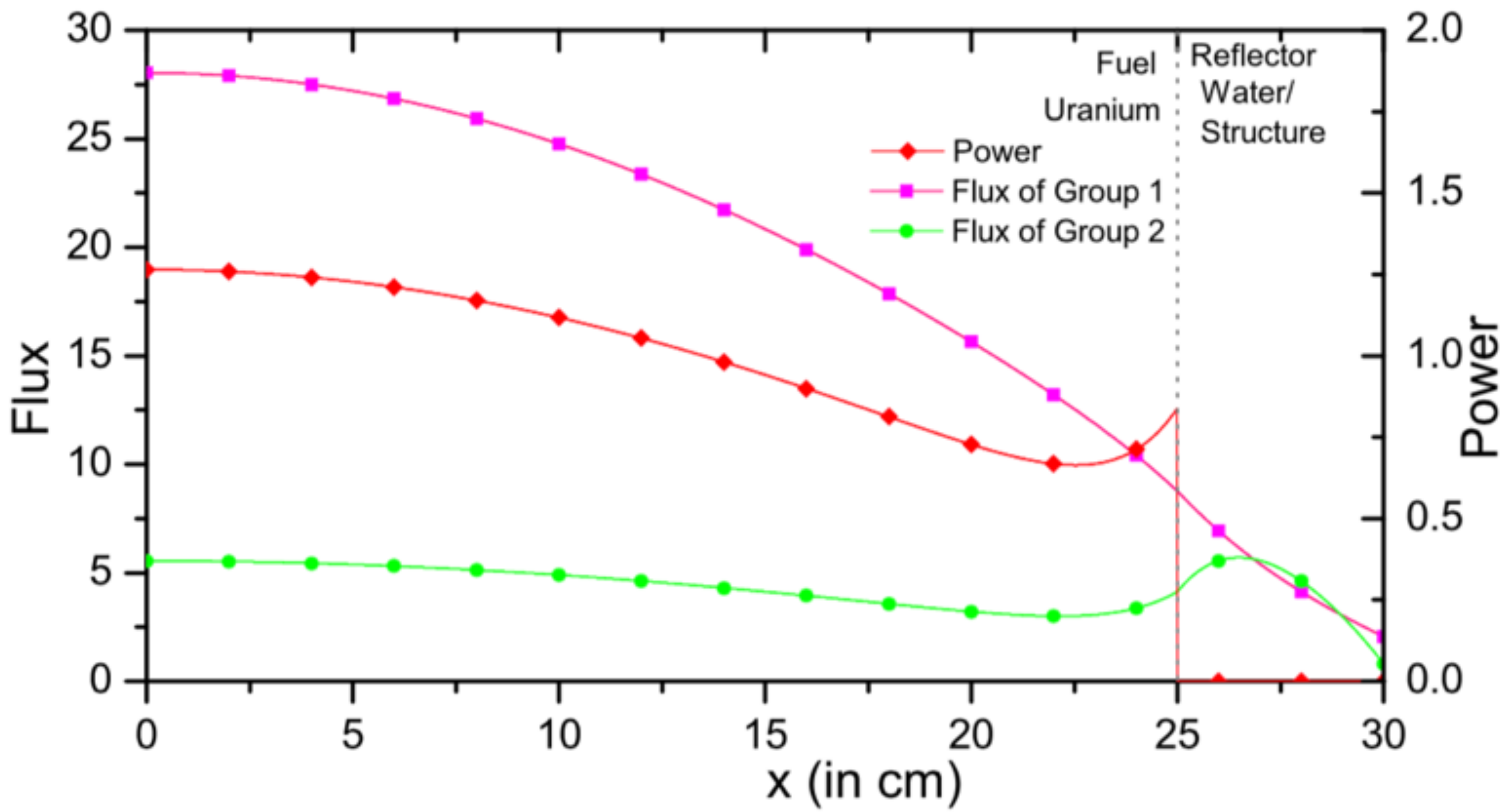
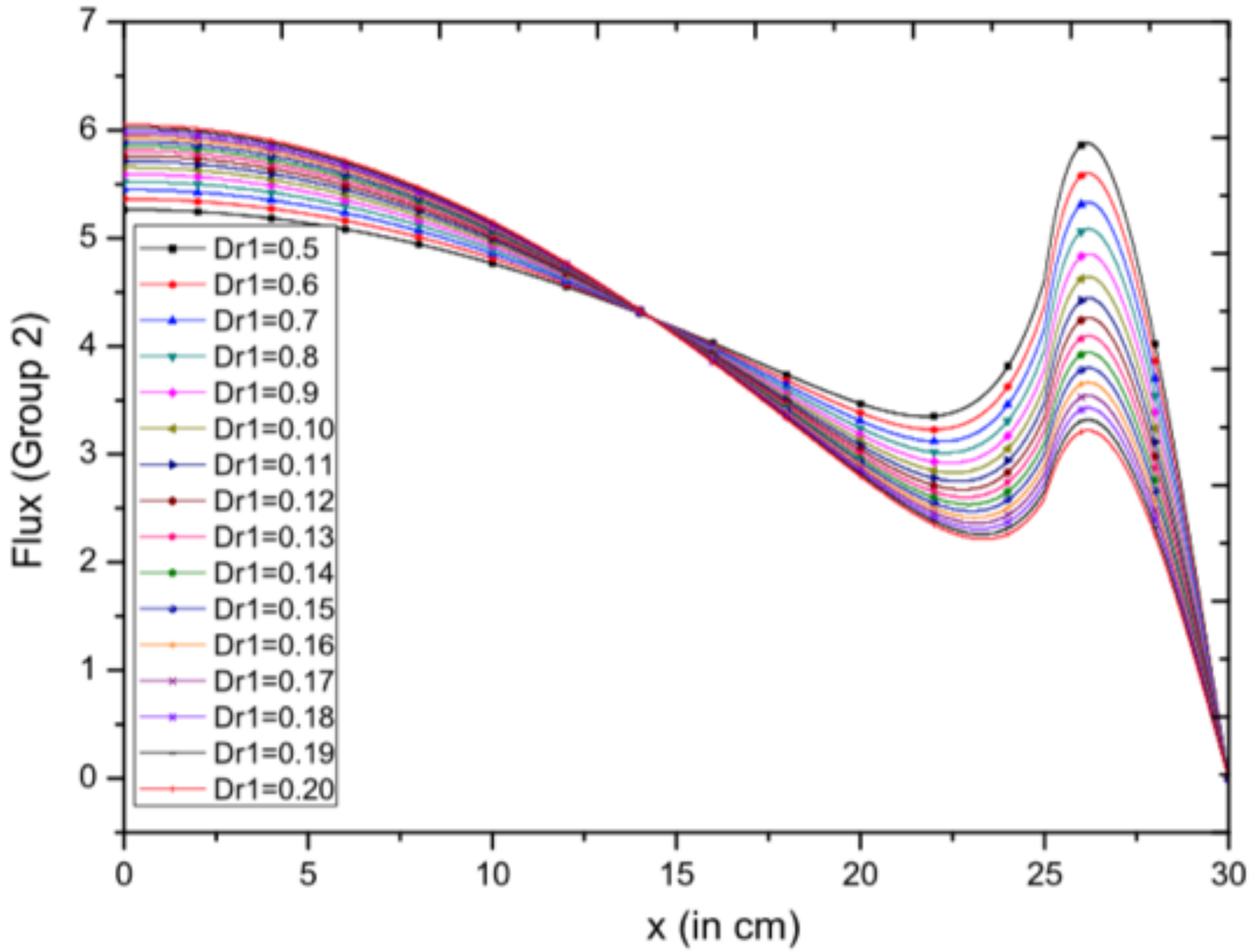
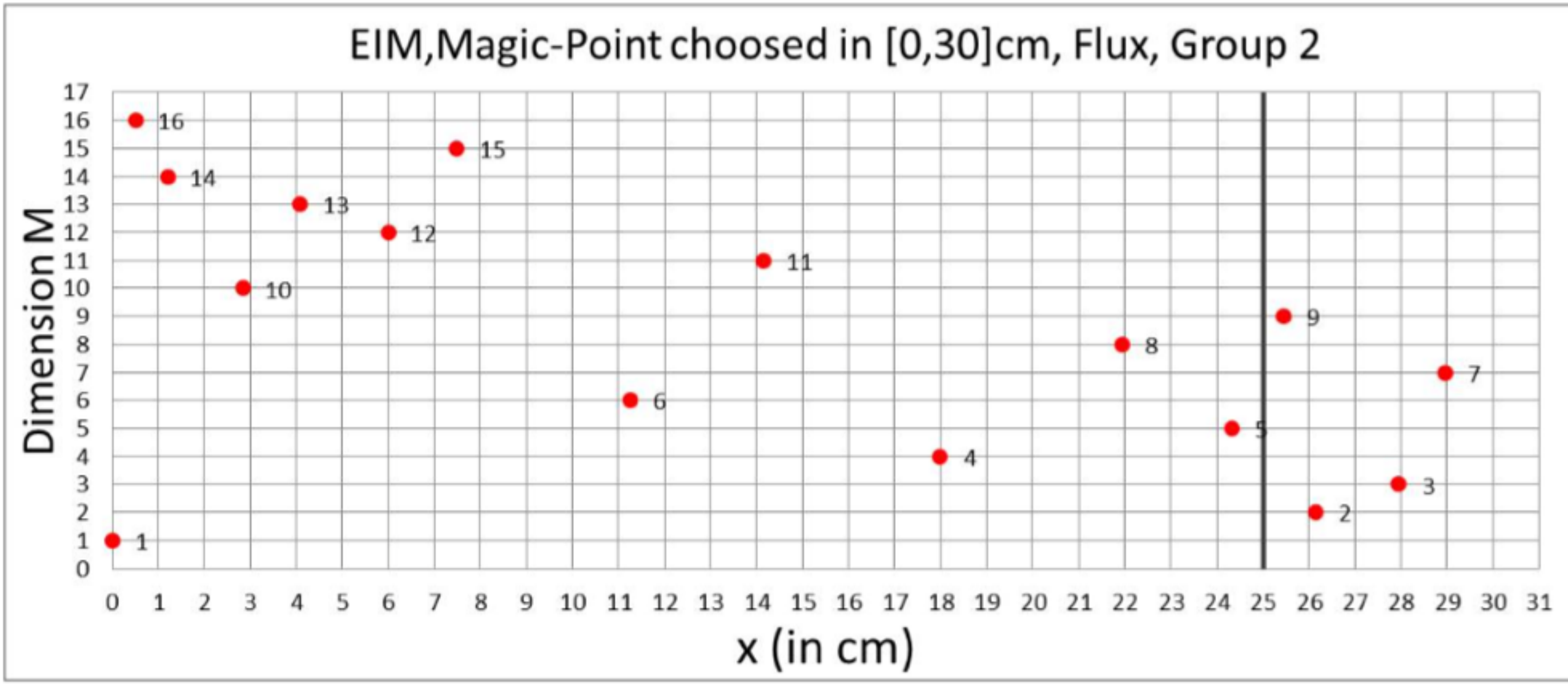


Figure 1: Flux and power distribution in the Core for the benchmark problem

In order you are convinced that the Kolmogorov dimension is small



First classical EIM



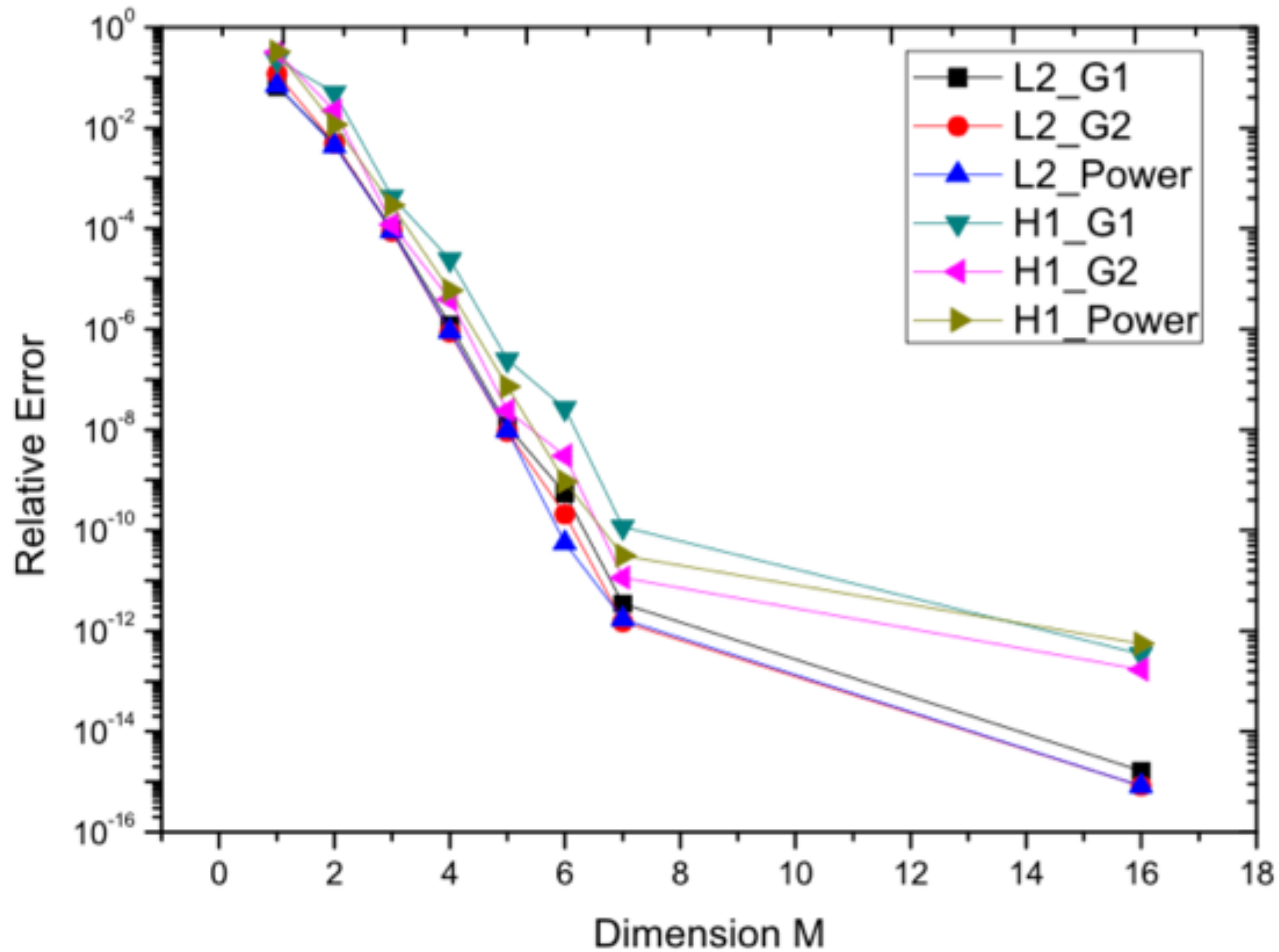
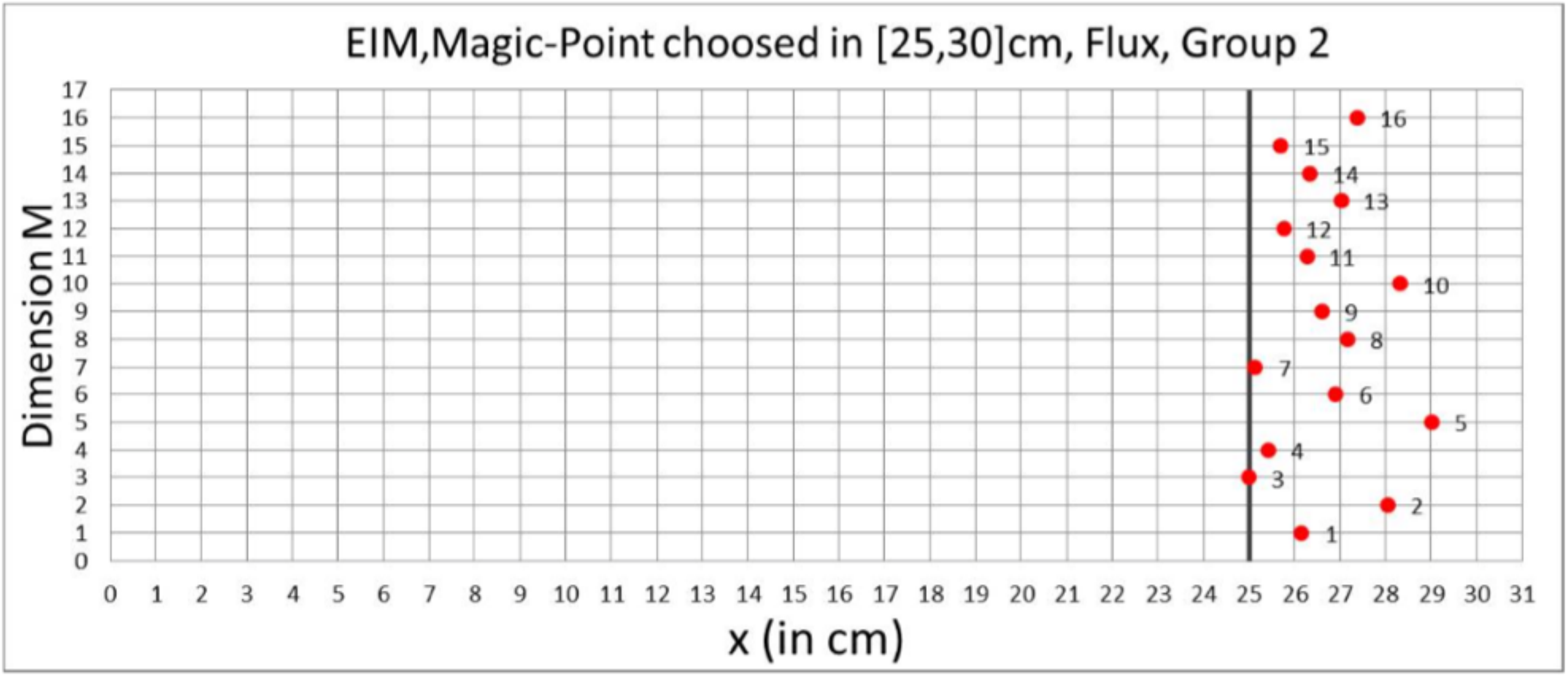
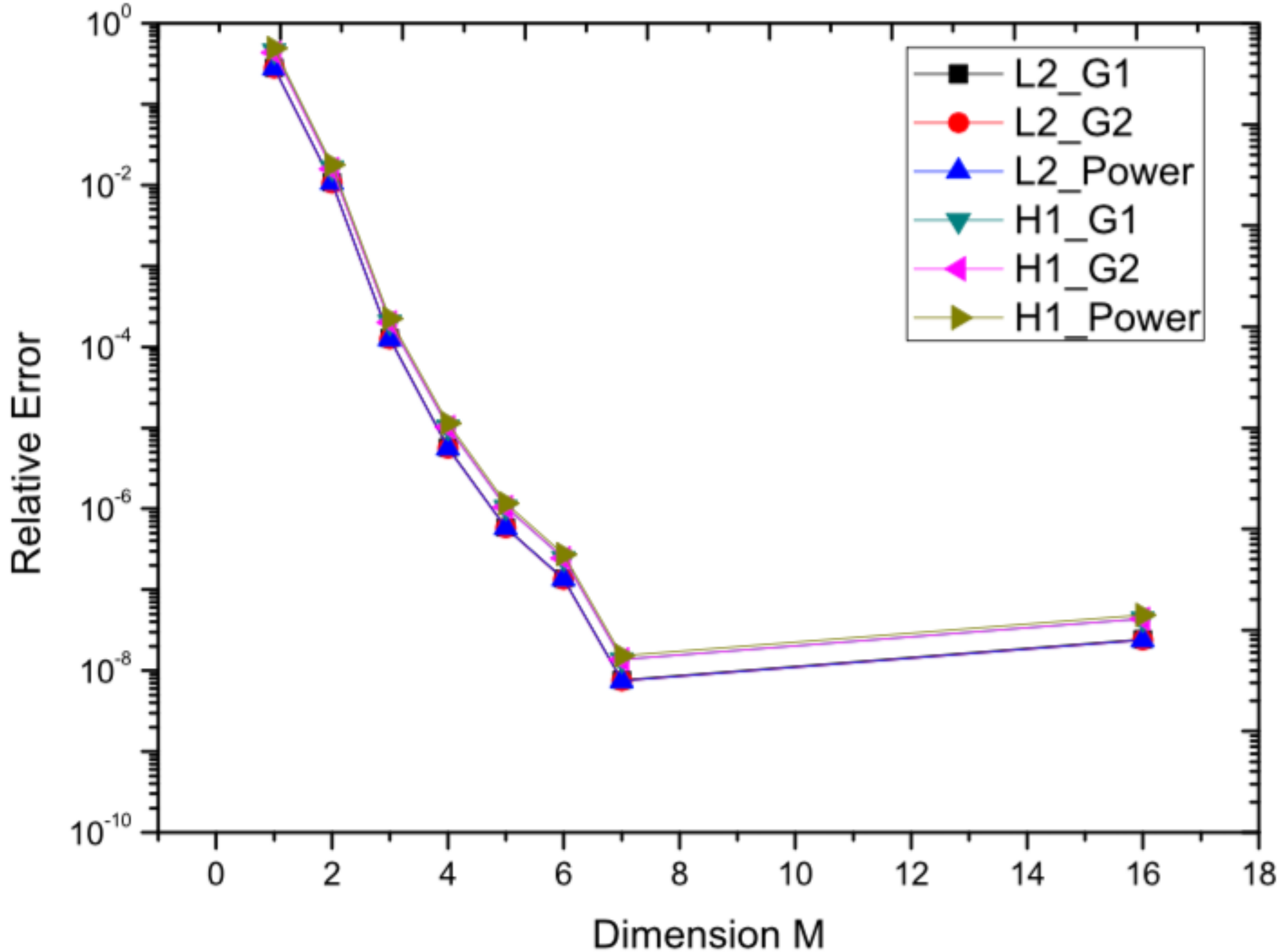


Figure 6: The relative error of different functions in L^2 and semi-H^1 norm versus the EIM dimension (Noise Free). The function used by EIM is the flux of group 2, the magic points are chosen from the whole area, [0, 30] cm. The interpolation for flux of group 1 and the power use the same coefficients solved by EIM for the flux of group 2.

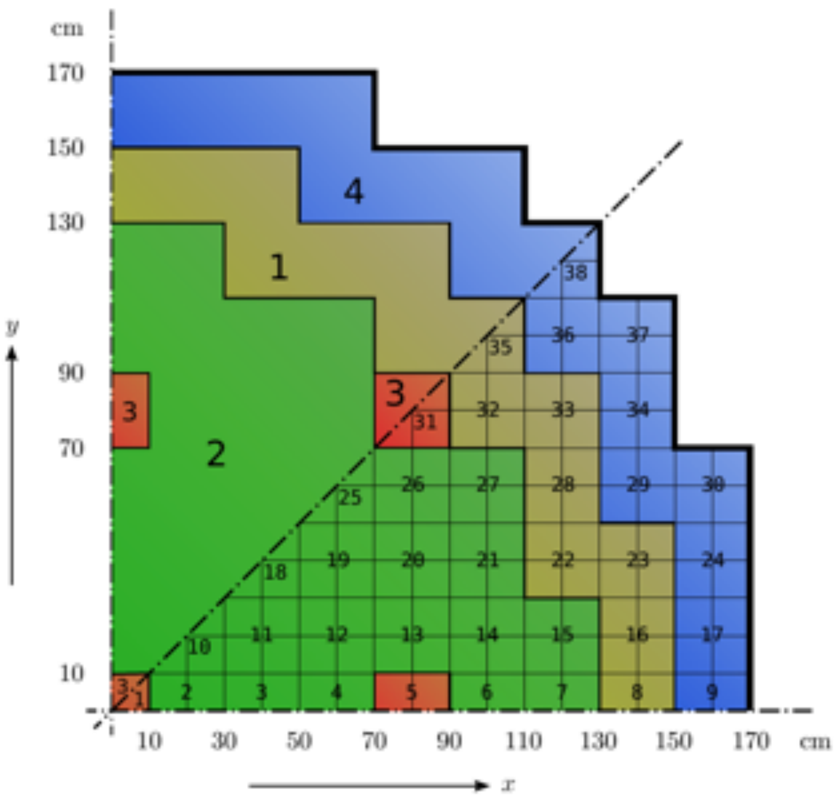
Then with a restriction on the position of the points

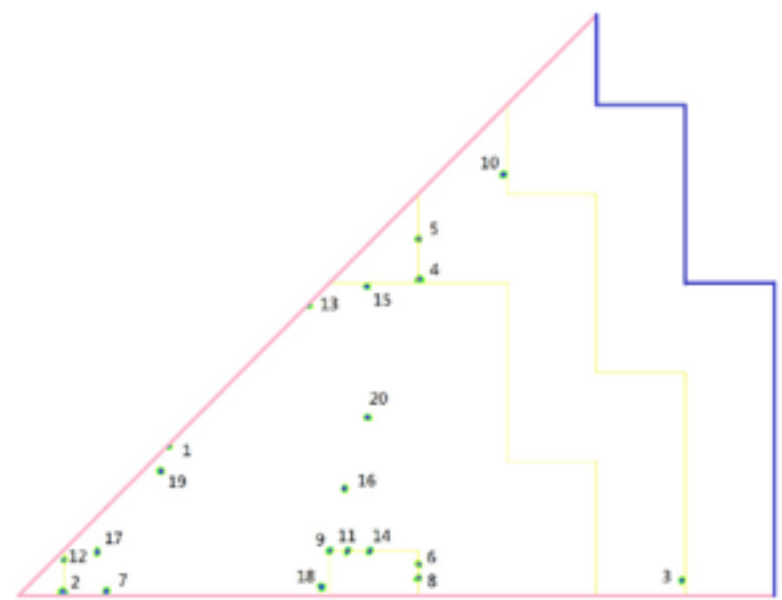




L2_G1
L2_G2
L2_Power
H1_G1
H1_G2
H1_Power
Relative Error
Dimension M

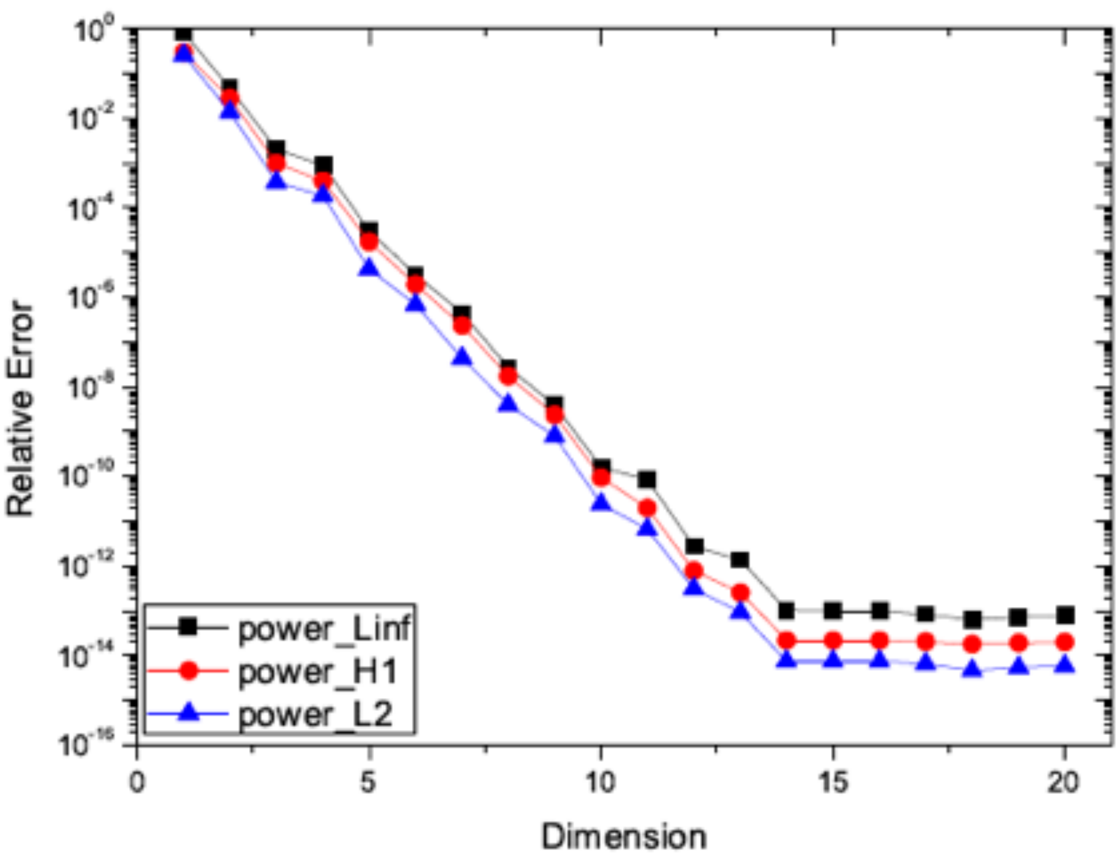
Similar results in two dimensions

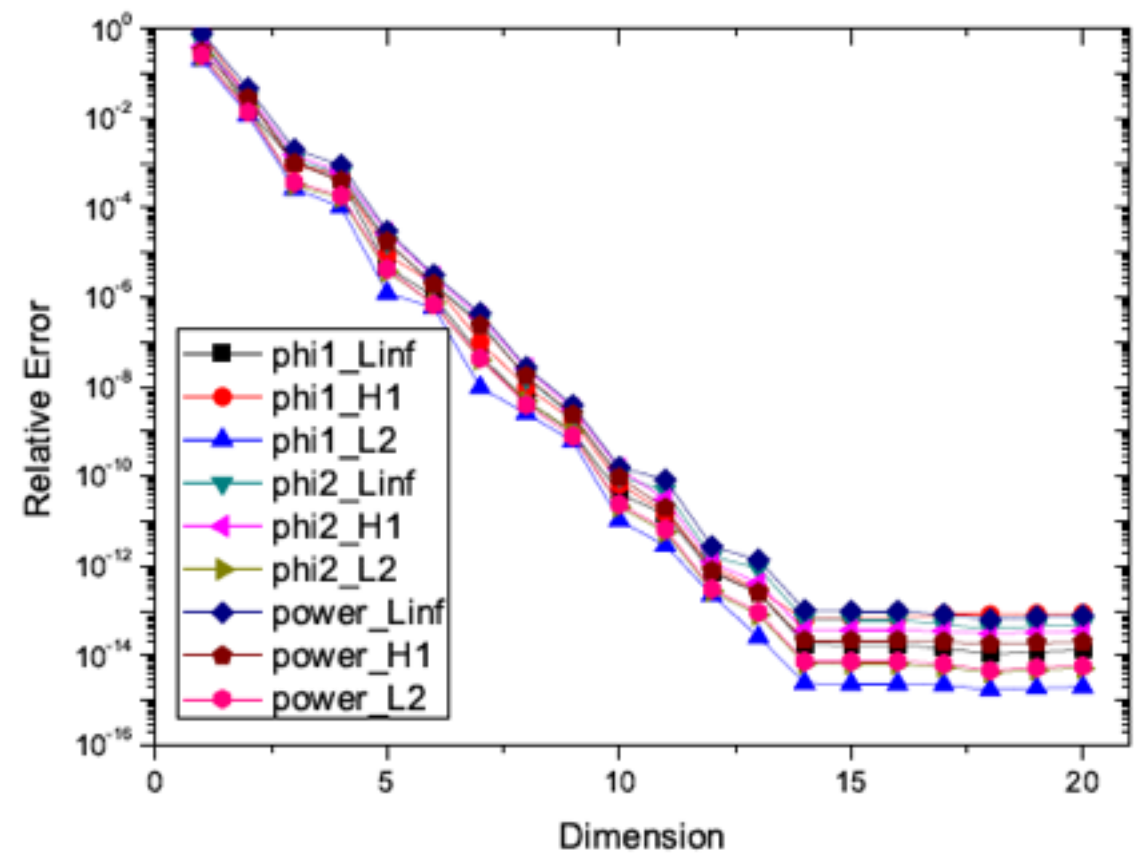




The first twenty interpolation points distribution (constrained in fuel region)

Similar results in two dimensions





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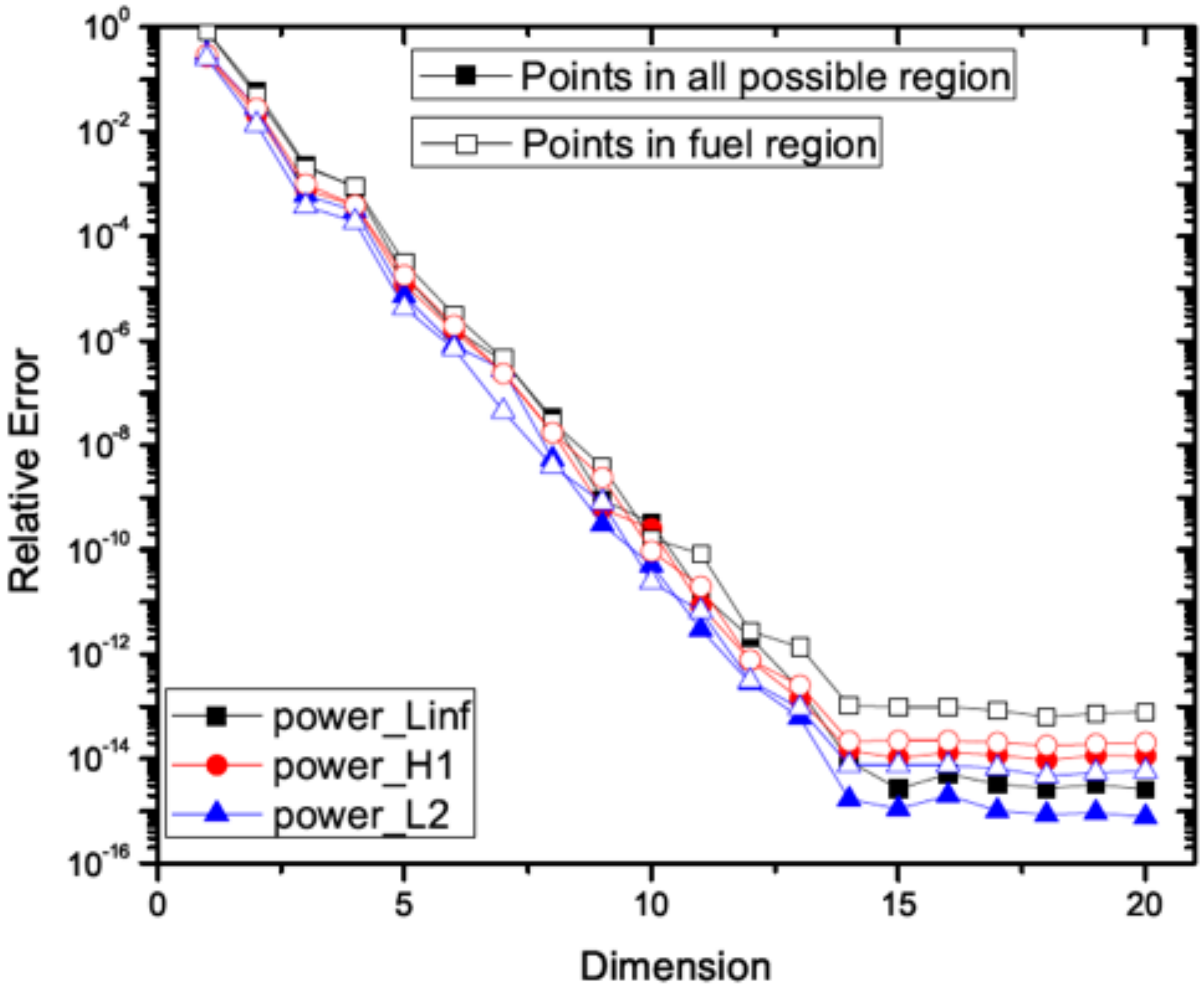
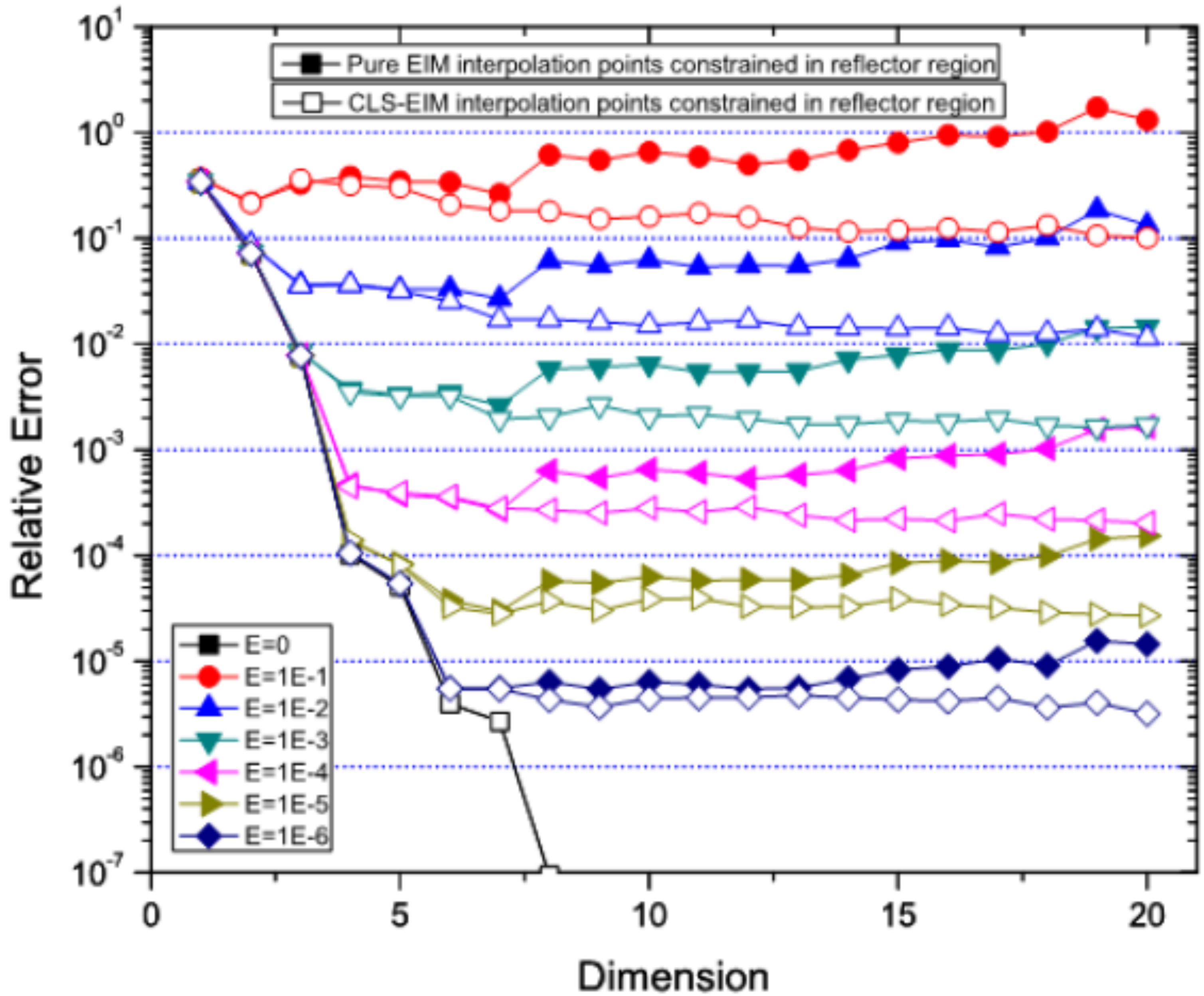
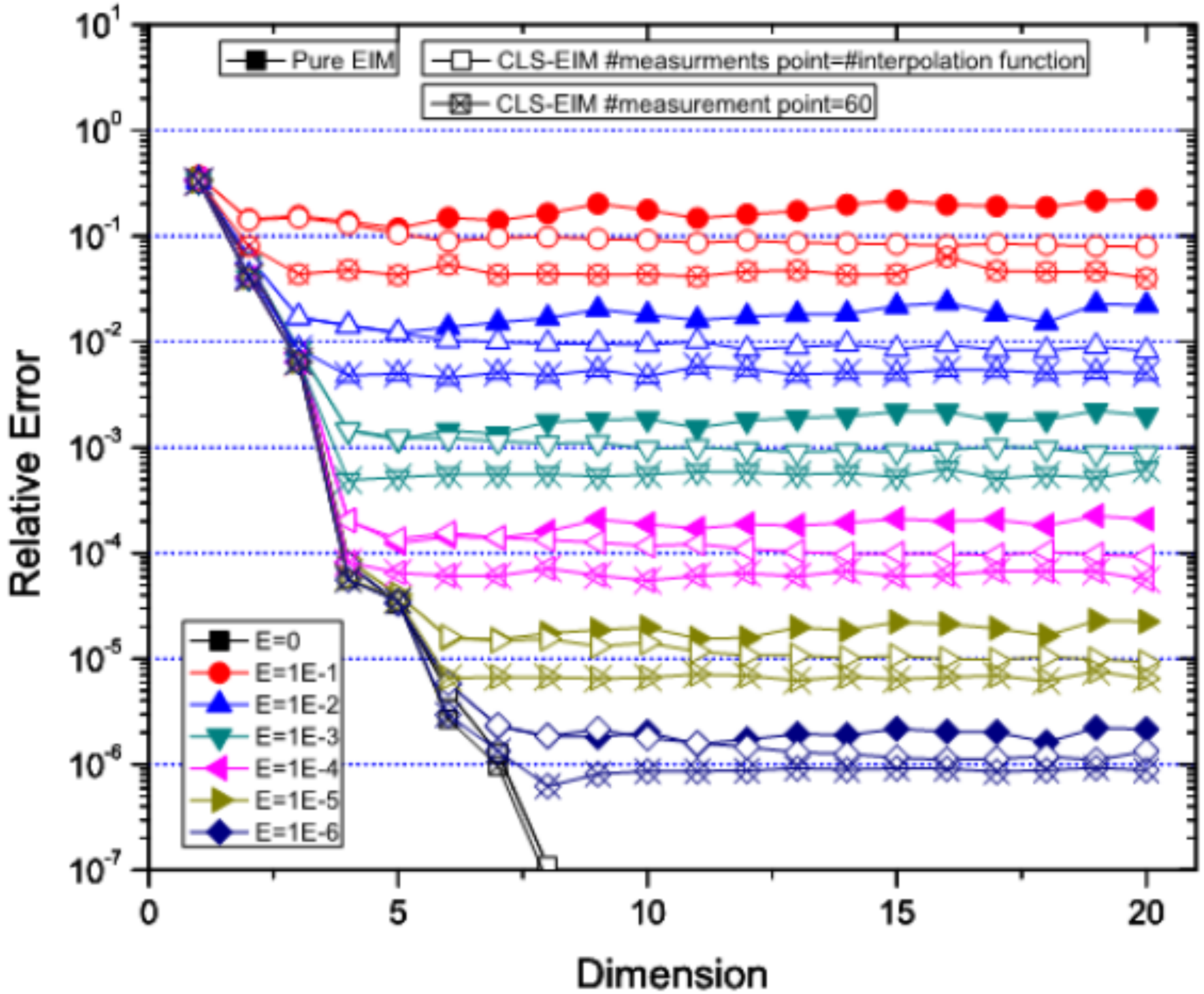


What about noisy data

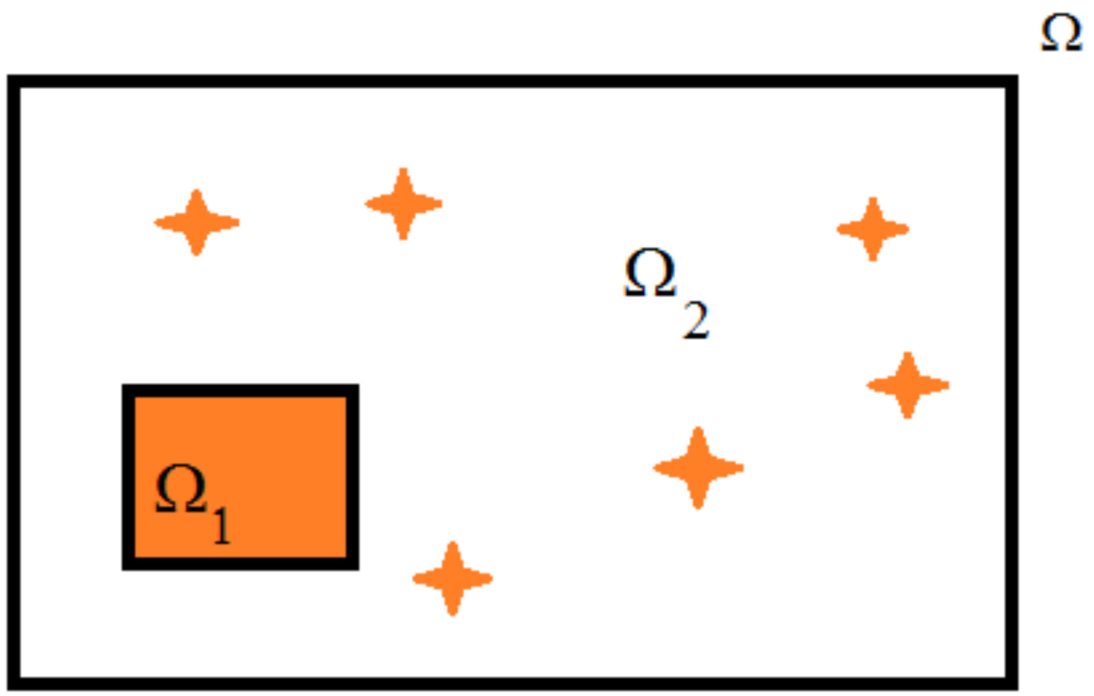


We clearly see the effect of the Lebesgue constant

What about noisy data :
how to minimize the effect of the Lebesgue constant



Application of GEIM in data assimilation and monitoring



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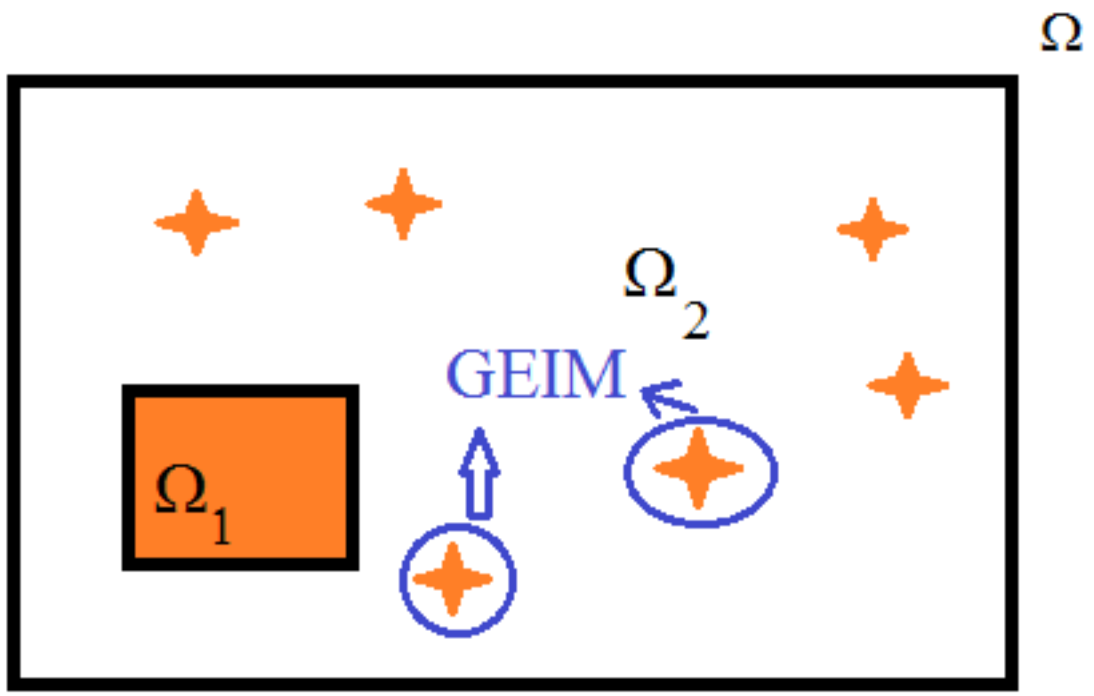

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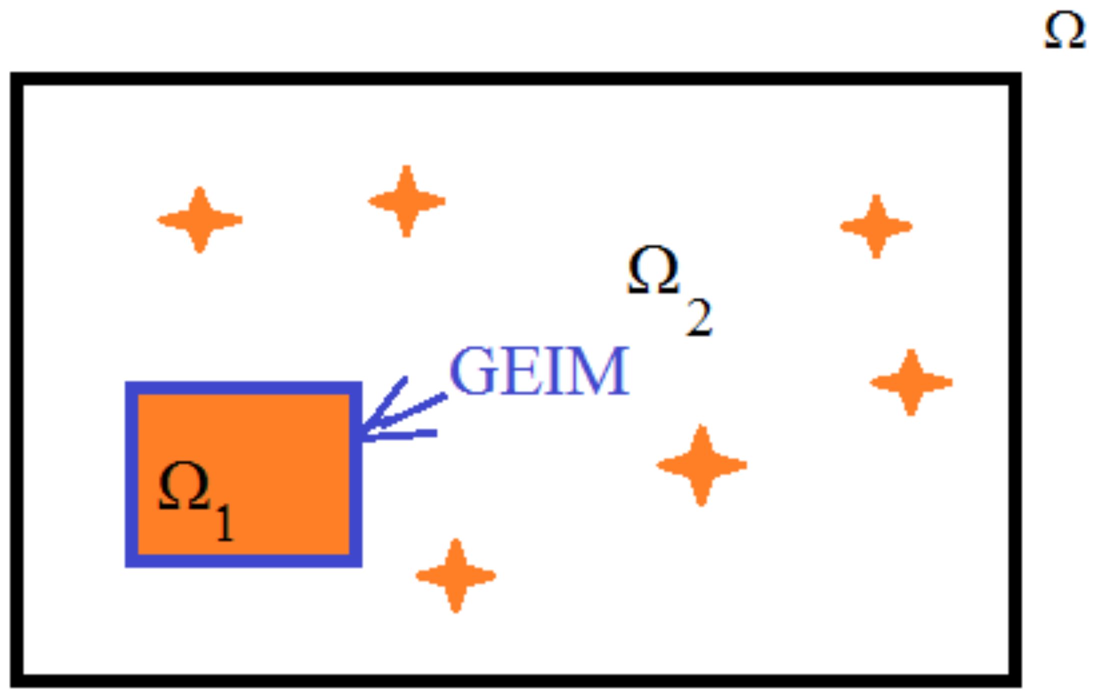

Incorporating the model error : Parametrized-Background Data-Weak (PBDW) formulation with A.T. Patera, J. D. Penn and M. Yano

The PBDW formulation integrates a parametrized mathematical model and M experimental observations associated with the configuration $\mathcal{C}$ to estimate the true field $u^{true}[\mathcal{C}]$ as well as any desired output $l^{out}(u^{true}[C]) \in C$ for given output functional l^{out}.

We first introduce a sequence of background spaces that reflect our (prior) best knowledge,

$$\mathcal{Z}_1 \subset \cdots \subset \mathcal{Z}_{N_{max}} \subset \mathcal{U};$$

here the second ellipsis indicates that we may consider the sequence of length N_{max} as resulting from a truncation of an infinite sequence. Our goal is to choose the background spaces such that

$$\lim_{N\to\infty} \inf_{w\in\mathcal{Z}_N} \|u^{\mathrm{true}}[\mathcal{C}] - w\| \leq \epsilon_{\mathcal{Z}} \quad \forall \mathcal{C} \in \mathcal{S},$$

In words, we choose the background spaces such that the most dominant physics that we anticipate to encounter for various system configurations is well represented for a relatively small N.

Incorporating the model error : Parametrized-Background Data-Weak (PBDW) formulation with A.T. Patera, J. D. Penn and M. Yano

We now characterize our data acquisition procedure. Given a system in configuration $\mathcal{C} \in \mathcal{S}$, we assume our observed data $y^{\rm obs}[\mathcal{C}] \in \mathbb{C}^M$ is of the form,

$$\forall m = 1, \ldots, M, \quad y_m^{\rm obs}[\mathcal{C}] = \ell_m^{\rm o}(u^{\rm true}[\mathcal{C}]) + e_m;$$

here $y_m^{\rm obs}[\mathcal{C}]$ is the value of the m-th observation, $\ell_m^{\rm o} \in \mathcal{U}'$ is the linear (and not antilinear) functional of the functional depends on the specific transducer used to acquire data.

Concerning the form of the noises $(e_m)_m$, we make the following three assumptions:

(A1) zero mean: $E[e_m] = 0$, $m = 1, \ldots, M$;
(A2) homoscedastic: $E[e_m^2] = \sigma^2$, $m = 1, \ldots, M$;
(A3) uncorrelated: $E[e_m e_n] = 0$, $m \neq n$.

We first associate with each observation functional $\ell_m^o \in \mathcal{U}'$ an observable function,

$$\forall m = 1, \ldots, M, \quad q_m = R_{\mathcal{U}} \ell_m^{\mathrm{o}},$$

the Riesz representation of the functional [1]. We then introduce hierarchical *observable spaces*,

$$\forall M = 1, \ldots, M_{\max}, \ldots, \quad \mathcal{U}_M = \mathrm{span}\{q_m\}_{m=1}^M;$$

We may now state the PBDW estimation statement: given a physical system in configuration $\mathcal{C} \in \mathcal{S}$, find $(u_{N,M}^*[\mathcal{C}] \in \mathcal{U}, z_{N,M}^*[\mathcal{C}] \in \mathcal{Z}_N, \eta_{N,M}^*[\mathcal{C}] \in \mathcal{U})$ such that

$$(u_{N,M}^*[\mathcal{C}], z_{N,M}^*[\mathcal{C}], \eta_{N,M}^*[\mathcal{C}]) = \underset{\substack{u_{N,M} \in \mathcal{U} \\ z_{N,M} \in \mathcal{Z}_N \\ \eta_{N,M} \in \mathcal{U}}}{\arg\inf} \; \|\eta_{N,M}\|^2 \tag{2}$$

subject to

$$\begin{aligned}
(u_{N,M}, v) &= (\eta_{N,M}, v) + (z_{N,M}, v) \quad \forall v \in \mathcal{U}, \\
(u_{N,M}, \phi) &= (u_M^{\mathrm{obs}}[\mathcal{C}], \phi) \quad \forall \phi \in \mathcal{U}_M.
\end{aligned}$$

We may readily derive the associated (reduced) Euler-Lagrange equations as a saddle problem [16]: given a physical system in configuration $\mathcal{C} \in \mathcal{S}$, find $(\eta_{N,M}^*[\mathcal{C}] \in \mathcal{U}_M, z_{N,M}^*[\mathcal{C}] \in \mathcal{Z}_N)$ such that

$$\begin{aligned}
(\eta_{N,M}^*[\mathcal{C}], q) + (z_{N,M}^*[\mathcal{C}], q) &= (u_M^{\mathrm{obs}}[\mathcal{C}], q) \quad \forall q \in \mathcal{U}_M, \\
(\eta_{N,M}^*[\mathcal{C}], p) \qquad\qquad\qquad &= 0 \quad \forall p \in \mathcal{Z}_N,
\end{aligned} \tag{3}$$

and set

$$u_{N,M}^*[\mathcal{C}] = \eta_{N,M}^*[\mathcal{C}] + z_{N,M}^*[\mathcal{C}]. \tag{4}$$

Algebraic Form: Offline-Online Computational Procedure

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^H & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}^*[\mathcal{C}] \\ \mathbf{z}^*[\mathcal{C}] \end{pmatrix} = \begin{pmatrix} y^{\mathrm{obs}}[\mathcal{C}] \\ 0 \end{pmatrix},$$

where

$$\mathbf{A} \equiv Q^\dagger U Q = LQ \in \mathbb{C}^{M\times M}$$
$$\mathbf{B} \equiv Q^\dagger U Z = LZ \in \mathbb{C}^{M\times N},$$

Analysis

Lemma 1. *The expectation of the norm of the state error may be decomposed into deterministic and stochastic components and is bounded by*

$$E[\|u^{\mathrm{true}}[\mathcal{C}] - u^{*}_{N,M}[\mathcal{C}]\|] \leq \|u^{\mathrm{true}}[\mathcal{C}] - u^{\mathrm{nf}}_{N,M}[\mathcal{C}]\| + E[\|u^{\mathrm{nf}}_{N,M}[\mathcal{C}] - u^{*}_{N,M}[\mathcal{C}]\|];$$

here $u^{\mathrm{true}}[\mathcal{C}]$ is the true deterministic state, $u^{*}_{N,M}[\mathcal{C}]$ is the PBDW estimate given by (3)*, $u^{\mathrm{nf}}_{N,M}[\mathcal{C}]$ is the noise-free estimate given by* (6)*, and E refers to expectation.*

Proposition 2. *The deterministic component of the error is bounded by*

$$\|u^{\mathrm{true}}[\mathcal{C}] - u^{\mathrm{nf}}_{N,M}[\mathcal{C}]\| \leq \left(1 + \frac{1}{\beta_{N,M}}\right) \inf_{q \in \mathcal{U}_M \cap \mathcal{Z}_N^{\perp}} \|\Pi_{\mathcal{Z}_N^{\perp}} u^{\mathrm{true}}[\mathcal{C}] - q\|,$$

where $\beta_{N,M}$ is the inf-sup constant given by

$$\beta_{N,M} \equiv \inf_{w \in \mathcal{Z}_N} \sup_{v \in \mathcal{U}_M} \frac{(w,v)}{\|w\|\|v\|};$$

here $u^{\mathrm{true}}[\mathcal{C}]$ is the true deterministic state, and $u^{\mathrm{nf}}_{N,M}[\mathcal{C}]$ is the noise-free estimate given by (6).

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Lemma 1. *The expectation of the norm of the state error may be decomposed into deterministic and stochastic components and is bounded by*

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Proposition 3. *Suppose the observation error* e *satisfies the assumptions (A1), (A2), and (A3). Then, the mean of the stochastic error is zero:*

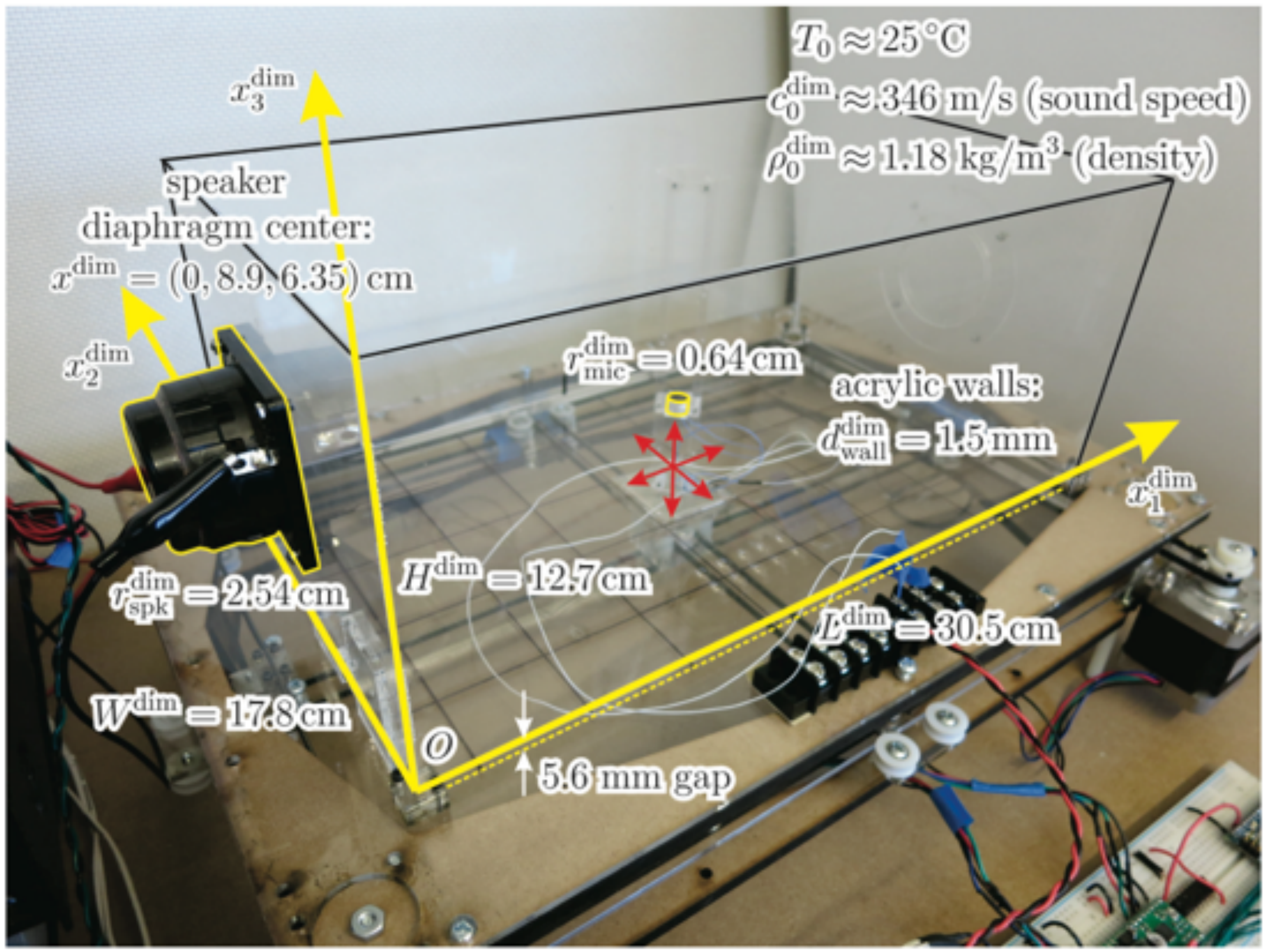
$$E\left[u^{\mathrm{nf}}_{N,M}[\mathcal{C}] - u^*_{N,M}[\mathcal{C}]\right] = 0; \tag{9}$$

here $u^{\mathrm{nf}}_{N,M}[\mathcal{C}]$ *is the noise-free estimate given by* (6), *and* $u^*_{N,M}[\mathcal{C}]$ *is the PBDW estimate given by* (3). *Moreover, the variance of the stochastic error is bounded by*

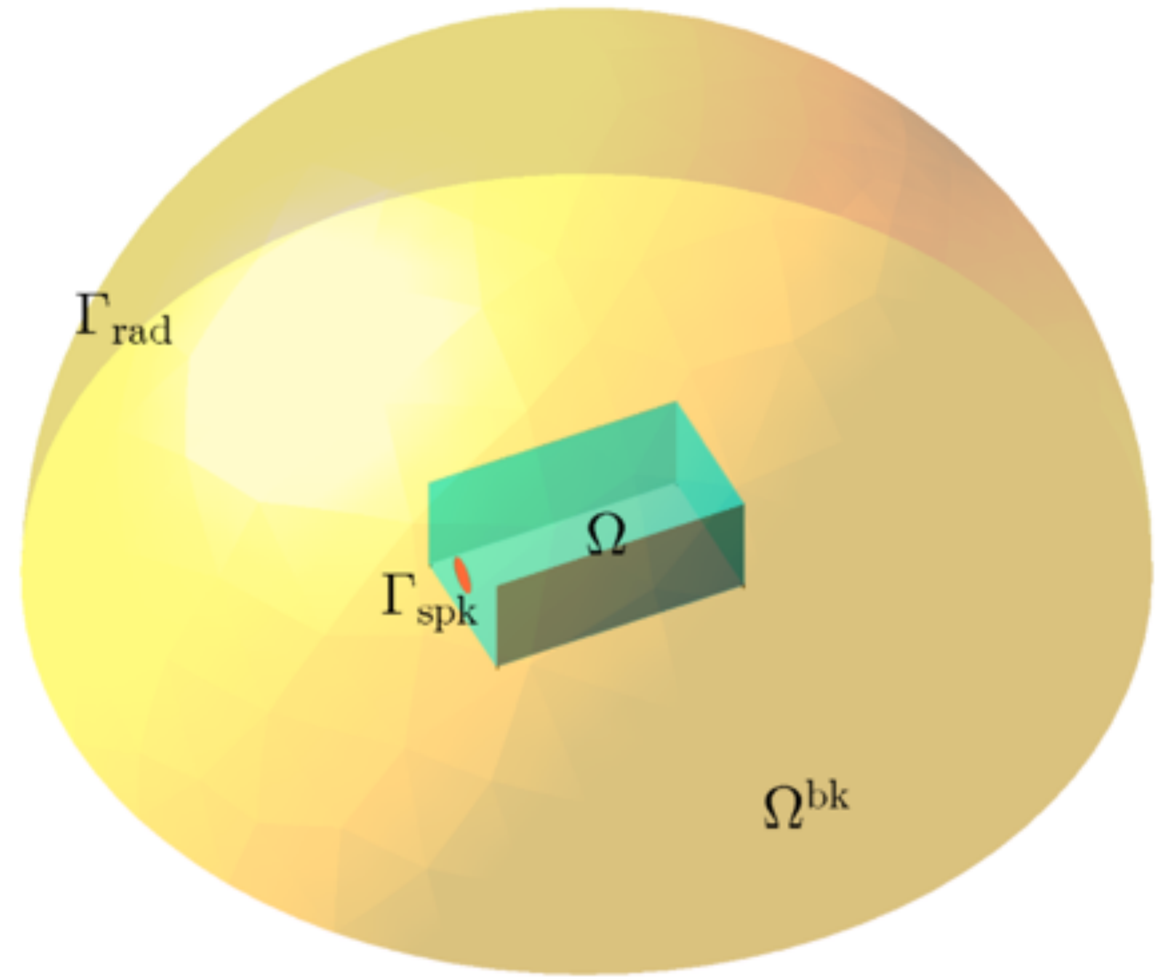
$$\sqrt{E\left[\|u^{\mathrm{nf}}_{N,M}[\mathcal{C}] - u^*_{N,M}[\mathcal{C}]\|^2\right]} \leq \sigma\left(1 + \frac{2}{\beta^2_{N,M}}\right)\sqrt{\mathrm{trace}(\mathbf{A}^{-1})}, \tag{10}$$

where $\mathbf{A} \equiv Q^\dagger U Q \in \mathbb{C}^{M\times M}$, $\beta_{N,M}$ *is the inf-sup constant defined in* (8), *and* σ^2 *is the variance of the measurement noise.*

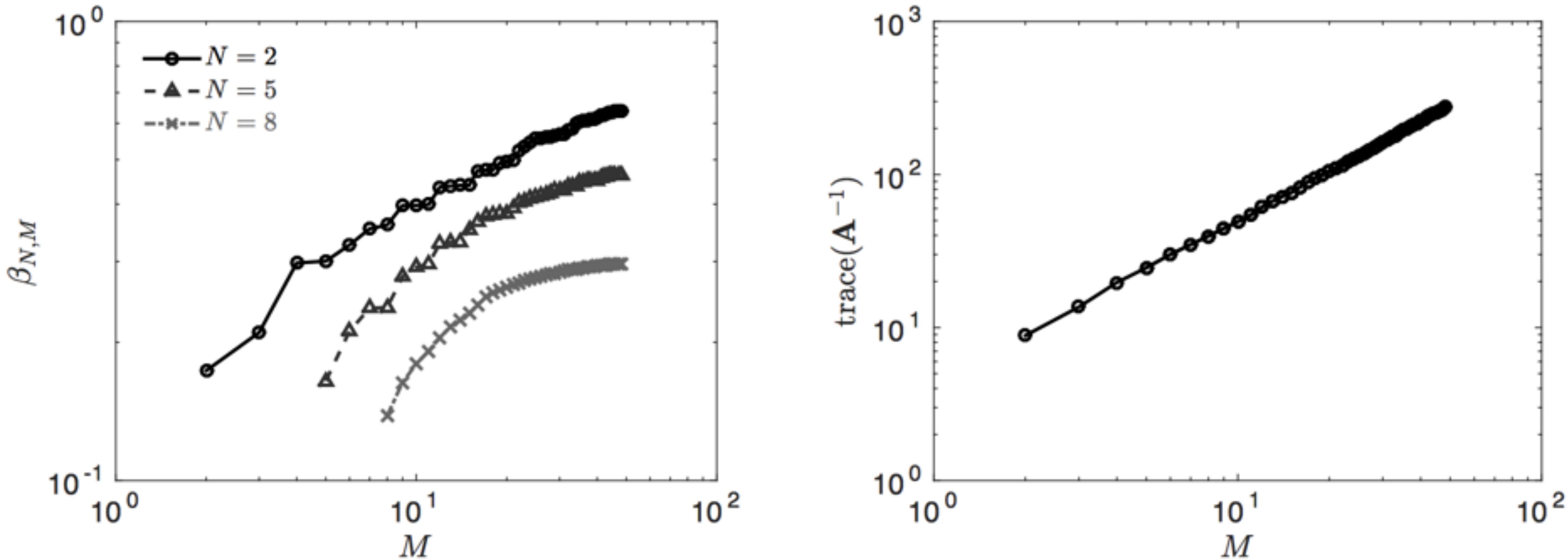
Experimental settings



(a) physical system



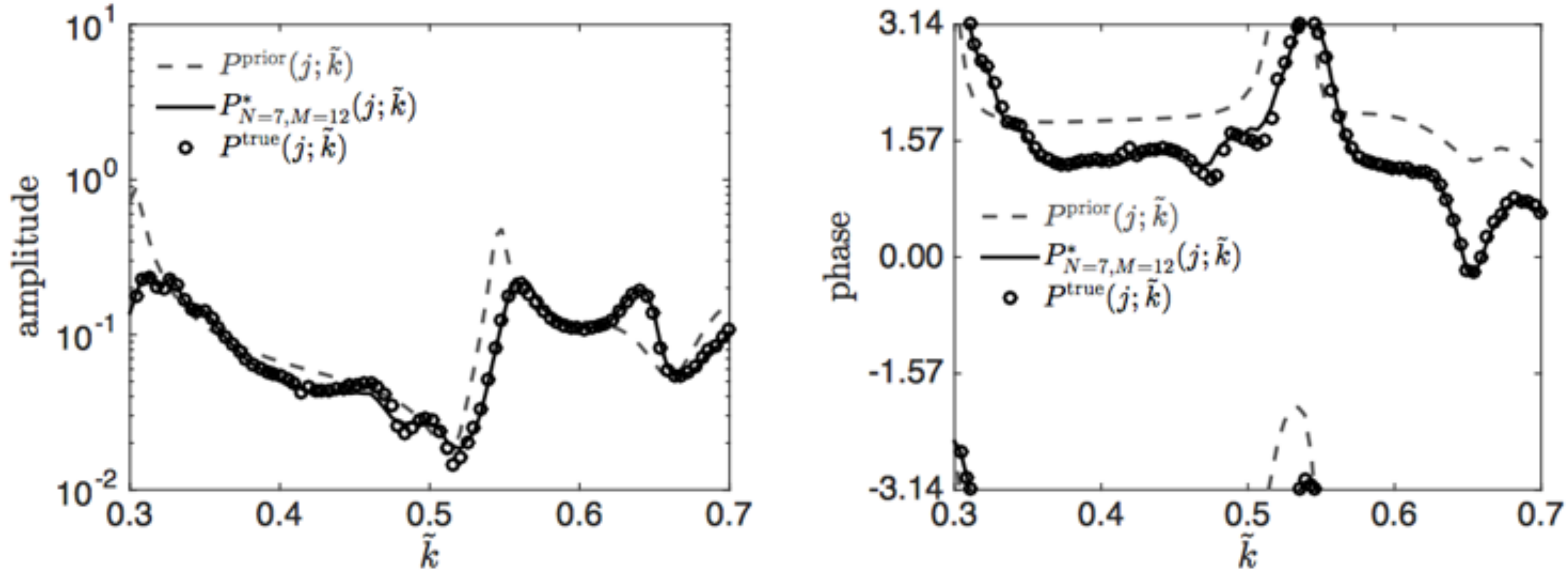
(b) computational domain



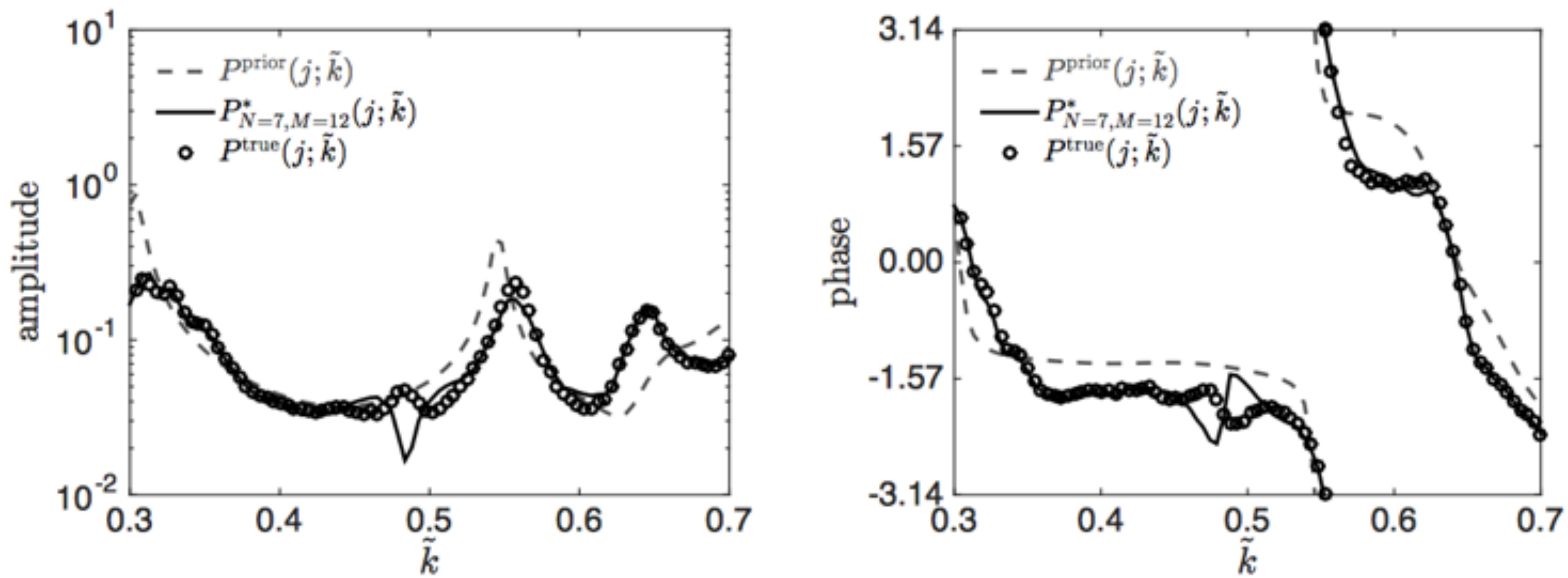
(a) inf-sup constant

(b) observation-conditioning metric

FIGURE 4. Behavior of the stability constant $\beta_{N,M}$ and the observation-conditioning metric $\mathrm{trace}(\mathbf{A}^{-1})$.



(a) $\xi_j^{\mathrm{c}} = (2.67, 2.67, 4.50)$



(b) $\xi_j^{\mathrm{c}} = (9.33, 2.67, 4.50)$

FIGURE 6. Frequency response (amplitude and phase) at the assessment center (a) $\xi_j^{\mathrm{c}} = (2.67, 2.67, 4, 50)$ and (b) $\xi_j^{\mathrm{c}} = (9.33, 2.67, 4.50)$.

Data-Driven Empirical Enrichment of the Background Space

We now devise a strategy to systematically incorporate the unmodeled physics identified by the update space $\mathcal{U}_M$ to augment the background space $\mathcal{Z}_N$ for subsequent data assimilation. The goal is to reduce the number of observations for future configurations. We consider the following algorithm:

(1) Find the configuration that maximizes the relative error (indicator):

$$\tilde{k}^* = \underset{\tilde{k}\in[0.3,0.7]}{\arg\sup} \frac{E_{\text{avg}}[\mathcal{C}_{\tilde{k}}](u_{N=N_{\max},M=12})}{E_{\text{avg}}[\mathcal{C}_{\tilde{k}}](u_{N=0,M=0})}.$$

(2) Compute the update state associated with the configuration $\mathcal{C}_{\tilde{k}^*}$,

$$\eta^*_{N=N_{\max}=8,M=M_{\max}=48}[\mathcal{C}_{\tilde{k}^*}].$$

(3) Construct the "augmented" best-knowledge space

$$\mathcal{Z}^{\text{aug}}_{N_{\max}+1} \equiv \text{span}\{\mathcal{Z}_{N_{\max}},\ \eta^*_{N=N_{\max}=8,M=M_{\max}=48}[\mathcal{C}_{\tilde{k}^*}]\};$$

note that $\eta^*_{N_{\max},M_{\max}}[\mathcal{C}_{\tilde{k}^*}] \in \mathcal{Z}^{\perp}_{N_{\max}} \cap \mathcal{U}_{M_{\max}}$ and hence $\eta^*_{N_{\max},M_{\max}}[\mathcal{C}_{\tilde{k}^*}]$ is orthogonal to $\mathcal{Z}_{N_{\max}}$.

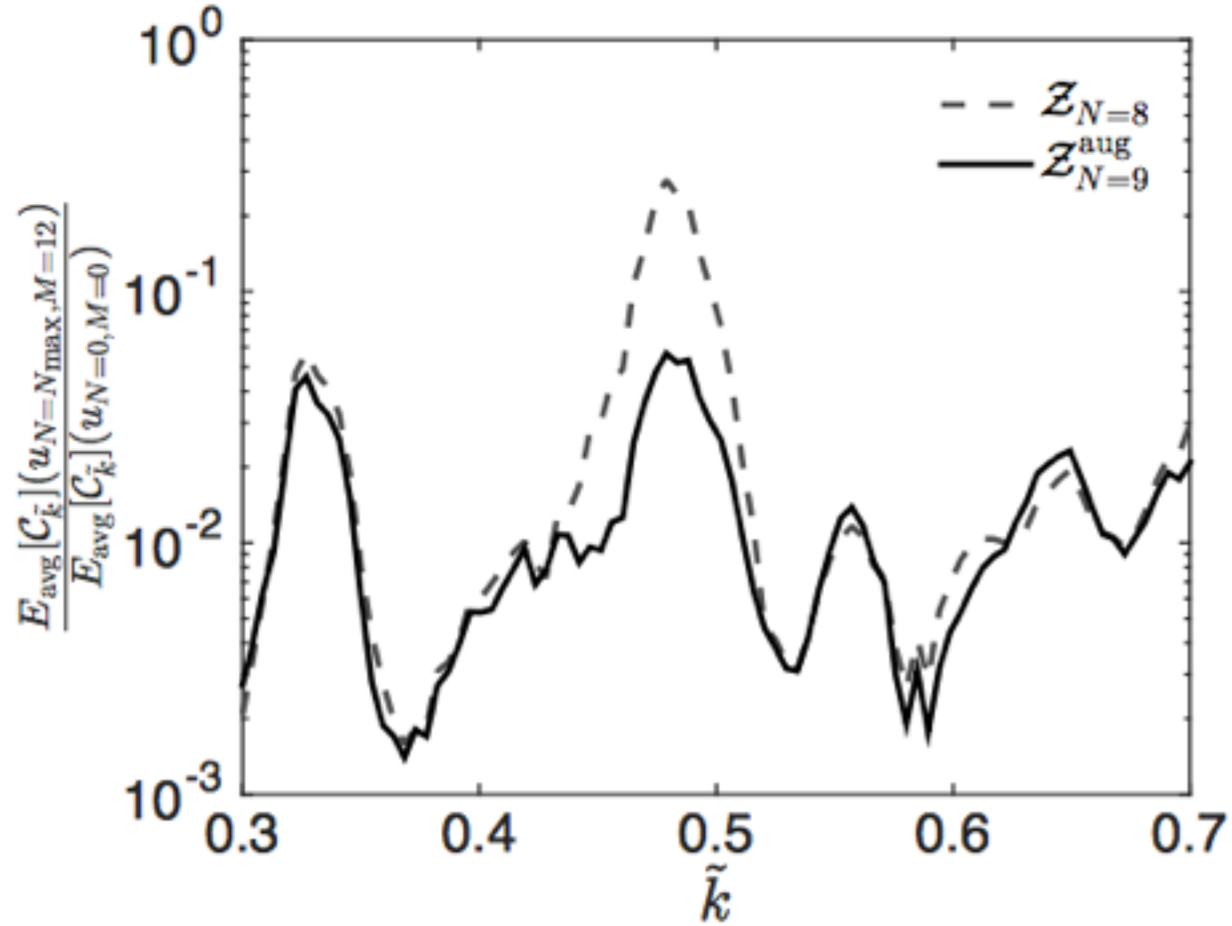


FIGURE 12. The relative error in the PBDW estimate for the original background space $\mathcal{Z}_{N=8}$ and the augmented background space $\mathcal{Z}_{N=9}^{\text{aug}}$ both using $M = 12$ observations.

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The PBDW formulation is endowed with the following characteristics:

- **Weak formulation.**
- **Actionable a priori theory.** The weak formulation facilitates the construction of a priori error estimates
- **Background space** that best reflects our (prior) best knowledge of the phenomenon under consideration
- **Design of quasi-optimal set of observations from a library of experimentally realizable observations** in order to maximize the stability of the data assimilation.
- **Correction of unmodeled physics with uncertainty.**
- **Online computational cost is O(M).** We may realize real-time state estimation
- **Simple non-intrusive implementation and generality.** The mathematical model appears only in the offline stage.

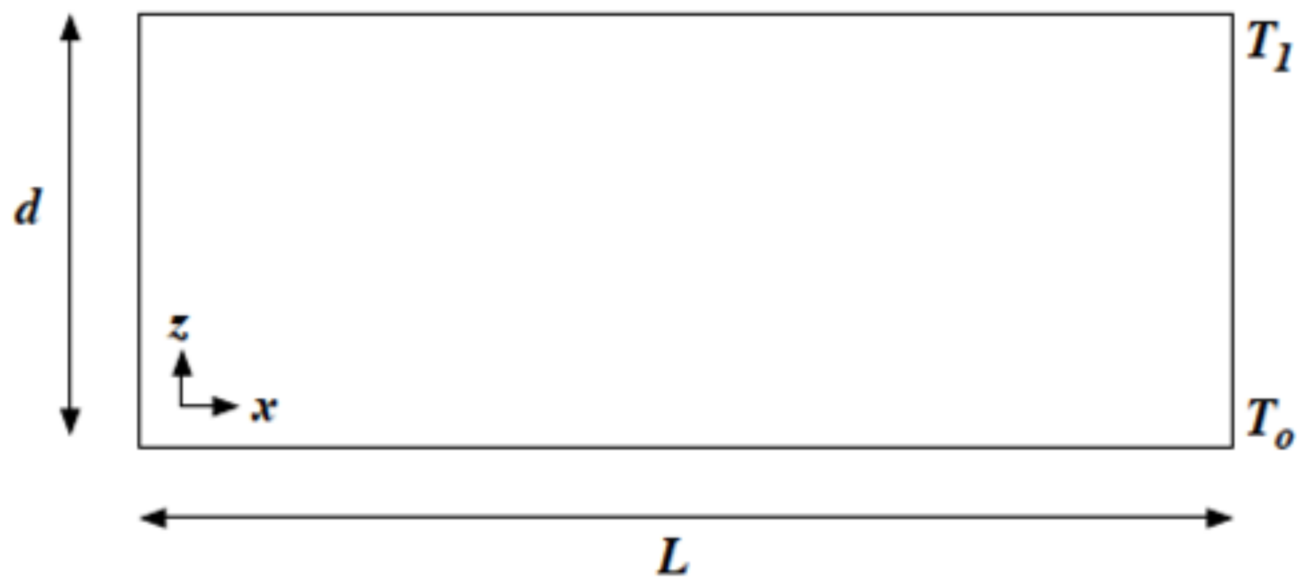
- Recently a work of P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk has analyzed this approach in terms of optimal recovery and our algorithm is optimal in this frame. They have extended it to the Multi-Space Case.

Reduced Basis method for a bifurcation study of a thermal convection problem

Henar Herrero, Yvon Maday, Francisco Pla

RB-4-RB : Formulation of the problem

A sketch of the domain and the physical situation is shown here



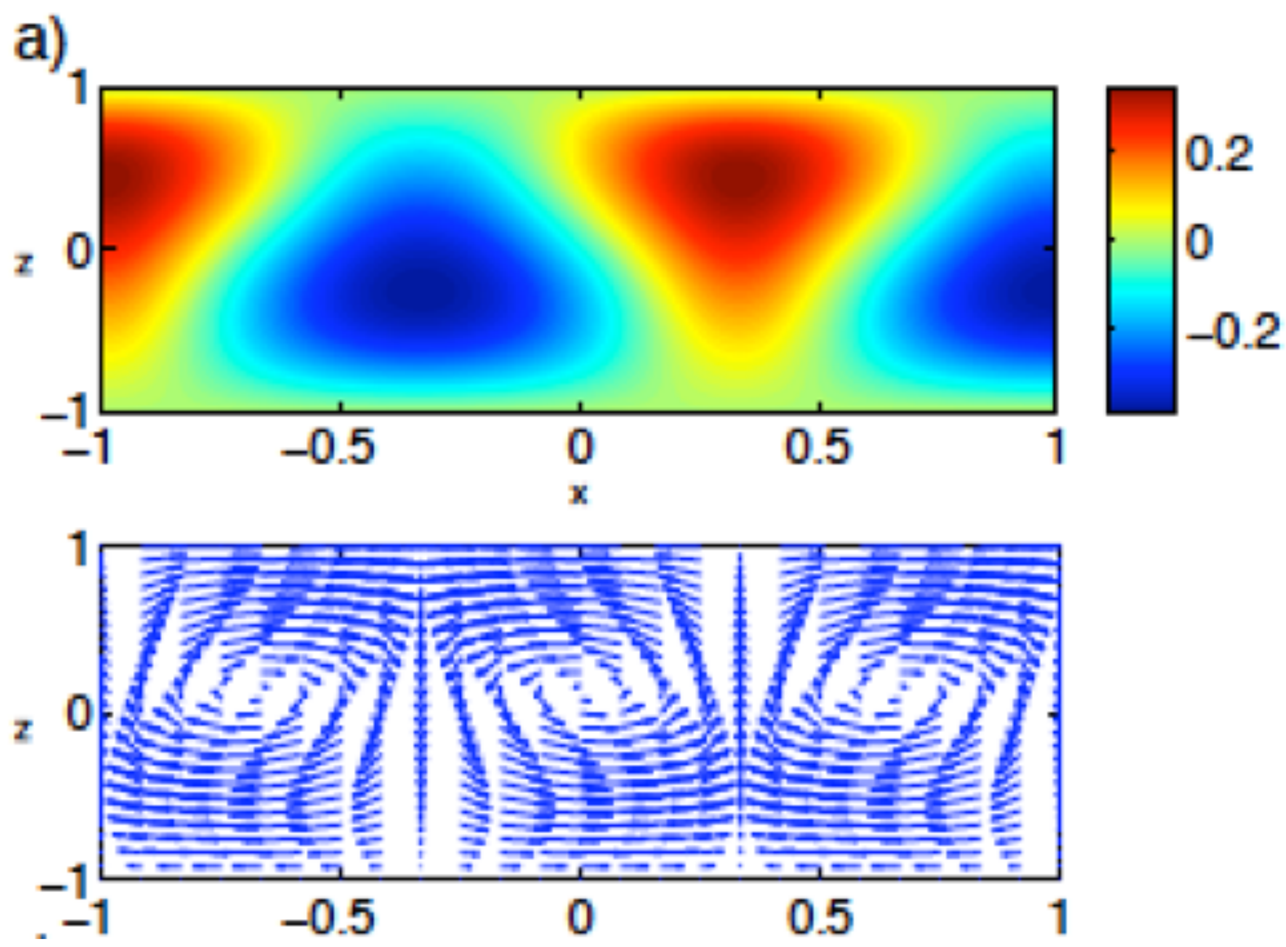
The domain is a rectangle of depth *d* and width *L*. The domain contains a fluid that is heated from below, so that on the bottom plate a temperature T_0 is imposed and on the upper plate the temperature is

$$T_1 = T_0 - \Delta T = T_0 - \beta d$$

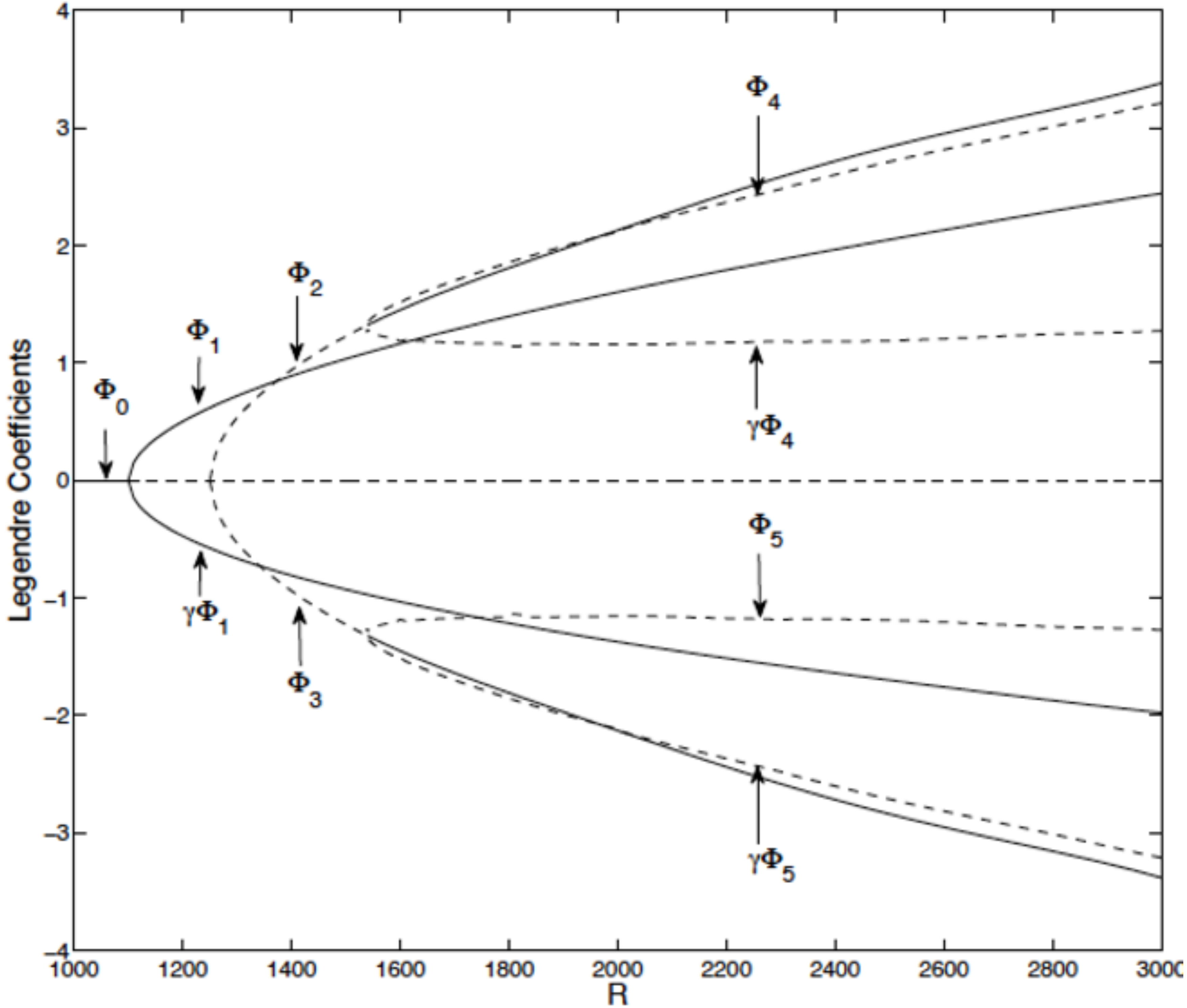
where β is the vertical temperature gradient.

Formulation of the problem

The equations governing the system are the incompressible Navier-Stokes equations with the Boussinesq approximation coupled with a heat equation



Bifurcation diagram



Stationary equations : Legendre collocation method

The Navier Stokes system with proper boundary conditions has a simple conductive solution $\mathbf{u}^c = \mathbf{0}$, $\theta^c = (1-z)/2$, $P^c = R(z-z^2/2)/2$. The stationary problem with the change of variables $\theta' = \theta - \theta^c$, $P' = P - P^c$ and dropping the primes to simplify notation, is the following,

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, && \text{in } \Omega, \\ R\,\theta \mathbf{e}_z - \nabla P + \Delta \mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} \cdot \nabla \theta - u_z &= \Delta \theta, && \text{in } \Omega. \end{aligned}$$

with the boundary conditions,

$$\begin{gathered} \mathbf{u} = \mathbf{0},\ \theta = 0 \ \text{ on } z = 0;\ \ \theta = \partial_z u_x = u_z = 0 \ \text{ on } z = 1, \\ \partial_x \theta = \partial_x u_z = u_x = 0, \ \text{ on } x = 0, \ \text{and on } x = \Gamma. \end{gathered}$$

The "standard" numerical method used here to solve it provided for different values of the Rayleigh number R is a Legendre spectral collocation method . The fields are expanded into Legendre polynomials, $U = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} L_i(x) L_j(z)$, where L_i is the Legendre polynomial of degree i.

Greedy procedure

We choose a value of the Rayleigh number that we name R_1, with its corresponding solution $\Phi(R_1)$, i.e. in this work it is the smallest value of R in the interval we consider. We normalize this stationary solution according to the L^2 scalar product:

$$\Psi_1 = \left(\psi_1^{\mathbf{u}} = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|_{L^2}}, \psi_1^{\theta} = \frac{\theta_1}{\|\theta_1\|_{L^2}}, \psi_1^{P} = \frac{P_1}{\|P_1\|_{L^2}}\right),$$

then we consider a first space $X_1 = \mathrm{span}\{\psi_1^{\mathbf{u}}\} \times \mathrm{span}\{\psi_1^{\theta}\} \times \mathrm{span}\{\psi_1^{P}\}$.

In this calculations we have considered expansions of order *n=35* in the *x*-direction and *m=13* in the *z*-direction.

We then choose R_2 where the Galerkin error when approaximated on

$$X_1 = \mathrm{span}\{\psi_1^{\mathbf{u}}\} \times \mathrm{span}\{\psi_1^{\theta}\} \times \mathrm{span}\{\psi_1^{P}\}$$

and the corresponding stationary solution is $\Phi(R_2)$. We orthonormalize both functions by Gram-Schmidt procedure in order to obtain a new Ψ_2 and we consider the second space $X_2 = \mathrm{span}\{\psi_1^{\mathbf{u}}, \psi_2^{\mathbf{u}}\} \times \mathrm{span}\{\psi_1^{\theta}, \psi_2^{\theta}\} \times \mathrm{span}\{\psi_1^{P}, \psi_2^{P}\}$.

An so on, until we reach a value $j = N < \mathrm{card}(\Xi_{trial})$ for which the stopping criterium $\epsilon_i^{(N)} \leq 10^{-7}$, $i = 1, 2$ is satisfied.

Greedy procedure

Therefore, we obtain the reduced basis $\{\Psi_1, \Psi_2, ..., \Psi_N\}$ and a corresponding discrete space $X_N \equiv X_N^{\mathbf{u}} \times X_N^{\theta} \times X_N^{P}$. For each branch we have constructed the reduced basis with solutions on that branch. We have constructed a reduced basis for each stable branche, i.e. for Φ_1 and $\gamma\Phi_1$. The branch for Φ_2 needs two reduced basis, one for the unstable part and another for the stable part. The same situation for Φ_3. The totally unstable branches need more care, and the interval has been divided in three parts, so that a reduced basis has been calculated in each part, i.e. $[1,539; 1,600]$, $[1,600; 2,000]$ and $[2,000; 3,000]$. Table 1 shows the number of snapshots used to calculate the reduced basis and the number of elements of the reduced basis in each branch of solutions.

Table 1: Number of snapshots in the trial set for the different reduced basis (RB) in each branch of solutions.

	Φ_1	Φ_2	Φ_2 (two sets)	Φ_4	Φ_4 (three sets)
$\#\Xi_{trial}$	23	22	29, 19	19	22, 26, 51
# RB	9	8	6, 6	10	6, 7, 7

Greedy procedure

Table 2: $\epsilon_1^{(j)}$, $\epsilon_2^{(j)}$, $j = 1, ..., N$ and the respective Rayleigh number R in which the maximum takes place for different dimensions j of the reduced basis space. R is in the interval $[1,102; 3,000]$ on the stable branch of solutions Φ_1.

j	$\epsilon_1^{(j)}$	$\epsilon_2^{(j)}$	R
1	0.809	0.236	1,102
2	0.085	0.0.009	3,000
3	0.008	0.001	1,500
4	0.003	$4.7{\cdot}10^{-4}$	2,200
5	$3.3{\cdot}10^{-4}$	$6.7{\cdot}10^{-5}$	1,110
6	$1.1{\cdot}10^{-4}$	$1.7{\cdot}10^{-5}$	2,700
7	$1.7{\cdot}10^{-5}$	$2.7{\cdot}10^{-6}$	1,900
8	$2.8{\cdot}10^{-6}$	$4.1 \cdot 10^{-7}$	1,800
9	$1.8{\cdot}10^{-7}$	$1.7 \cdot 10^{-8}$	1,300

Greedy procedure

Table 3: $\epsilon_1^{(j)}$, $\epsilon_2^{(j)}$, $j = 1, ..., N$ and the respective Rayleigh number R in which the maximum takes place for different dimensions j of the reduced basis space. R is in the interval $[1, 253; 3, 000]$ on the branch of solutions Φ_2.

j	$\epsilon_1^{(j)}$	$\epsilon_2^{(j)}$	R
1	0.748	0.588	1,253
2	0.056	0.015	3,000
3	0.007	0.001	1,600
4	0.002	$2.5{\cdot}10^{-4}$	2,200
5	$1.1{\cdot}10^{-4}$	$3.5{\cdot}10^{-5}$	1,300
6	$2.6{\cdot}10^{-5}$	$7.4{\cdot}10^{-6}$	2,700
7	$4.6{\cdot}10^{-6}$	$7.5{\cdot}10^{-7}$	1,400
8	$7.0{\cdot}10^{-7}$	$9.6 \cdot 10^{-8}$	1,260

Galerkin procedure for each branch

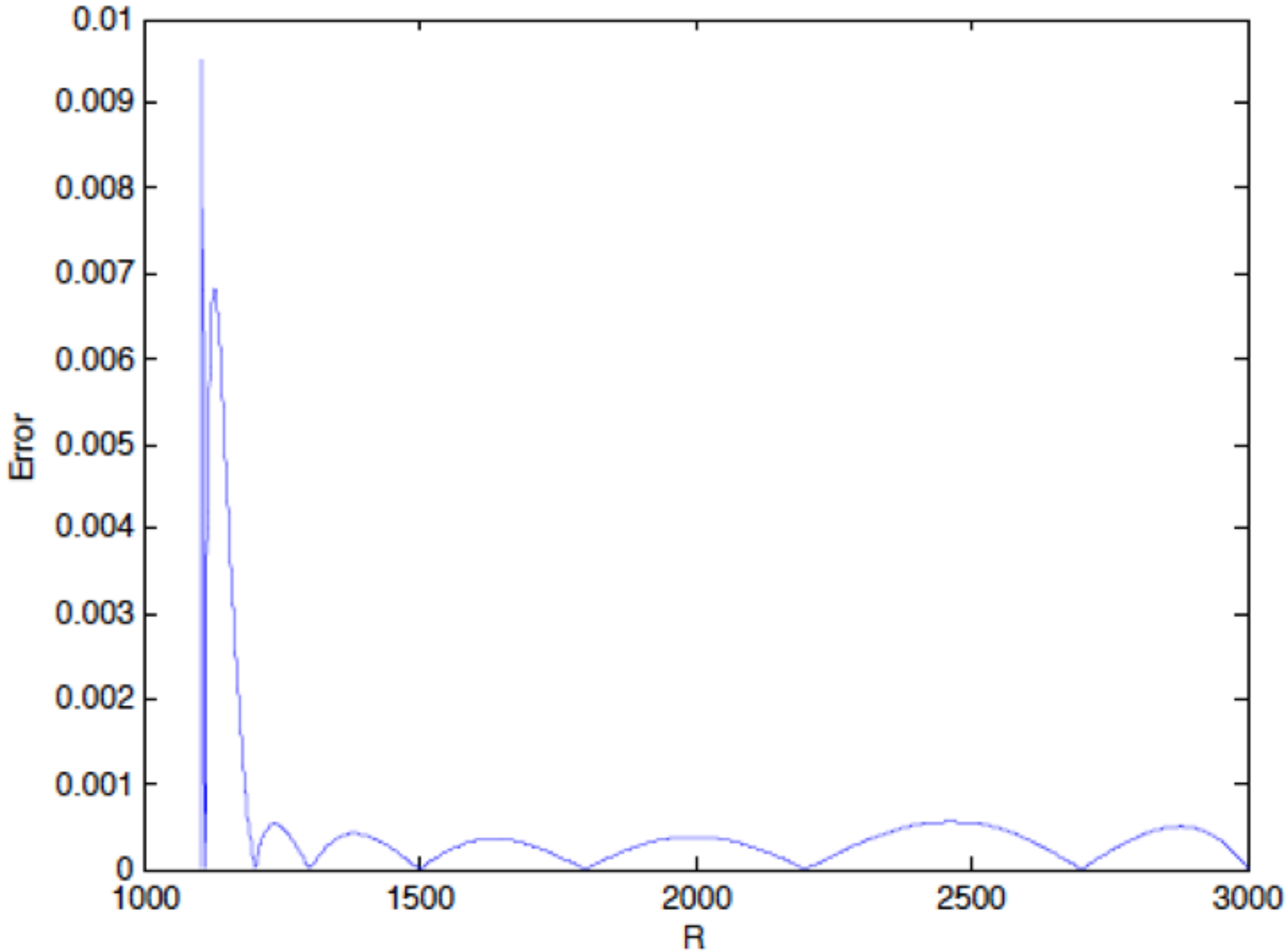


Figure 6: Norm of the difference between the stationary solution obtained with Legendre collocation and with a post-processed reduced basis method based on Legendre collocation for the stable branch Φ_1 in the interval of R [1,101; 3,000].

Galerkin procedure for each branch

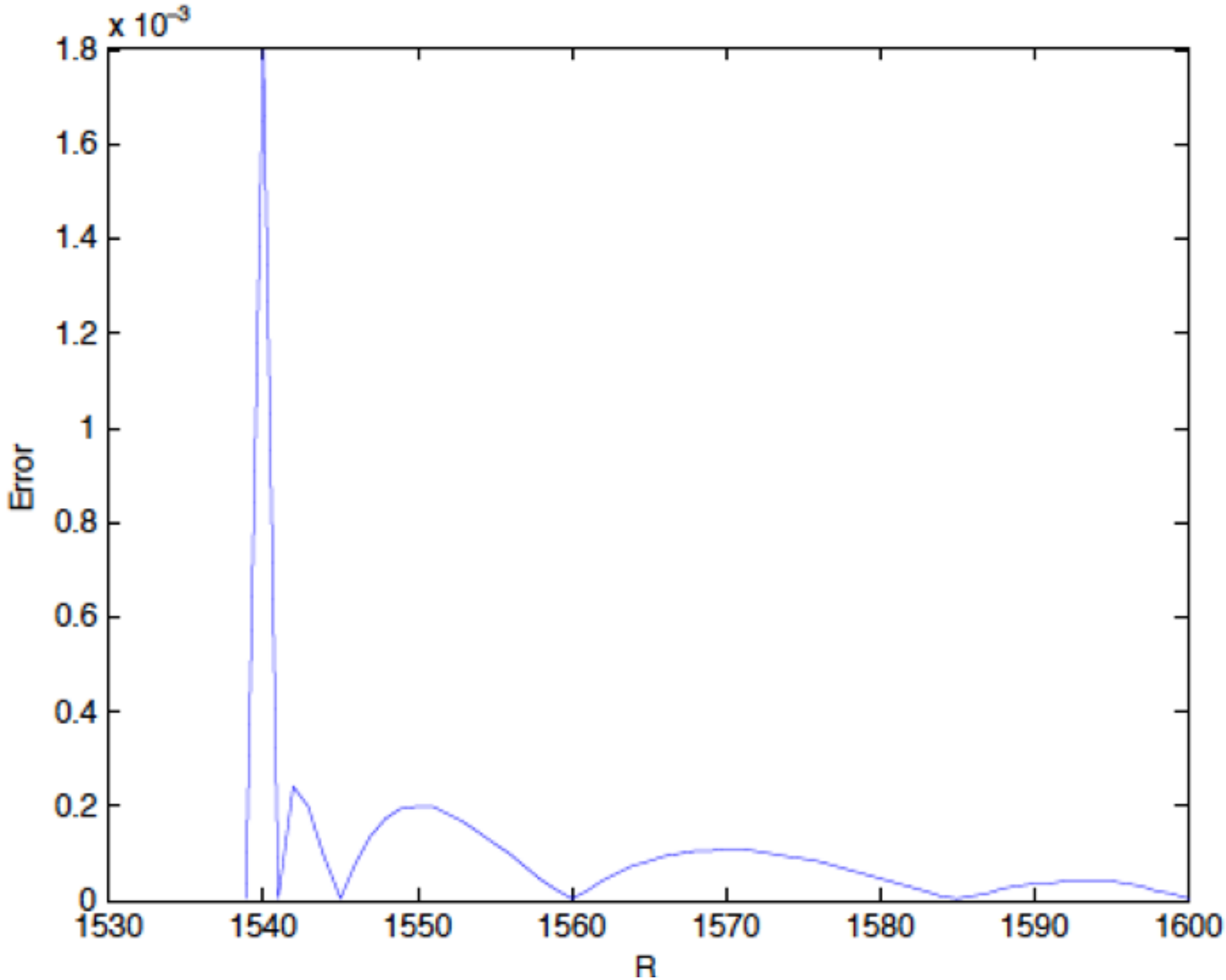


Figure 9: Norm of the difference between the stationary solution obtained with Legendre collocation and with a post-processed reduced basis method based on Legendre collocation for the first part of the unstable branch of Φ_4 in the interval of R [1,539; 1,600].

Galerkin procedure for each branch

The reduced basis method is supported by standard discretizations. The *off-line* work for the calculation of the solutions to construct the reduced basis needs these standard methods. But, once the work of the standard method is done, the use of the reduced basis has several advantages.

The size of the matrices after the discretization is very small. For a single value of the Rayleigh number R the size of the matrices that appear after the discretizations are 2,016 in Legendre collocation with expansions of order 13×35, whereas in the case of the reduced basis with 8 elements the size of matrices are 16. A factor of 126 in the size of the matrices for each value of R.

The behavior of the Newton method for the nonlinearity is improved with respect to standard methods. In the Legendre collocation method, for instance, we obtain the first solution in the branch in the interval $[1,101; 3,000]$ near $R = 1,101$. To obtain the solution at $R = 3,000$ we need to calculate the solution at $R = 1,102$, take this solution as initial guess for $R = 1,110$ and calculate the solutions increasing the value of R in steps of 10 till $R = 3,000$. Sometimes the steps of increase on R can be larger. So, it is not possible to jump from $R = 1,101$ till $R = 3,000$ with Legendre collocation.

In the reduced basis this is not the case, the solution can be directly calculated for any value of R. The reason for this behavior must be that nothing drive the solutions to be attracted by a different branch since there is not unexpected elements in the basis set.

This is reflected in the computational cost in time, it is 122 s for Legendre collocation and it is 6 s for reduced basis. Therefore the reduction is of a factor of 20 in time.

A new concept of reduced basis approximation for convection dominated problems

Nicolas Cagniart, Yvon Maday, Benjamin Stamm

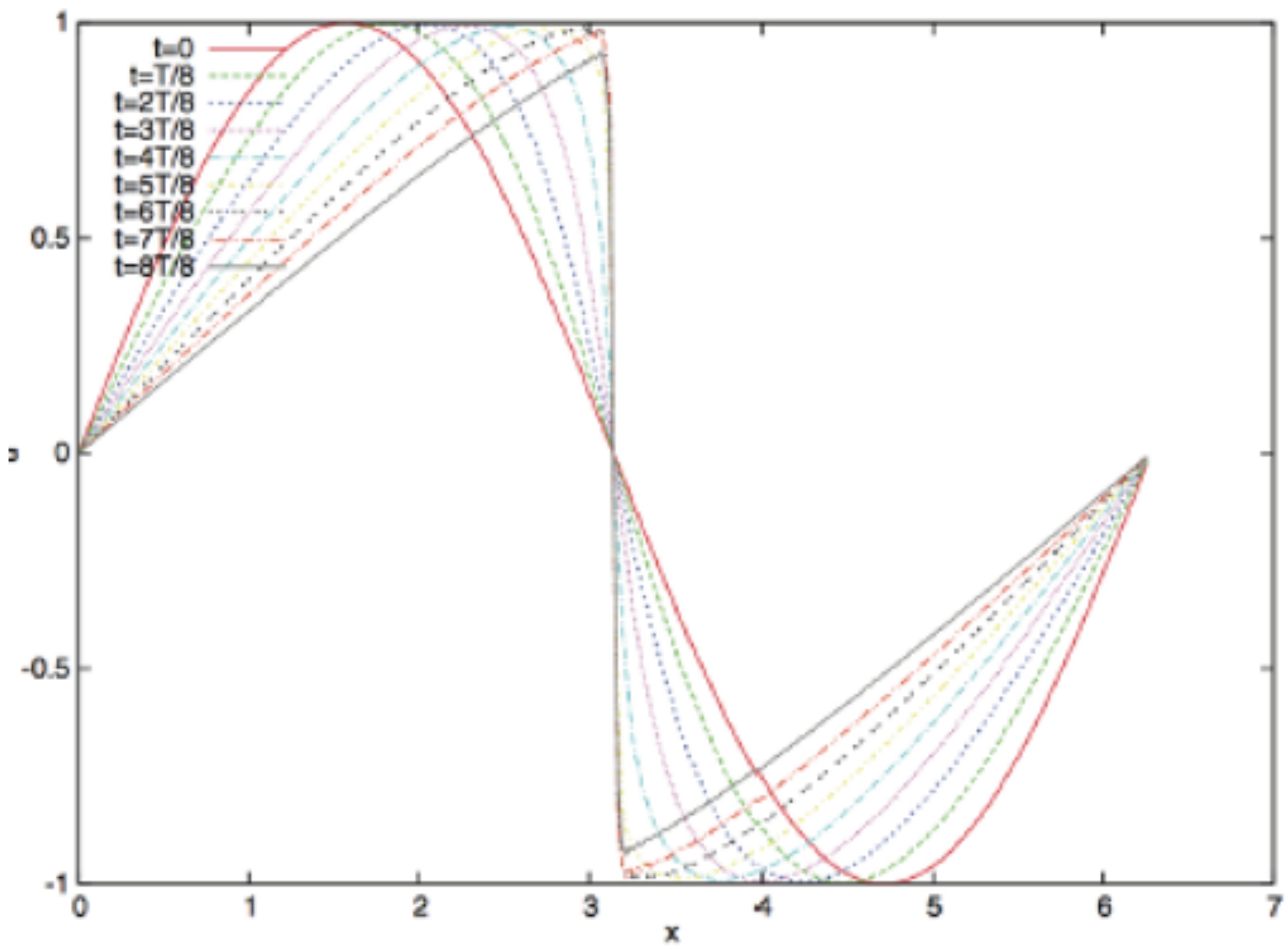
And if the Kolmogorov n-width is not small ?

Try to better look at the set of solutions

A case where the dimension is not small

1D viscous Burger equation

$$u_t + \nu u u_x - \epsilon u_{xx} = 0$$



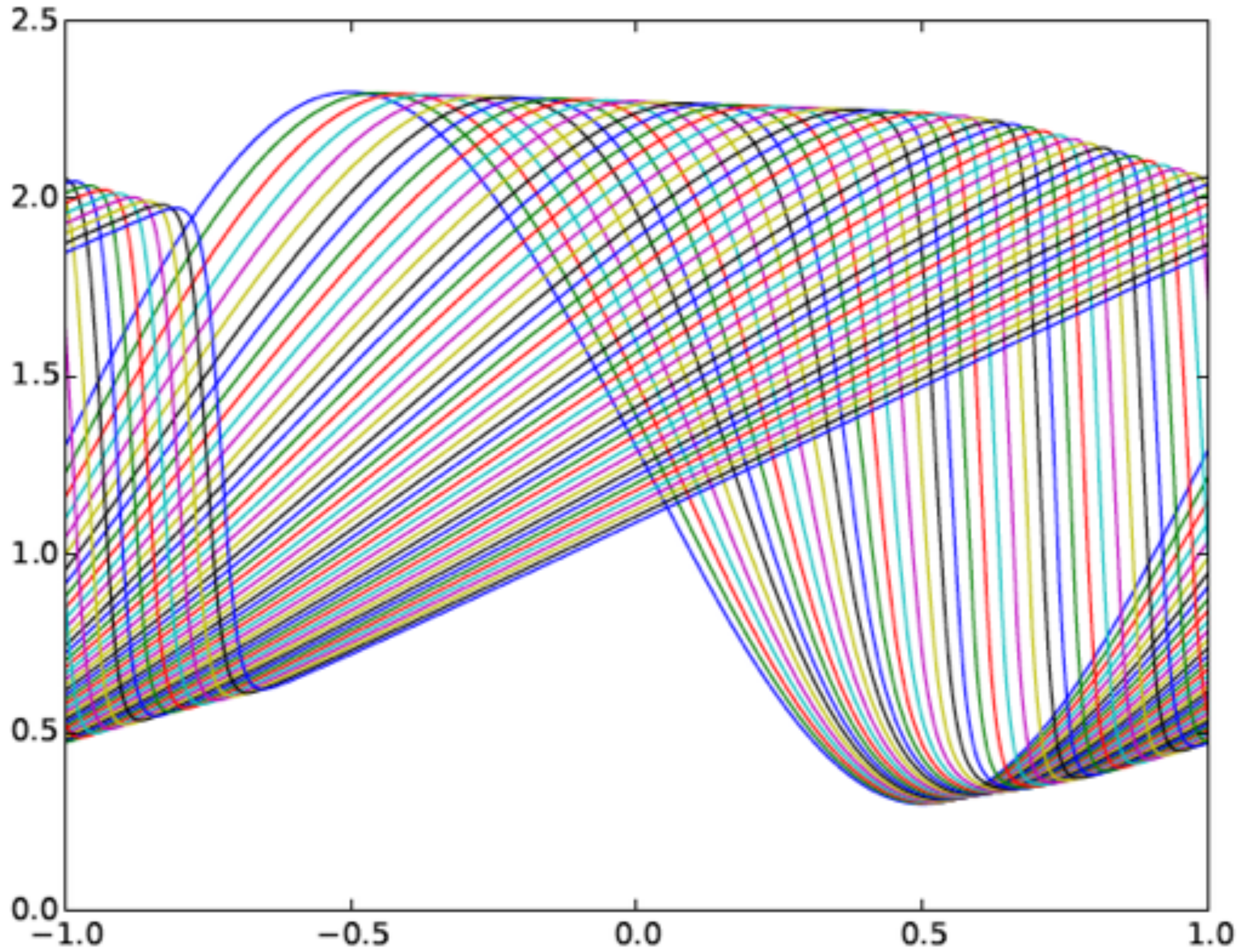
Snapshots of the solution to the unsteady viscous burger equation with

$$u_0 = sin(x), \nu = 4, \epsilon = 0.04$$

A case where the dimension is not small

1D viscous Burger equation

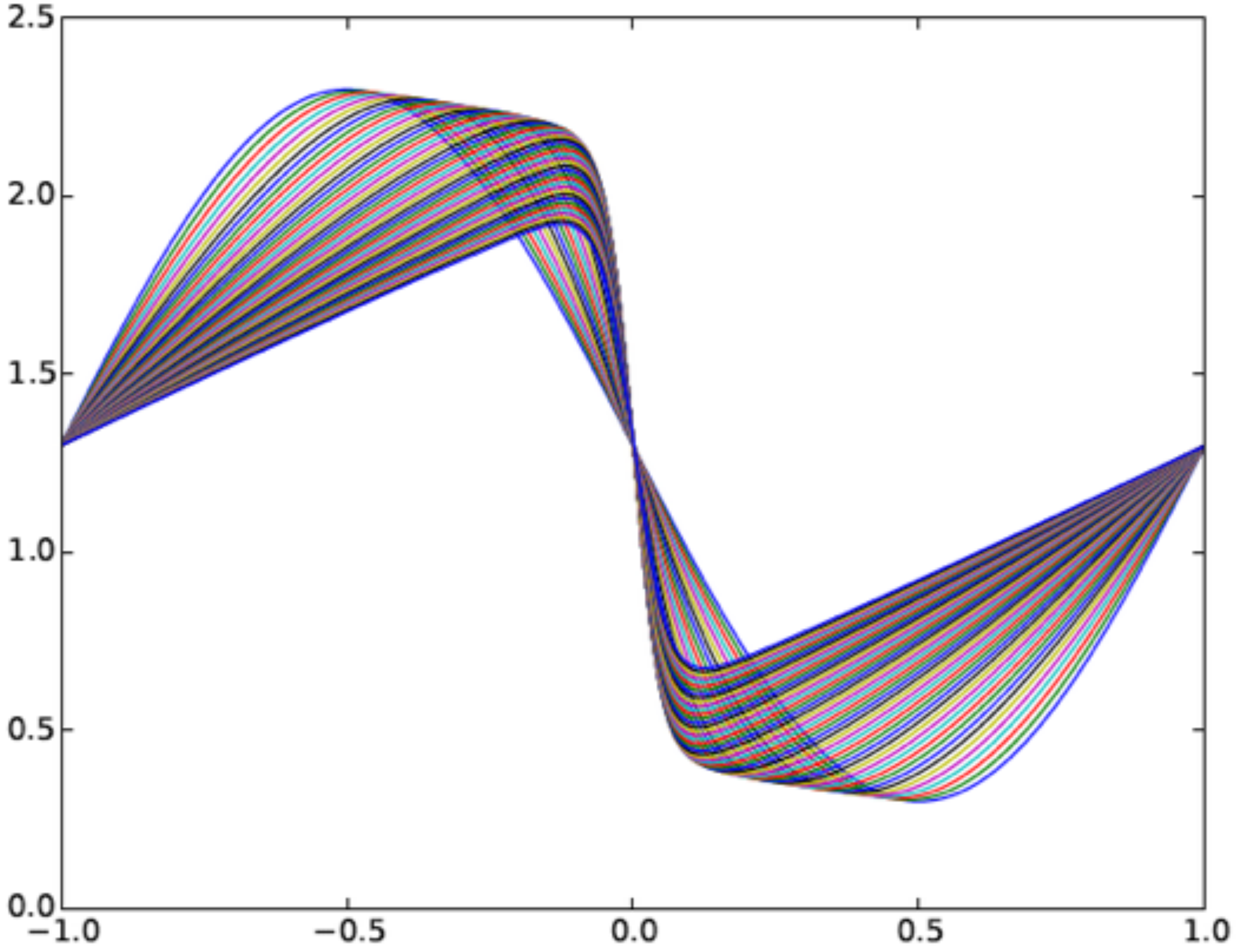
$$u_t + \nu u u_x - \epsilon u_{xx} = 0$$



Snapshots of the solution to the unsteady viscous burger equation with

$$u_0 = \lambda + sin(x), \nu = 4, \epsilon = 0.04$$

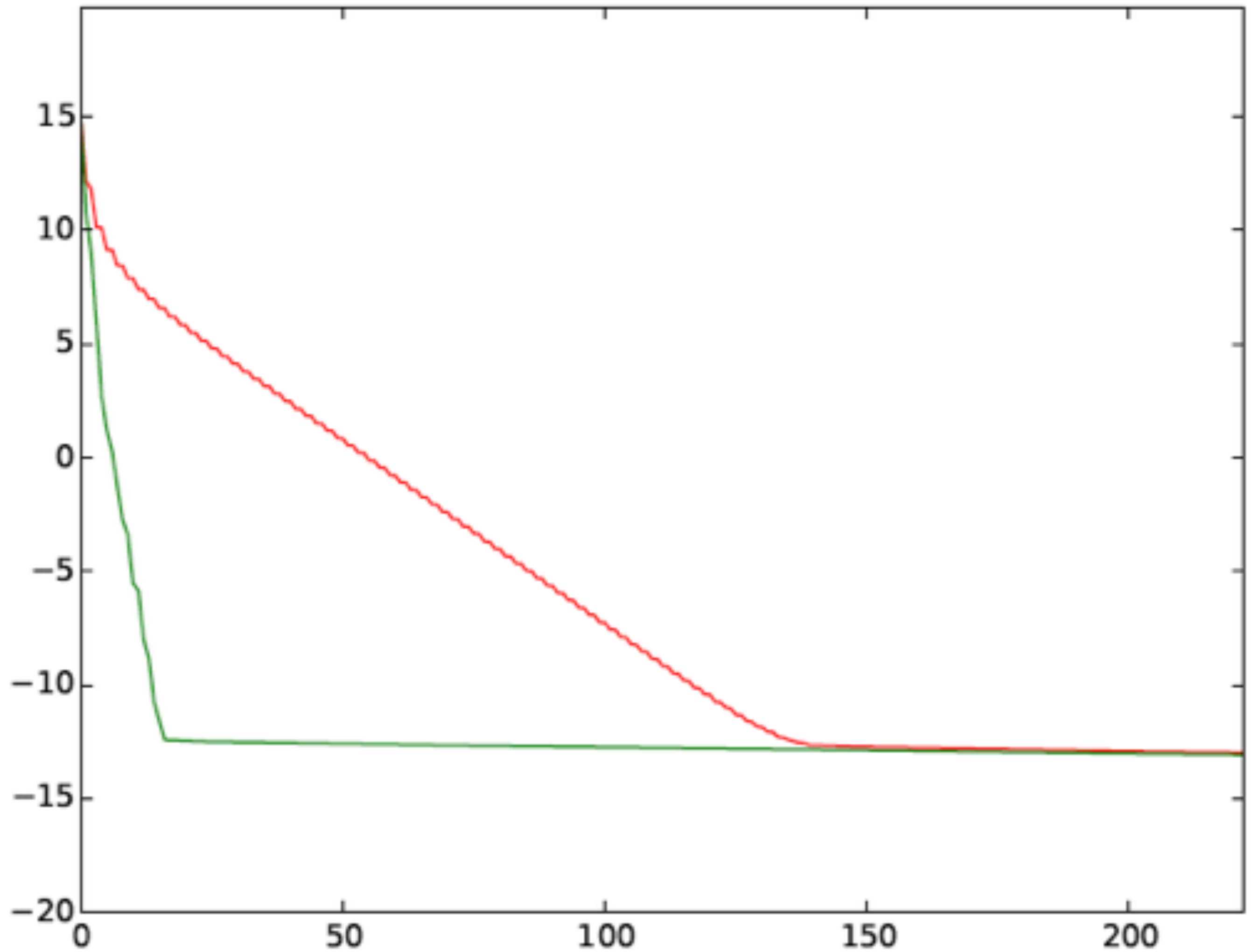
Nevertheless … all solutions look alike



what we have done above is to “center” the solutions

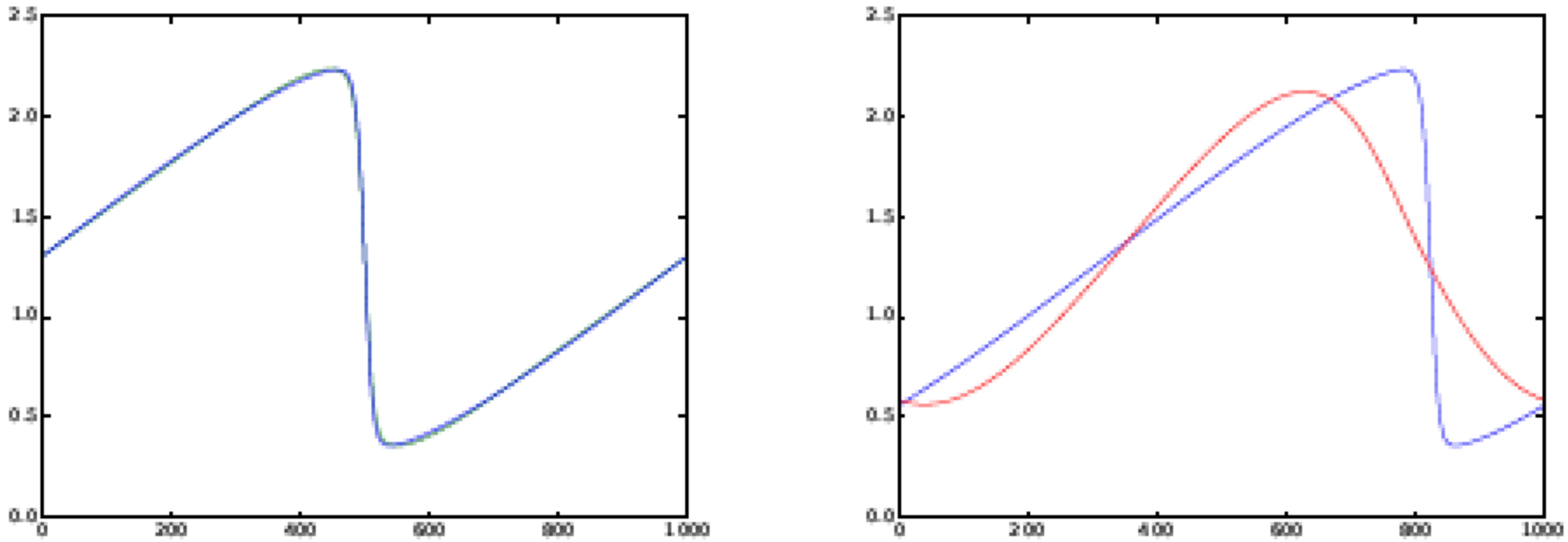
these are the $u(x - \gamma_n, t_n)$

and the dimension diminishes largely !!



Eigenvalues of the POD decomposition of the original set of snapshots (in red) and of the centered set of snapshots (in green)

and the dimension diminishes largely !!



<u>Reconstruction</u> of a snapshot (blue) using 3 POD modes. Left figure is in the centered case. Right figure is the uncentered case

The discrete scheme we want to mimic

$$u^{n+1}(\cdot,\mu) = u^n(\cdot,\mu) - dt\nu u^n(\cdot,\mu)u^n_x(\cdot,\mu) + dt\epsilon u^n_{xx}(\cdot,\mu)$$

leads to

At each time step, we are looking for a set of reduced coordinates $\{\alpha_i^n\}$ and a translation parameter γ^n such that our "true" solution $u^{\mathcal{N}}(\cdot,\mu,t^n)$ is well apprxoximated by :

$$u_N(\cdot,\mu,t^n) = \sum_i \alpha_i^n(\mu)\Phi_i(\cdot-\gamma^n) \tag{1}$$

We replace the search of the real γ and αs as in the usual galerkin method. That is, we are trying to minimize the residual.

$$\min_{\gamma^{n+1}\in\Omega} \min_{\alpha^{n+1}\in\mathbb{R}} \left\| \sum_i \alpha_i^{n+1}\Phi_i(\cdot-\gamma^{n+1}) - u^n - dt\nu u^n u^n_x + dt\epsilon u^n_{xx} \right\|_2 \tag{2}$$

Complexity .. how to compute and solve ??

We have assumed periodic boundary conditions (for the sake of simplicity) thus we need to compute online the following terms :

$$\begin{cases} \forall \Delta\gamma,\ \forall i,j, & \int_\Omega \Phi_i(\cdot - \Delta\gamma)\Phi_j(\cdot) \\ \forall \Delta\gamma,\ \forall i,j,p & \int_\Omega \Phi_i(\cdot - \Delta\gamma)\Phi_j(\cdot)(\Phi_p)_x(\cdot) \\ \forall \Delta\gamma,\ \forall i,j, & \int_\Omega (\Phi_i)_x(\cdot - \Delta\gamma)(\Phi_j)_x(\cdot) \end{cases}$$

for unknown value (to be optimized) of $\Delta\gamma$

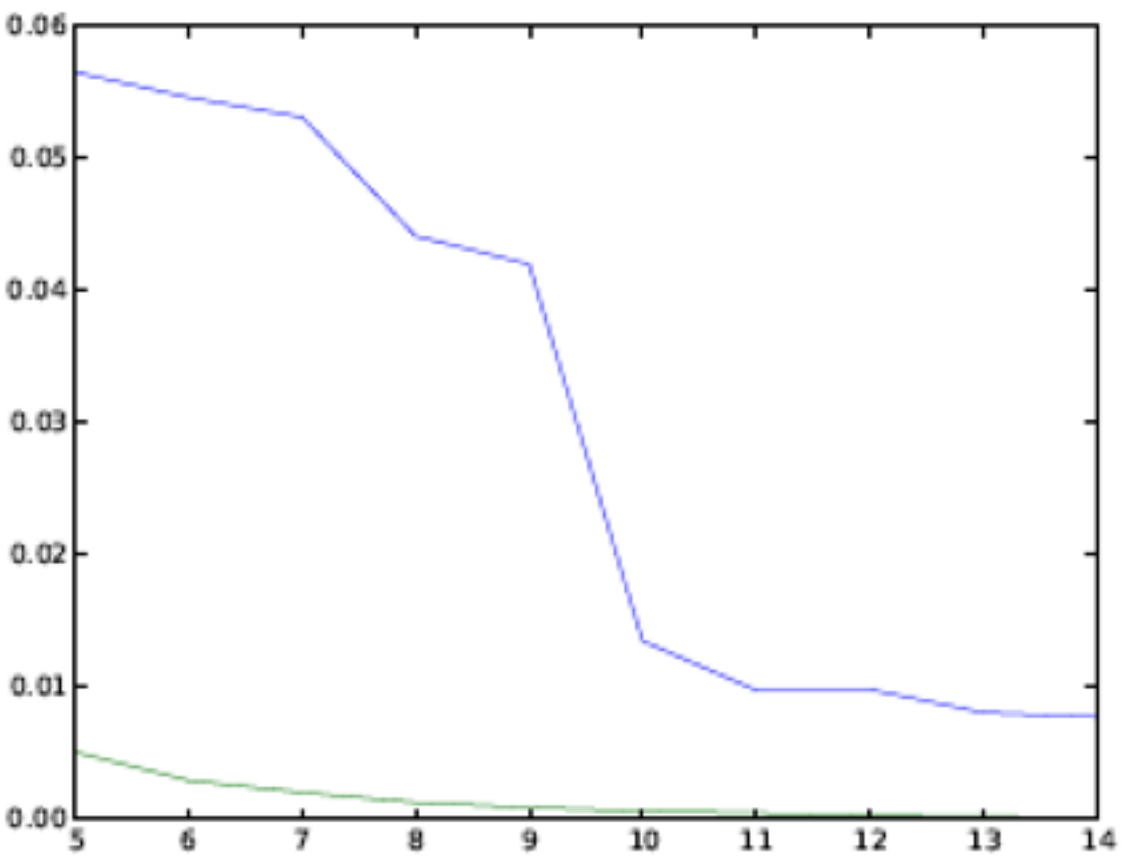
For a sufficiently small time step, we expect $\Delta\gamma$ to be of order $\delta t * c$ where c is some local characteristic velocity. We have chosen the following method:

- precompute the scalar products for a predefined small set of discrete values of $\Delta\gamma$ in $[-\delta t * c_{max}, \delta t * c_{max}]$, where c_{max} is the maximum expected shock speed during the simulation.

- using the regularity of the scalar product, we use spline interpolation to get and approximate value for any $\Delta\gamma$ in $[-\delta t * c_{max}, \delta t * c_{max}]$

The RB discrete scheme thus iterates between the evaluation of the proper best translation γ^{n+1} and the proper definition of the coefficient α_i^{n+1}

$$\sum_i \alpha_i^{n+1}\Phi_i(\cdot - \gamma^{n+1})$$



Mean reconstruction error w.r.t number of POD basis used

extensions .. what needs to be done

- non periodic (superposition of basic space and convection space)
- higher dimensions (POD representation of the "translations") see [1)]
- better fitting (add the derivatives of the POD functions)
- better fitting (replace least square with L^1 , see [2)] approximation)

1) Iollo, A., Lombardi, D.: Advection modes by optimal mass transfer. Physical Review E 89(2), 022923 (2014)

2) Technical paper: Roxana Crisovan, Rémi Abgrall, David Amsallem
Robust Model Reduction by L1-norm Minimization and Approximation via Dictionaries:
Application to Linear and Nonlinear Hyperbolic Problems