Fluid-Structure Interactions by Monolithic Methods

Olivier Pironneau[1] with Frédéric Hecht

[1]University of Paris VI (UPMC), Laboratoire J.-L. Lions (LJLL)
http://ann.jussieu.fr/pironneau

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My contribution: a robust and fast solver for parameter identification

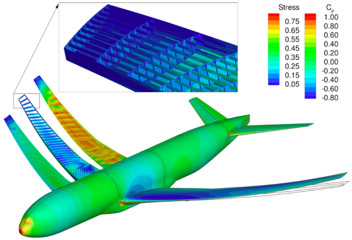
- My solution: Formulate the equation for the solid in the moving domain
- Tools: obtain an energy estimate
- Issue: powerful mesh generators turn out to be game changers.

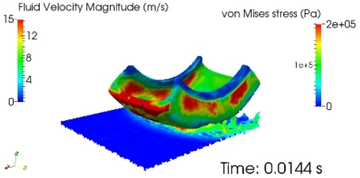
Part I: Introduction

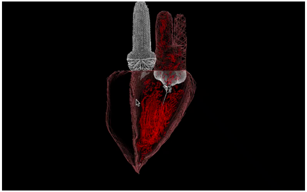
Who is Monolithic?

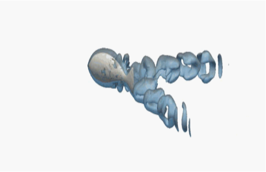
Johachim Martin et al (U of Michigan)

INSA - Lyon & Michelin









B. Griffith, C. Peskin et al.

M. Bergman and A. Iollo

Immersed Boundary Method (IBM) is Monolithic

- Lucia Gastaldi, Daniele Boffi & Nicola Cavallini!
- Solid $\mathcal{B}$ volume/surface/curve in a fluid Ω. $\delta\rho = \rho^s - \rho^f$.

Let $X(s,t)$ be the position at t of a point s at $t=0$ in the solid.

$$
\begin{aligned}
&\rho \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) \\
&\qquad - (\operatorname{div}\mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\mathbf{X}(\cdot,t))) = 0 && \forall \mathbf{v} \in H_0^1(\Omega)^d \\
&(\operatorname{div}\mathbf{u}(t), q) = 0 && \forall q \in L_0^2(\Omega) \\
&\delta\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2} \mathbf{Y}\, ds + \kappa \int_{\mathcal{B}} \nabla_s \mathbf{X} \nabla_s \mathbf{Y}\, ds \\
&\qquad - \mathbf{c}(\boldsymbol{\lambda}, \mathbf{Y}) = 0 && \forall \mathbf{Y} \in H^1(\mathcal{B})^d \\
&\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}(t)}{\partial t}\right) = 0 && \forall \boldsymbol{\mu} \in \Lambda
\end{aligned}
$$

- Existence, stability, convergence, stationary error estimate if $h_{\mathcal{B}} > C h_{\Omega}$
- Solid is sum of fluid + elastic material. Regularity of X?

arxiv.org/abs/1407.5184

Two Fluids by Level Sets is Monolithic

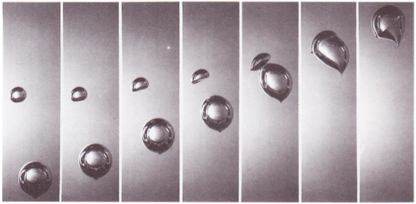
In the solid $(\rho^s, \mu^s) >> (\rho^f, \mu^f)$

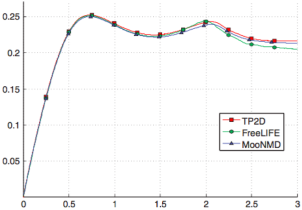
$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot(\mu \mathrm{D}\mathbf{u} - pI) = \rho\vec{g} + f, \quad \nabla\cdot\mathbf{u} = 0$$

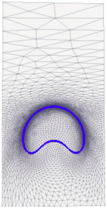
- The interface can be tracted by a level set $\partial_t\phi + \mathbf{u}\nabla\phi = 0$ and

$$\rho = \rho^f \mathbf{1}_{\{\phi(x)<-\epsilon\}} + \rho^s \mathbf{1}_{\{\phi(x)>\epsilon\}} + (\rho^s - \rho^f)(1 + \frac{\phi}{\epsilon} + \frac{1}{\pi}\sin\frac{\phi}{\epsilon})\mathbf{1}_{\{|\phi(x)|\le\epsilon\}}.$$

- At any interface there is continuity of velocity and normal stress built in.

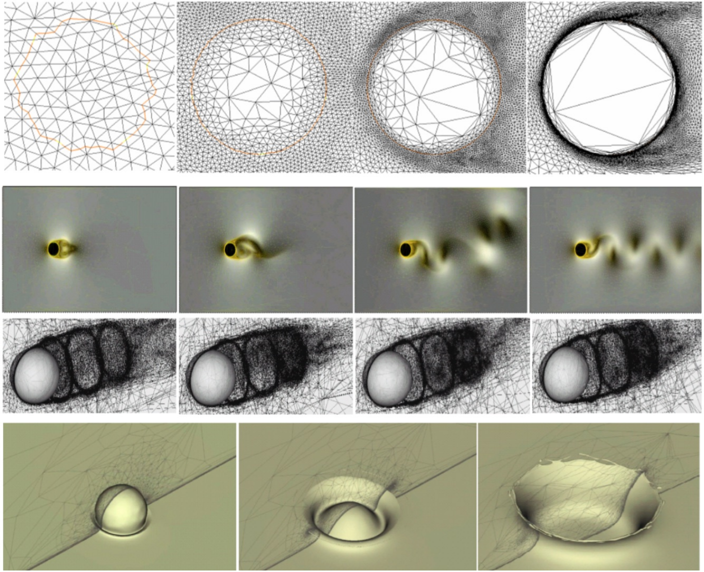






A gallery of fluid motions (2003) Velocity vs time of a rising bubble with $(\rho^s, \mu^s) = 1000(\rho^f, \mu^f)$ (Hysing-Yamaguchi-Otsuka-Marrouf-Th. Coupez)

Everything is in a Good Mesh (Thierry Coupez)



Math for the Two Fluids Case

Regularity

$$\mathbf{u} \in C^0(H^1_0 \cap W^{1,\infty}), \;\; \partial_t\phi + \mathbf{u}\nabla\phi = 0 \Rightarrow \phi \in C^0(L^2)$$

If $\phi \in L^4(W^{1,4})$ then $(\mathbf{u}, p)$ exists in $L^2(H^1_0) \cap C^0(L^2) \;\times\; L^2(L^2)$ and

$$\rho_\phi(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \nabla\cdot(\mu\mathrm{D}\mathbf{u} - pI) = \rho_\phi\vec{g}, \;\; \nabla\cdot\mathbf{u} = 0$$

- **No proof** (except $T << 1$ by G.H. Cottet et al.) for problems coupled by

$$\rho = \rho^f \mathbf{1}_{\{\phi(x)<-\epsilon\}} + \rho^s \mathbf{1}_{\{\phi(x)>\epsilon\}} + \ldots$$

- Marrouf-Bernardi show convergence of a Characteristic-Galerkin scheme + P^2/P^1 with error $O_\epsilon(h)$ and CFL $\delta t < C_\epsilon h$ if

$$\phi \in C^0(W^{2,\infty}), \;\; \mathbf{u} \in W^{1,\infty}(]0, T[\times\Omega) \cap H^2(L^2) \cap C^0(H^2), \;\; p \in L^\infty(H^2)$$

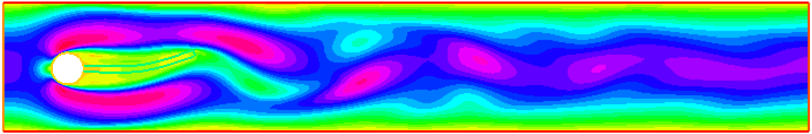
Part II: Eulerian Formulation

Eulerian Formulation

Result: Navier-Stokes and hyperelastic incompressible Mooney-Rivlin material in one formulation (2D): find $\mathbf{u}, p, \Omega_t$, such that for all $\hat{\mathbf{u}}, \hat{p}$

$$\boxed{\begin{aligned}&\int_{\Omega_t}[\rho\mathbb{D}_t\mathbf{u}\cdot\hat{\mathbf{u}} - p\nabla\cdot\hat{\mathbf{u}} - \hat{p}\nabla\cdot\mathbf{u}] + \int_{\Omega_t^f}\frac{\nu}{2}\mathrm{D}\mathbf{u}:\mathrm{D}\hat{\mathbf{u}}\\ &\quad + \int_{\Omega_t^s} c_1(\mathrm{D}\mathbf{d} - \nabla\mathbf{d}\nabla^t\mathbf{d}):\mathrm{D}\hat{\mathbf{u}} = \int_{\Omega_t} f\hat{\mathbf{u}}\end{aligned}}$$

$\mathbf{d}$ = solid displacement $\mathbb{D}_t\mathbf{d} := \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \mathbf{u}$ ⚠ and $\mathrm{D}u = \nabla\mathbf{u} + \nabla^t\mathbf{u}$



Open Movie See Dunne [1], Rannacher-Turek[2]

Notations and Conservation laws

- Ω^f_t, Ω^s_t fluid and solid regions at time t and $\Omega_t = \Omega^f_t \cup \Omega^s_t$.
- $x = \mathbf{X}(x_0, t)$ Lagrangian position at t of x_0;
- $\mathbf{d} = \mathbf{X}(x_0, t) - x_0, \quad \mathbf{u} = \partial_t \mathbf{X}, \quad \mathbf{F} = \nabla^t_{x_0} \mathbf{X} = ((\partial_{x_{0_j}} \mathbf{X}_i)), \quad J = \det_{\mathbf{F}}$
  Conservation of momentum $\rho \mathbb{D}_t \mathbf{u} = f + \nabla \cdot \sigma$
- σ = stress tensor, $\rho(x, t)$ = density.

  Mass conservation: $$\frac{d}{dt}(J\rho) = 0$$
- Incompressibility $\Rightarrow \ \nabla \cdot \mathbf{u} = 0, \ J = 1$ so if ρ_0 is piecewise constant $\Rightarrow$

$$\rho = \rho^s_0 \mathbf{1}_{\Omega^s_t} + \rho^f_0 \mathbf{1}_{\Omega^f_t}, \quad \rho \mathbb{D}_t \mathbf{u} = f + \nabla \cdot \sigma \ \text{ in } \ \Omega, \quad \Omega^s_t = \mathbf{X}(\Omega^s_0, t)$$

State law

- Incompressible flow $\sigma = -p^f \mathbf{I} + \mu^f \mathrm{D}\mathbf{u}$ with $\mathrm{D}\mathbf{u} := \nabla \mathbf{u} + \nabla^t \mathbf{u}$
- Hyperelastic incompressible solid $\sigma = -p^s \mathbf{I} + \partial_{\mathbf{F}} \Psi \mathbf{F}^T$ where Mooney-Rivlin Helmholtz potential: $\Psi(\mathbf{F}) = c_1 \mathrm{tr}_{\mathbf{F}^T\mathbf{F}} + c_2(\mathrm{tr}_{(\mathbf{F}^T\mathbf{F})^2} - \mathrm{tr}^2_{\mathbf{F}^T\mathbf{F}})$.

Details (Two Dimensional Only)

- **Proposition**:

For a Mooney-Rivlin 2D hyperelastic incompressible material, for some α, α',

$$\partial_{\mathbf{F}}\Psi\mathbf{F}^T = 2c_1(\mathbf{I}-\nabla\mathbf{d})^{-T}(\mathbf{I}-\nabla\mathbf{d})^{-1} + \alpha'\mathbf{I} = 2c_1(\mathrm{D}\mathbf{d} - \nabla\mathbf{d}\nabla^t\mathbf{d}) + \alpha\mathbf{I}$$

Proof

- $\mathrm{tr}_{\mathbf{F}^T\mathbf{F}} = \sum_{m,n} F^2_{m,n}$, hence $\partial_{\mathbf{F}}\mathrm{tr}_{\mathbf{F}^T\mathbf{F}} = 2\mathbf{F}$
- $\mathrm{tr}_{(\mathbf{F}^T\mathbf{F})^2} = \sum_{n,m,p,i} F_{n,i}F_{n,m}F_{p,m}F_{p,i}$, hence $\partial_{\mathbf{F}}\mathrm{tr}_{(\mathbf{F}^T\mathbf{F})^2} = 4\mathbf{F}\mathbf{F}^T\mathbf{F}$

$$\Psi(\mathbf{F}) = c_1\mathrm{tr}_{\mathbf{F}^T\mathbf{F}} + c_2(\mathrm{tr}_{(\mathbf{F}^T\mathbf{F})^2} - \mathrm{tr}^2_{\mathbf{F}^T\mathbf{F}}) \Rightarrow \partial_{\mathbf{F}}\Psi = 2c_1\mathbf{F} + c_2(4\mathbf{F}\mathbf{F}^T\mathbf{F} - 4\mathrm{tr}_{\mathbf{F}^T\mathbf{F}}\mathbf{F})$$

Let $B := \mathbf{F}\mathbf{F}^T$, $b :=\mathrm{det}_B$, $c := \mathrm{tr}_B = \mathrm{tr}_{\mathbf{F}^T\mathbf{F}}$, $\Rightarrow \partial_{\mathbf{F}}\Psi\mathbf{F}^T = (2c_1 - 4c_2c)B + 4c_2B^2$, where c_1, c_2 may depend on b, c. Now by Cayley-Hamilton: $B^2 = cB - b\mathbf{I}$ so

$$\partial_{\mathbf{F}}\Psi\mathbf{F}^T = 2c_1\mathbf{F}\mathbf{F}^T - 4c_2\mathrm{det}_{\mathbf{F}\mathbf{F}^T}\mathbf{I}$$

$$\mathbf{F}_{ji} = \partial_{x_{0i}}\mathbf{d}_j + \delta_{ij} = \partial_{x_{0i}}X_k\partial_{x_k}\mathbf{d}_j + \delta_{ij} = \mathbf{F}^T\nabla\mathbf{d} + \mathbf{I} \ \Rightarrow\ \mathbf{F} = (\mathbf{I}-\nabla\mathbf{d})^{-T}$$

Now, $B = c\mathbf{I} - bB^{-1} = c\mathbf{I} - b(\mathbf{I} - \nabla\mathbf{d} - \nabla^t\mathbf{d} + \nabla^t\mathbf{d}\nabla\mathbf{d})$ so

$$\begin{aligned}\partial_{\mathbf{F}}\Psi\mathbf{F}^T &= (2c_1(c-b) - 4c_2b)\mathbf{I} + 2c_1b(\nabla\mathbf{d} + \nabla^t\mathbf{d} - \nabla^t\mathbf{d}\nabla\mathbf{d})\\ &= 2c_1(\nabla\mathbf{d} + \nabla^t\mathbf{d} - \nabla^t\mathbf{d}\nabla\mathbf{d}) + \alpha\mathbf{I}\end{aligned}$$

Stability of the Continuous Problem

Proposition

$$\frac{d}{dt}\int_{\Omega_t}\frac{\rho}{2}|\mathbf{u}|^2+\frac{\nu}{2}\int_{\Omega_t^f}|\mathrm{D}\mathbf{u}|^2+\frac{d}{dt}\int_{\Omega_\mathbf{0}^s}\Psi(\mathbf{I}+\nabla_{x_\mathbf{0}}^T\mathbf{d})=\int_{\Omega_t}f\cdot\mathbf{u}$$

Proof. Recall the formulation

$$\int_{\Omega_t}\left[\rho\mathbb{D}_t\mathbf{u}\cdot\hat{\mathbf{u}}-p\nabla\cdot\hat{\mathbf{u}}-\hat{p}\nabla\cdot\mathbf{u}\right]+\int_{\Omega_t^f}\frac{\nu}{2}\mathrm{D}\mathbf{u}:\mathrm{D}\hat{\mathbf{u}}+\int_{\Omega_t^s}2c_1(\mathrm{D}\mathbf{d}-\nabla\mathbf{d}\nabla^t\mathbf{d}):\mathrm{D}\hat{\mathbf{u}}=\int_{\Omega_t}f\hat{\mathbf{u}}$$

Choosing $\hat{\mathbf{u}}=\mathbf{u}$, $\hat{p}=-p$ will give the proposition provided

$$2c_1\int_{\Omega_t^s}(\mathrm{D}\mathbf{d}-\nabla\mathbf{d}\nabla^t\mathbf{d}):\mathrm{D}(\mathbb{D}_t\mathbf{d})=\frac{d}{dt}\int_{\Omega_\mathbf{0}^s}\Psi(\nabla_{x_\mathbf{0}}\mathbf{X}).$$ By definition

$$\int_{\Omega_t^s}2c_1(\mathrm{D}\mathbf{d}-\nabla\mathbf{d}\nabla^t\mathbf{d}):\mathrm{D}\hat{\mathbf{u}}=\int_{\Omega_t^s}(\partial_\mathbf{F}\Psi(\mathbf{F})\mathbf{F}^T-\alpha\mathbf{I}):\nabla\hat{\mathbf{u}}=\int_{\Omega_\mathbf{0}^s}\partial_\mathbf{F}\Psi(\mathbf{F}):\nabla_{x_\mathbf{0}}\hat{\mathbf{u}}$$

Now as $\frac{d}{dt}\Psi(\mathbf{F})=\partial_\mathbf{F}\Psi(\mathbf{F}):\partial_t\mathbf{F}$ and $\mathbf{u}=\mathbb{D}_t\mathbf{d}\ \Rightarrow\ \nabla_{x_\mathbf{0}}\mathbf{u}=\nabla_{x_\mathbf{0}}\partial_t\mathbf{d}=\partial_t\mathbf{F}$,

$$\int_{\Omega_t^s}2c_1(\mathrm{D}\mathbf{d}-\nabla\mathbf{d}\nabla^t\mathbf{d}):\mathrm{D}\mathbf{u}=\int_{\Omega_\mathbf{0}^s}\frac{d}{dt}\Psi(\mathbf{F})=\frac{d}{dt}\int_{\Omega_\mathbf{0}^s}\Psi(\mathbf{I}+\nabla_{x_\mathbf{0}}\mathbf{d}^T).$$

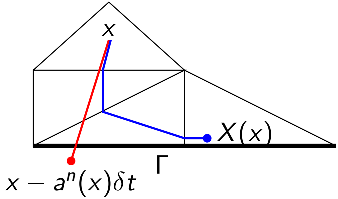
First order discretization of the total derivative:

$$(\partial_t u + a\cdot\nabla u)|_{x,(m+1)\delta t} = \frac{u^{n+1}(x) - u^n(x - a^n(x)\delta t)}{\delta t} + O(\delta t)$$

$$= \frac{u^{n+1} - u^n o \mathbb{Y}}{\delta t} + O(\delta t)$$

with $\mathbb{Y}(x) = \mathcal{X}_{a^n}(m\delta t)$ and

$$\frac{d\mathcal{X}}{d\tau}(\tau) = a^n(\mathcal{X}(\tau)), \quad \mathcal{X}((m+1)\delta t) = x$$



Second order approximation (Boukir-Maday-Metivet[1])

$$\left(\partial_t u + a\cdot\nabla u\right)|_{x,(m+1)\delta t} \approx \frac{3u^{n+1} - 4u^n o\mathbb{Y}_1^n + u^{n-1} o\mathbb{Y}_2^{n-1}}{2\delta t} + O(\delta t^2)$$

with $\mathbb{Y}_{k+1}^{n-k}(x) = \mathcal{X}_{a^{*\ m+\frac{1}{2}}}((m-k)\delta t),\ k = 0, 1$ and $a^{*\ m+\frac{1}{2}} = 2a^n - a^{n-1}$

O. Pironneau. On the Transport-Diffusion Algorithm and its Applications to the Navier-Stokes Eqs. Numer. Math., 38:309-312, 1982.

K. Boukir, Y. Maday, B. Metivet, A high order characteristics method for the incompressible Navier Stokes equations, Comp. Methods in Applied Mathematics and Engineering 116 (1994), 211-218

Time Discretization of the Eulerian Monolithic Formulation

Problem: Find $\mathbf{u}^{n+1} \in \mathbf{H}^1_0(\Omega_{n+1}), p \in L^2(\Omega_{n+1})$, $\Omega^r_{n+1} \subset \mathcal{R}^2$, $r = s, f$, such that $\Omega_{n+1} = \Omega^f_{n+1} \cup \Omega^s_{n+1}$,

$$\int_{\Omega_{n+1}} \Big[\rho_{n+1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n \circ \mathbb{Y}^{n+1}}{\delta t} \cdot \hat{\mathbf{u}} - p^{n+1} \nabla \cdot \hat{\mathbf{u}} - \hat{p} \nabla \cdot \mathbf{u}^{n+1}$$
$$+ \mathbf{1}_{\Omega^f_{n+1}} \frac{\mu^f}{2} \mathrm{D}\mathbf{u}^{n+1} : \mathrm{D}\hat{\mathbf{u}}$$
$$+ c_1 \mathbf{1}_{\Omega^s_{n+1}} J^{-1}_{n+1} [(\mathbf{I} - \nabla \mathbf{d}^{n+1})^{-T} (\mathbf{I} - \nabla \mathbf{d}^{n+1})^{-1}] : \mathrm{D}\hat{\mathbf{u}}\Big] = \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}},$$

$$\Omega_{n+1} = (\mathbb{Y}^{n+1})^{-1}(\Omega_n) = \{x : \ \mathbb{Y}^{n+1}(x) := x - \mathbf{u}^{n+1}(x)\delta t \in \Omega_n\}$$
$$\mathbf{d}^{n+1} = \mathbf{d}^n \circ \mathbb{Y}^{n+1} + \delta t \mathbf{u}^{n+1}, \quad J^{-1}_{n+1} = \det{}_{\mathbf{I} - \nabla \mathbf{d}^{n+1}}$$

$\forall \hat{\mathbf{u}} \in \mathbf{H}^1_0(\Omega_{n+1})$, $\forall \hat{p} \in L^2(\Omega_{n+1})$

Fixed Point Loop

1. Set $\rho=\rho_n,\ \Omega=\Omega_n, \mathbf{u}=\mathbf{u}^n, \mathbb{Y}(x)=x-\mathbf{u}\delta t, b=b_n, c=c_n$
2. Solve

$$\int_\Omega [\rho\frac{\mathbf{u}^{n+1}-\mathbf{u}^n\circ\mathbb{Y}}{\delta t}\cdot\hat{\mathbf{u}}-p^{n+1}\nabla\cdot\hat{\mathbf{u}}-\hat{p}\nabla\cdot\mathbf{u}^{n+1}]+\int_{\Omega^f}\frac{\mu^f}{2}\mathrm{D}\mathbf{u}^{n+1}:\mathrm{D}\hat{\mathbf{u}}$$
$$+\int_{\Omega^s}\delta t\Big[c_1(\mathrm{D}\mathbf{u}^{n+1}-\nabla\tilde{\mathbf{d}}^n\nabla^T\mathbf{u}^{n+1}-\nabla\mathbf{u}^{n+1}\nabla^T\tilde{\mathbf{d}}^n+\delta t\nabla\mathbf{u}\nabla^T\mathbf{u}):\mathrm{D}\hat{\mathbf{u}}\Big]$$
$$+\int_{\Omega^s}\Big[c_1(\mathrm{D}\tilde{\mathbf{d}}^n-\nabla\tilde{\mathbf{d}}^n\nabla^T\tilde{\mathbf{d}}^n):\mathrm{D}\hat{\mathbf{u}}\Big]=\int_\Omega f\cdot\hat{\mathbf{u}}$$

3. Set $\mathbf{u}=\mathbf{u}^{n+1}, \mathbb{Y}(x)=x-\mathbf{u}\delta t, \Omega^r=\mathbb{Y}^{-1}(\Omega^r_n),\ r=s,f$, update ρ by Ω.
4. If not converged return to Step 2.

From the definition of $\mathbf{d}^{n+1}$ we have

$$\nabla\mathbf{d}^{n+1}=\nabla\mathbb{Y}^{n+1}\nabla\mathbf{d}^n\circ\mathbb{Y}^{n+1}+\nabla\mathbf{u}^{n+1}\delta t=(\mathbf{I}-\delta t\nabla\mathbf{u}^{n+1})\nabla\mathbf{d}^n\circ\mathbb{Y}^{n+1}+\nabla\mathbf{u}^{n+1}\delta t$$

Hence

$$\mathbf{I}-\nabla\mathbf{d}^{n+1}=(\mathbf{I}-\nabla\mathbf{u}^{n+1}\delta t)(\mathbf{I}-\nabla\tilde{\mathbf{d}}^n)$$

The identity (in 2D only) $(\mathbf{I}-\nabla\mathbf{u}^{n+1}\delta t)^{-1}=\mathbb{J}^{-1}_{n+1}(\mathbf{I}+\nabla\mathbf{u}^{n+1}\delta t)$ completes the proof

Spatial Discretization with Finite Elements

Discretization in space by the Finite Element Method leads to find $\mathbf{u}_h^{n+1}, p_h^{n+1} \in V_{0h} \times Q_h$ such that for all $\hat{\mathbf{u}}_h, \hat{p}_h \in V_{0h} \times Q_h$,

$$\int_{\Omega_{n+1}} \Big[\rho_{n+1} \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n \circ \mathbb{Y}^{n+1}}{\delta t} \cdot \hat{\mathbf{u}}_h - p_h^{n+1} \nabla \cdot \hat{\mathbf{u}}_h - \hat{p}_h \nabla \cdot \mathbf{u}_h^{n+1}$$
$$+ \mathbf{1}_{\Omega^f_{n+1}} \frac{\mu^f}{2} \mathrm{D}\mathbf{u}^{n+1} : \mathrm{D}\hat{\mathbf{u}}$$
$$+ c_1 \mathbf{1}_{\Omega^s_{n+1}} [\mathrm{D}(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1}) - \nabla^T(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1}) \nabla(\tilde{\mathbf{d}}^n + \delta t \mathbf{u}^{n+1})] : \mathrm{D}\hat{\mathbf{u}}\Big]$$
$$= \int_{\Omega_{n+1}} \mathbf{f} \cdot \hat{\mathbf{u}}_h, , \quad \Omega_{n+1} = (\mathbb{Y}^{n+1})^{-1}(\Omega_n) = \{x : \ \mathbb{Y}^{n+1}(x) \in \Omega_n\}$$

with $\mathbf{d}_h$ updated by $\mathbf{d}_h^{n+1} = \tilde{\mathbf{d}}_h^n + \delta t \mathbf{u}_h^{n+1}$ where $\tilde{\mathbf{d}}_h^n = \mathbf{d}_h^n \circ \mathbb{Y}^{n+1}$ and where

$$\mathbb{Y}^{n+1}(x) = x - \mathbf{u}_h^{n+1}(x)\delta t$$

The proof for conservation of energy in the spatially continuous case will work for the discrete case if

$$\mathbf{X}^n = \mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}.$$

This means that $\mathbf{d}[i] = \mathbf{d}|_{q_i} \ \Rightarrow \ \mathbf{d}^{n+1}[i] = \mathbf{d}^n[i] + \mathbf{u}_h^{n+1}[i]\delta t.$

Implementation with FreeFem++

```
fespace Wh(th,[P2,P2,P1,P1]);
Wh [u,v,p,pp],[uh,vh,ph,pph];

macro div(u,v) ( dx(u)+dy(v) ) // EOM
macro DD(u,v)  [[2*dx(u),div(v,u)],[div(v,u),2*dy(v)]] // EOM
macro Grad(u,v)[[dx(u),dy(u)],[dx(v),dy(v)]] // EOM

problem aa([u,v,p,pp],[uh,vh,ph,pph]) =
    int2d(th,beam)( rhos*[u,v]'*[uh,vh]/dt - div(uh,vh)*pp  - div(u,v)*pph
                + penal*pp*pph+ penal*p*ph
        +dt*c1*trace(DD(uh,vh)*(DD(u,v)  -Grad(u,v)*Grad(d1,d2)' - Grad(d1,d2)*Grad(u,v)')))
    + int2d(th,beam) ( g*vh*rhos+c1*trace(DD(uh,vh)*(DD(d1,d2) - Grad(d1,d2)*Grad(d1,d2)'))
    -  rhos*[usold,vsold]'*[uh,vh]/dt )
    +  int2d(th,fluid)(rhof*[u,v]'*[uh,vh]/dt- div(uh,vh)*p -div(u,v)*ph
                + penal*p*ph + penal*pp*pph
    + nu/2*trace(DD(uh,vh)'*DD(u,v)))
    -  int2d(th,fluid)(-g*vh*rhof+rhof
        *[convect([uold,vold],-dt,uold),convect([uold,vold],-dt,vold)]'*[uh,vh]/dt)
    + int1d(th,2)(g*uh*y*rhof) + on(1,4, u=0,v=0) +  on(3,u=Ubar*y*(H-y)*6/H/H,v=0) ;
```

Parallel Implementation with FreeFem++

```
int L2=(mpirank==0)?2:-1, beammpi = mpirank*2+1, fluidmpi=mpirank*2;
varf GStokesl([u,v,p,pp],[uh,vh,ph,pph]) =
        int2d(th,beammpi)( rhos*[u,v]'*[uh,vh]/dt - div(uh,vh)*pp  - div(u,v)*pph
                   + penal*pp*pph+ penal*p*ph
           +dt*c1*trace(DD(uh,vh)*(DD(u,v)-Grad(u,v)*Grad(d1,d2)'-Grad(d1,d2)*Grad(u,v)')))
        +  int2d(th,fluidmpi)(rhof*[u,v]'*[uh,vh]/dt- div(uh,vh)*p -div(u,v)*ph
           + penal*p*ph + penal*pp*pph + nu/2*trace(DD(uh,vh)'*DD(u,v)))
        + on(1,4,3, u=0,v=0) ;

varf RHS([u,v,p,pp],[uh,vh,ph,pph]) =
 int2d(th,beammpi) ( -g*vh*rhos - c1*trace(DD(uh,vh)*(DD(d1,d2) - Grad(d1,d2)*Grad(d1,d2)'))
                   + rhos*[usold,vsold]'*[uh,vh]/dt )
+  int2d(th,fluidmpi)(-g*vh*rhof+rhof
    *[convect([uold,vold],-dt,uold),convect([uold,vold],-dt,vold)]'*[uh,vh]/dt)
 + int1d(th,L2)(g*uh*y*rhof)
+ on(1,4, u=0,v=0) +  on(3,u=Ubar*y*(H-y)*6/H/H,v=0) ;
```

Parallel Implementation with FreeFem++

```
mesh thold;
for(int n=1;n<NN;n++){
    thold=th;
    thsold=ths;
    dd1=d1;dd2=d2;
    {
        Wh [w1,w2,wp,wpp];
        if(mpirank==0){
            int[int] nupart(th.nt);
            nupart=0;
            if(mpisize>1)
                scotch(nupart, th, mpisize);
            th=change(th,fregion= nupart[nuTriangle]*2+(region==beam));
        }
        broadcast(processor(0),th);
        matrix Al=GStokesl(Wh,Wh,solver=sparsesolver,master=-1);
        real[int] bl=RHS(0,Wh);
        w1[]=Al^-1*bl;
        [u,v,p,pp]=[w1,w2,wp,wpp];
    }
    th=thold;
```

The Turek-Dunne-Rannacher Test Case

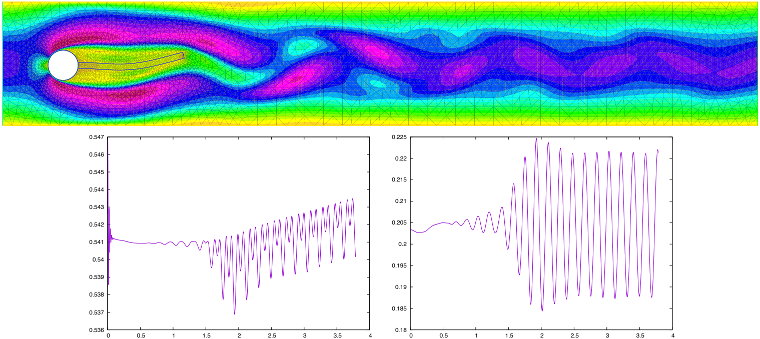


Figure: *On the left (resp. right) the x (resp. y) position of the upper right corner of the flagella versus time. The frequency is around* $5.4s^{-1}$ *while* $6.7s^{-1}$ *in [?] and the amplitude around 0.017 compared to 0.013 in [?]. The mesh has 2500 vertices and the time step is 0.005.*

RUN

Stability Estimate of the Time Discretized Problem

Proposition The Lagrangian map $\mathbf{X}^n : \Omega_0 \mapsto \Omega_n$ satisfies $\mathbf{X}^{n+1} = (\mathbb{Y}^{n+1})^{-1} \circ \mathbf{X}^n$, $n \geq 1$ and $\mathbf{F}^n := \nabla_{x_0}^T \mathbf{X}^n = (\mathbf{I} - \nabla \mathbf{d}^n)^{-T}$. *Proof*
Notice that $\mathbb{Y}^1(\mathbb{Y}^2(..\mathbb{Y}^{n-1}(\mathbb{Y}^n(\Omega_n))..)) = \Omega_0$ Hence

$$\mathbf{X}^{n+1} = [\mathbb{Y}^1(\mathbb{Y}^2(..\mathbb{Y}^n(\mathbb{Y}^{n+1})))]^{-1} = (\mathbb{Y}^{n+1})^{-1} \circ \mathbf{X}^n.$$

By definition of $\mathbf{d}^{n+1}$ in (1),

$$\begin{aligned} \mathbf{d}^{n+1}(\mathbf{X}^{n+1}(x_0)) &= \mathbf{d}^n(\mathbb{Y}^{n+1}(\mathbf{X}^{n+1}(x_0))) + \mathbf{u}^{n+1}(\mathbf{X}^{n+1}(x_0))\delta t \\ &= \mathbf{d}^n(\mathbf{X}^n(x_0)) + \mathbf{u}^{n+1}(\mathbf{X}^{n+1}(x_0))\delta t, \end{aligned}$$

so $\mathbf{X}^{n+1}(x_0) = \mathbf{d}^{n+1}(\mathbf{X}^{n+1}(x_0)) + x_0$ and therefore

$$\begin{aligned} \mathbf{F}^{n+1} &= \nabla_{x_0}^T(\mathbf{d}^{n+1}((\mathbf{X}^{n+1}(x_0))) + x_0), \\ &= \nabla \mathbf{d}^{n+1\,T} \mathbf{F}^{n+1} + \mathbf{I} \;\Rightarrow\; \mathbf{F}^{n+1} = (\mathbf{I} - \nabla \mathbf{d}^{n+1})^{-T} \end{aligned}$$

Stability Estimate (II)

Lemma

$$\int_{\Omega^s_{n+1}} c_1[J_{n+1}^{-1}[(\mathbf{I}-\nabla \mathbf{d}^{n+1})^{-T}(\mathbf{I}-\nabla \mathbf{d}^{n+1})^{-1}] : \mathrm{D}\hat{\mathbf{u}} = \int_{\Omega^s_{\mathbf{0}}} \partial_{\mathbf{F}}\Psi^{n+1} : \nabla_{x_\mathbf{0}} \hat{\mathbf{u}}$$

Proof

From Proposition 21,

$$\begin{aligned}
\int_{\Omega^s_{n+1}} & c_1[J_{n+1}^{-1}[(\mathbf{I}-\nabla \mathbf{d}^{n+1})^{-T}(\mathbf{I}-\nabla \mathbf{d}^{n+1})^{-1}] : \mathrm{D}\hat{\mathbf{u}} \\
&= \int_{\Omega^s_{n+1}} c_1 J_{n+1}^{-1}[\mathbf{F}^{n+1}{\mathbf{F}^{n+1}}^T] : \mathrm{D}\hat{\mathbf{u}} \\
&= \int_{\Omega^s_{\mathbf{0}}} c_1 \mathbf{F}^{n+1} : \mathrm{D}_{x_\mathbf{0}}\hat{\mathbf{u}} = \frac{1}{2}\int_{\Omega^s_{\mathbf{0}}} \partial_{\mathbf{F}}\Psi^{n+1} : \mathrm{D}_{x_\mathbf{0}}\hat{\mathbf{u}} \qquad (1)
\end{aligned}$$

Stability Estimate (III)

Theorem

When $f = 0$ and ρ is constant in each domain $\Omega_n^{s,f}$,

$$\int_{\Omega_n} \frac{\rho^n}{2}|\mathbf{u}^n|^2 + \delta t \sum_{k=1}^{n} \int_{\Omega_k^f} \frac{\nu}{2}|\mathrm{D}\mathbf{u}^k|^2 + \int_{\Omega_0^s} \Psi^n \leq \int_{\Omega_0} \frac{\rho^0}{2}|\mathbf{u}^0|^2 + \int_{\Omega_0^s} \Psi^0$$

Proof Let $r = s$ or f. Let us choose $\hat{\mathbf{u}} = \mathbf{u}^{n+1}$. By Schwartz inequality

$$\int_{\Omega_{n+1}^r} (\rho_n^r \mathbf{u}^n) \circ \mathbb{Y}^{n+1} \cdot \mathbf{u}^{n+1} \leq \left(\int_{\Omega_{n+1}^r} (\sqrt{\rho}_n^r \mathbf{u}^n)^2 \circ \mathbb{Y}^{n+1})\right)^{\frac{1}{2}} \left(\int_{\Omega_{n+1}^r} \rho_{n+1}^r {\mathbf{u}^{n+1}}^2\right)^{\frac{1}{2}}$$

because $\rho_n^r \circ \mathbb{Y}^{n+1}(x) = \rho_{n+1}^r(x),\ x \in \Omega_{n+1}^r$, so by a change of variable

$$\int_{\Omega_{n+1}^r} \rho_{n+1}^r (\mathbf{u}^n \circ \mathbb{Y}^{n+1})^2 = \int_{\Omega_{n+1}^r} (\sqrt{\rho_n^r}\mathbf{u}^n)^2 \circ \mathbb{Y}^{n+1} = \int_{\Omega_n^r} \rho_n^r {\mathbf{u}^n}^2.$$

Consequently, using $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$,

Conservation of Energy

We need $\mathbf{X}^n = \mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}$.

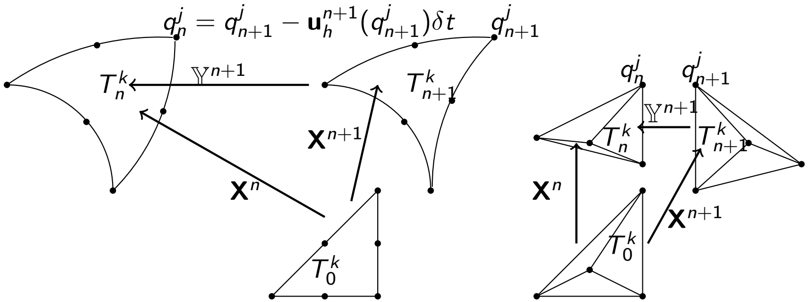


Figure: Sketch to understand if $\mathbf{X}^n = \mathbb{Y}^{n+1}\mathbf{X}^{n+1}$; on the left the case of P^2-isoparametric element for the velocities and on the right the case of the $P^1 - double$ element where each triangle is divided into four subtriangles on which the velocities are P^1 and continuous. The inner vertex used to construct the fluid mesh will be moved also by $\mathbb{Y}$ but $\mathbf{X}^{n+1} \circ \mathbb{Y}^{n+1}$ remains linear and for each triangle $T_n^k = \mathbb{Y}^{n+1}(T_{n+1}^k)$.

- Extension to compressible follows the same procedure
- Extension to 3D also doable but $\mathbf{F}\mathbf{F}^T$ has a third invariant, e.g. its norm

Part III: Proof of Concept Tests

- All examples implemented with `freefem++`

Numerical Tests

- Large displacement rod test (incompressible)

$$E = 2.15,\ \sigma = 0.29,\ \mu = \frac{E}{2(1+\sigma)},\ \rho^s = 1,\ c_1 = \frac{\mu}{2},\ f = -0.02,\ T = 50,\ \delta t = 1.$$

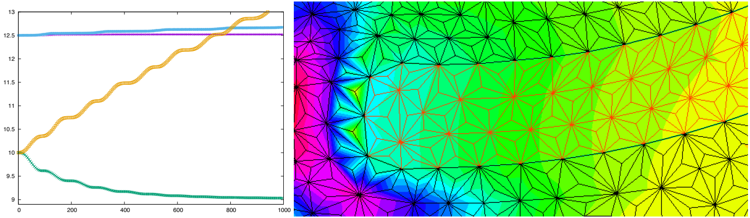


Figure: Energy (bllue and magenta) and surface (yellow and green) vs time for the **d**-scheme, j=1,2.

RUN **F**-scheme RUN **d**-scheme

Compression Stockings

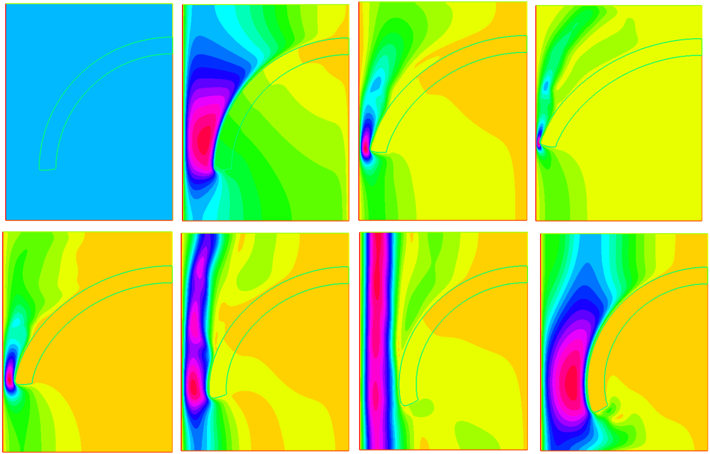
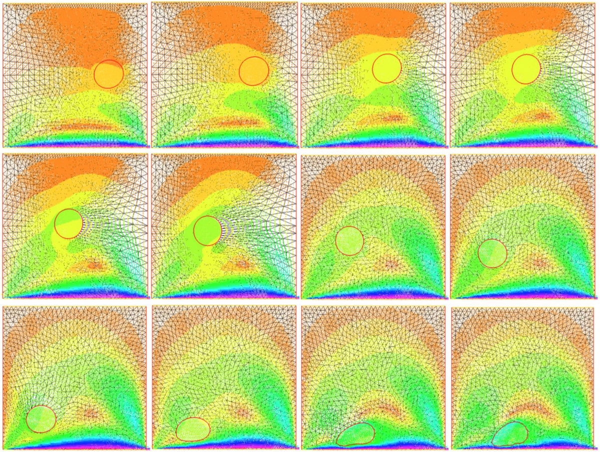


Figure: *Vein valve in a pulsating flow*

RUN Closing blood flow valve

Ball in a Rotating Fluid



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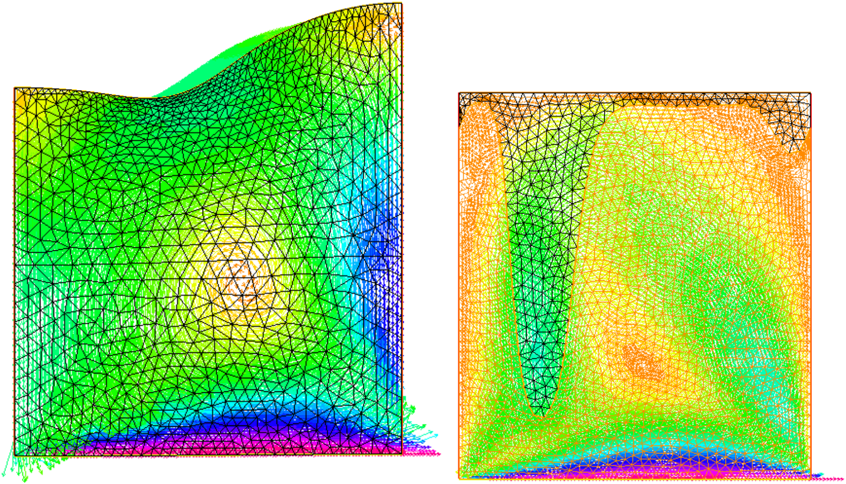
Free Surface Flow with the Same Code



T=100, T=100

Validation with a Rotating Disk

A disk of radius $r_s = 1.5$ within a disk of radius R=3. $c_1 = 0.833$. The outer disk is filled by a fluid with $\nu = 0.1$, the velocity of the fluid is due toa rotation on the outer boundary of magnitude 3. $t \in (0, 10.)$; 80 time steps.
As everything is axisymmetric the computation can be done with the reduced problem

$$\rho\partial_t v - \frac{1}{r}\partial_r[\xi r \partial_r v] + \xi\frac{v}{r^2} = 0$$

with $\rho = \mathbf{1}_{|x|\le r_s}\rho^s + \mathbf{1}_{|x|>r_s}\rho^f$, $\xi = \mathbf{1}_{|x|<=r_s}2c_1 + \mathbf{1}_{|x|>r_s}\nu)$, and with $v_{|x|=R} = 3$.

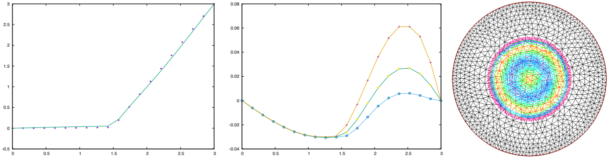


Figure: *Left: comparison at $T = 5$ of the vertical velocity on the horizontal axis for $x \in (0, R)$, after 100 time steps for the coarsest triangulation compared with the axisymmetric 1D solution. Center for 3 triangulations, errors versus x: on the right the velocity in the solid is shown.*

Inverse Problems

$$\min_v \{ \int_{[0,T]\times\Gamma_{top}} (d_2 - d_2^0)^2 \ : \ \text{FSI equations for } p, \mathbf{u}, \mathbf{d}, \text{ functions of } v \}$$

1. $d^0(x,t)$ is obtained by a reference computation with known parameters.
2. Optimization done by using the Stochastic Optimization module CMAES[1]
3. Recovering the bottom velocity u_b and the structure density ρ_s.

	u_b	ρ_s
(R)eference	1.0	1.5
(S)tart	0.5	1.
(30i)terations	1.04	1.502

4. Recover the fluid viscosity ν : (R): 0.1, (S): 0.05, (5i): 0.098
5. Recover one Lamé coefficient μ : (R): 8.333, (S): 4,166, (4i): 9.50

[1] Hansen, N., Mueller, S. D. and Koumoutsakos, P. (2003). Reducing the time complexity of the derandomized evolution strategy with covariance matrix adaptation (CMA-ES). Evolutionary Computation, 11(1), 1-18.

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Conclusion and Perspectives

1. Unconditionally stable algorithms are still in wants
2. Monolithic Methods have an edge (IBM in particular)
3. Difficult: Free boundary and fluid mixing are sub-problems!
4. `freefem++` is useful to prototype new ideas.

Many things to do:

1. Extend to 3D elasticity.
2. Find stability conditions
3. More inverse problems

Thanks for the invitation!

These slides : https://dl.dropboxusercontent.com/u/6801560/FSIConfSpore.pdf

Small Displacement Linear Elasticity

Conservation law for a deformable structure in the initial configuration Ω^0:

$$\rho_s(\ddot{\mathbf{d}}\cdot\hat{\mathbf{d}})_{|\Omega_s^0} + a^0(\mathbf{d},\hat{\mathbf{d}}) = \int_{\Gamma_N^0} g\cdot\hat{\mathbf{d}}, \quad \forall\hat{\mathbf{d}}|_{\Gamma_t^D} = 0; \quad \mathbf{d}|_{\Gamma_t^D} = \mathbf{d}_t^D$$

$$\rho_s(\mathbf{d}\cdot\hat{\mathbf{d}})_{|\Omega_s^t} = \int_{\Omega^t}\rho_s\mathbf{d}\cdot\hat{\mathbf{d}}, \quad \mathrm{D} = \frac{1}{2}(\nabla\mathbf{d}+\nabla^t\mathbf{d}), \quad a^t(\mathbf{d},\hat{\mathbf{d}}\Big) = \int_{\Omega^t}\sigma(\mathbf{d}):\mathrm{D}\hat{\mathbf{d}}$$

ρ_s=solid density, σ=stress tensor, g=surface forces, Γ_t^D the clamped surface.

$$\sigma_s = \lambda_s\mathbf{I}\nabla\cdot\mathbf{d} + \mu_s(\nabla\mathbf{d}+\nabla^t\mathbf{d})$$

Numerical Scheme $O(\delta t)$

$$\rho_s(\frac{\mathbf{d}^{n+1}-2\mathbf{d}^n+\mathbf{d}^{n-1}}{\delta t^2},\hat{\mathbf{d}})_{|\Omega_s^0} + a_s^0(\mathbf{d}^{n+1},\hat{\mathbf{d}}) = \int_{\Gamma_N^0} g\cdot\hat{\mathbf{d}}, \quad \Leftrightarrow$$

$$\rho_s(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\delta t},\hat{\mathbf{u}})_{|\Omega_s^0} + a_s^0(\mathbf{d}^n+\delta t\mathbf{u}^{n+1},\hat{\mathbf{u}}) = \int_{\Gamma_N^0} g\cdot\hat{\mathbf{u}}, \quad \mathbf{d}^{n+1} = \mathbf{d}^n+\delta t\mathbf{u}^{n+1}$$