Godunov Type Schemes for Low Froude Flows with Coriolis Term

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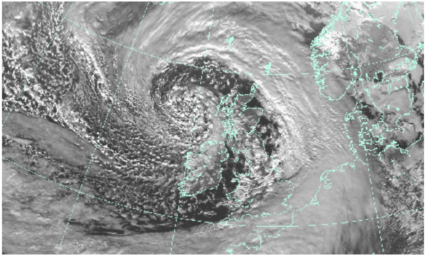
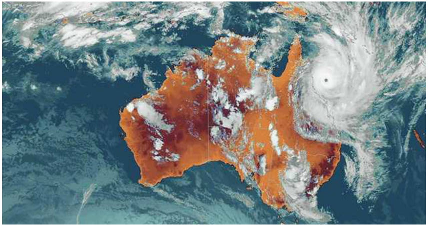
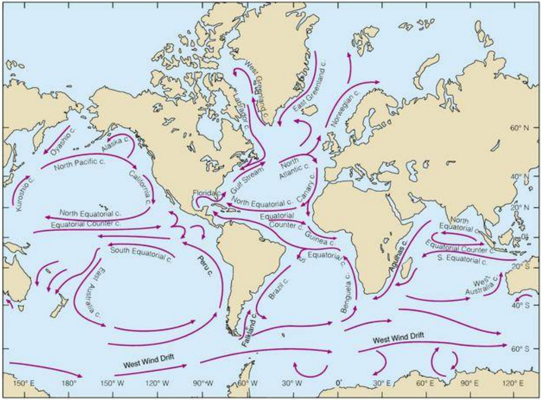
Coriolis effect in the atmosphere





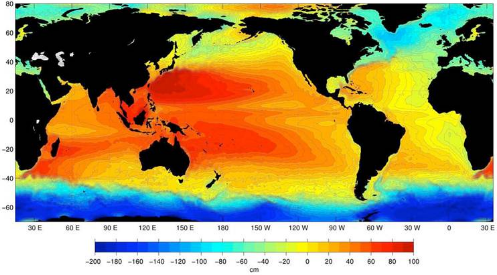
Met Office / Bureau of Meteorology

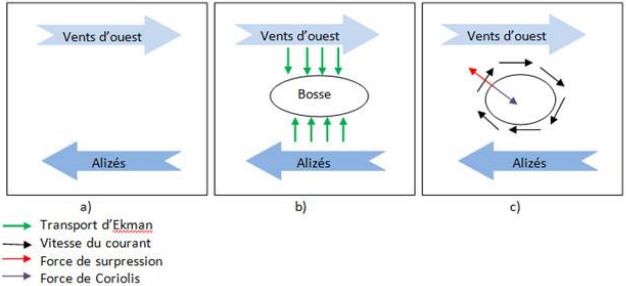
Coriolis effect in the ocean



http://www.emse.fr/ bouchardon/

Coriolis effect in the ocean





Shallow Water Equations with Coriolis Force

- Equations

$$\partial_t H + \nabla \cdot (H\mathbf{U}) = 0$$

$$\partial_t (H\mathbf{U}) + \nabla \cdot (H\mathbf{U} \otimes \mathbf{U}) + \nabla \frac{gH^2}{2} = -gH\,\nabla B - 2\Omega \times (H\mathbf{U})$$

- Source terms
  - Topography
  - Coriolis Force

Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t\,\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$S_t\,\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{F_r^2}\,\nabla\frac{h^2}{2}$$

$$= -\frac{1}{F_r^2}\,h\nabla b - \frac{1}{R_o}\,2\omega\times(h\mathbf{u})$$

- Dimensionless Numbers

$$S_t = \frac{L}{UT}\,,\quad F_r = \frac{U}{\sqrt{gH}}\,,\quad R_o = \frac{U}{\|\Omega\|\,L}$$

  - Strouhal : Advection vs. Non stationarity
  - Froude : Advection vs. Pressure Gradient
  - Rossby : Advection vs. Rotation

Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t\,\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$S_t\,\partial_t\,(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{F_r^2}\,\nabla\frac{h^2}{2} = -\frac{1}{F_r^2}\,h\nabla b - \frac{1}{R_o}\,2\omega\times(h\mathbf{u})$$

- Typical values in lakes or ocean

$$U = 1m/s\,,\ L = 10\text{–}10^3 km\,,\ H = 10\text{–}10^3 m\,,\ \|\Omega\| = 10^{-4} rad/s$$

  - Lakes or Oceanic bay : $F_r \approx 10^{-1}$, $R_o \approx 1$
  - Deep Ocean : $F_r \approx 10^{-2}$, $R_o \approx 10^{-2}$

Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t\,\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$S_t\,\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{F_r^2}\,\nabla\frac{h^2}{2} = -\frac{1}{F_r^2}\,h\nabla b - \frac{1}{R_o}\,2\omega\times(h\mathbf{u})$$

- Lake at rest

$$\nabla(h+b) = 0\,,\ \mathbf{u} = 0$$

A fast and stable WB scheme with hydrostatic reconstruction for SW flows
ABBKP, SIAM JSC, 2004.

Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t\,\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$S_t\,\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{F_r^2}\nabla\frac{h^2}{2} = -\frac{1}{F_r^2}\,h\nabla b - \frac{1}{R_o}\,2\omega\times(h\mathbf{u})$$

- Lake at rest

$$\nabla(h+b) = 0\,,\ \mathbf{u} = 0$$

- Geostrophic Equilibrium

$$\nabla h + 2\omega\times\mathbf{u} = 0\,,\ \nabla\cdot\mathbf{u} = 0$$

Shallow Water Equations with Coriolis Force

- Dimensionless Equations

$$S_t\,\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$S_t\,\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \frac{1}{F_r^2}\nabla\frac{h^2}{2} = -\frac{1}{F_r^2}h\nabla b - \frac{1}{R_o}2\omega\times(h\mathbf{u})$$

- Lake at rest

$$\nabla(h+b) = 0\,,\ \mathbf{u} = 0$$

- Geostrophic Equilibrium

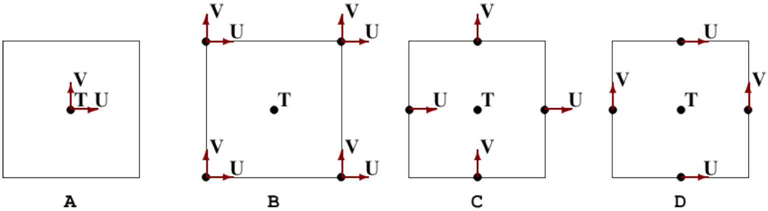
$$\nabla h + 2\omega\times\mathbf{u} = 0\,,\ \nabla\cdot\mathbf{u} = 0$$

Numerical Simulations

- Stability of the scheme
  - OK for classical treatment of the homogeneous part
  - Modifications due to source terms
  - Linear and/or non linear studies
- Hability to preserve stationary states
  - Kernels of the continuous and discrete space operators
  - Impact on transient and long time results
- Accuracy at Low Froude / Low Rossby
  - Stability of the kernel of the discrete space operator
  - Spurious numerical waves
- Dispersion laws
  - Linear case
  - More important for high order schemes

Finite Difference Approach

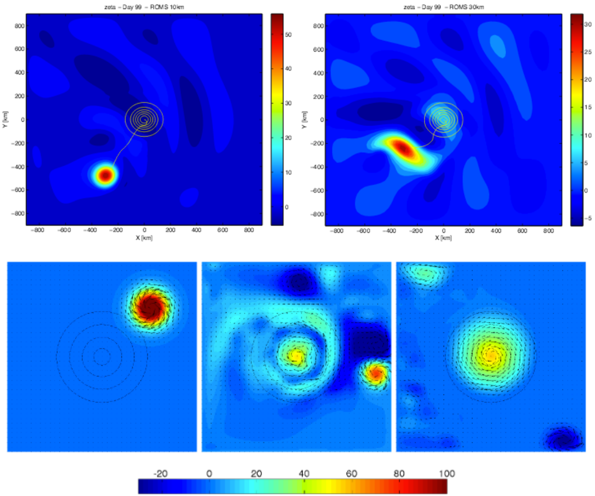
- ROMS, NEMO, HYCOM...
- Semi-implicit in time schemes
- Arakawa grids (MCP, 1977)



On the Approx. of Coriolis Terms in C-Grid Model,
Nechaev et al., AMS, 2004.

Num. Represent. of Geostrophic Modes on Arbitrarily Structured C-Grids,
Thuburn et al., JCP, 2009.

Finite Difference Approach



ANR COMODO test cases

Galerkin Framework

- Semi-implicit in time schemes
- Many possible choices for the FE-DG element

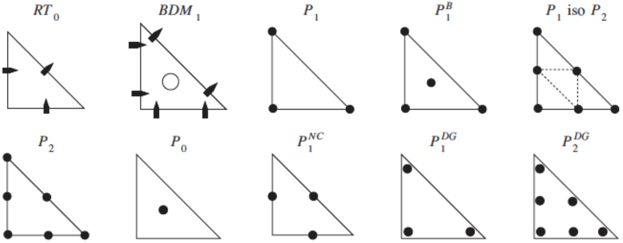


Fig. 1. Typical node locations represented by the symbols • for the $RT_0, BDM_1, P_1, P_1^B, P_1$ iso $P_2, P_2, P_0, P_1^{NC}, P_1^{DG}$ and P_2^{DG} finite elements.

Spurious Inertial Oscillations in SW Models,
LeRoux, JCP, 2012.

Galerkin Framework

- Study of the kernel of space operator / Fourier analysis
  - Not the same number of velocity and pressure points
  - Spurious inertial oscillations

	FE pair	(p, q)	nr	Geostrophic		Inertial ($\pm f$, mult.)	Spurious η modes	Inertia-gravity	
				0	$O(h^{\zeta})$			$O(1)$	$O(\frac{1}{h})$
1	P_1-P_1	(1,1)	3	**1**			**Yes**	2	
	$P_1^B-P_1$	(3,1)	7	**1**		**4** (2,2)	No	2	
	P_1 iso P_2-P_1	(4,1)	9	**1**		**6** (2,3)	No	2	
	P_2-P_0	(4,2)	10		**2**	**4** (2,2)	No	2	2
	P_2-P_1	(4,1)	9	**1**		**6** (2,3)	No	2	
	$P_1^{NC}-P_0$	(3,2)	8	**2**		**2** (2,1)	No	2	2
	$P_1^{NC}-P_1$	(3,1)	7	**1**		**4** (2,2)	No	2	
	P_0-P_1	(2,1)	5	**1**		**2** (2,1)	No	2	
	$P_1^{DG}-P_1$	(6,1)	13	**1**		**10** (2,5)	No	2	
	$P_1^{DG}-P_2$	(6,4)	16	**4**		**4** (2,2)	No	2	6
2	RT_0-P_0	(3,2)	5	**1**			No	2	2
	RT_0-P_1	(3,1)	4	**2**			**Yes**	2	
	BDM_1-P_0	(6,2)	8	**4**			No	2	2
	BDM_1-P_1	(6,1)	7	**3**	**2**		**Yes**	2	

"We recommend to employ the same finite-element bases leading to $p = q$ [...]
to approximate surface-elevation and velocity."

Finite Volume Framework

- WB Schemes with Coriolis terms

Frontal Geostrophic Adjustment in 1D-Rotating SW
Bouchut et al., JFM, 2004.

WB FV Evolution Galerkin Methods for the SWE
Lukacova et al., JCP, 2007.

FV Simulation of the Geostrophic Adjustment in a Rotating SW System
Castro et al., SIAM JSC, 2008

Preservation of the Discrete Geostrophic Equilibrium in SW Flows
AKNV, FVCA VI Proc., 2011.

WB Schemes for the SW Equations with Coriolis Forces
Chertock et al., preprint, 2016.

Finite Volume Framework

- WB Schemes with Coriolis terms : $\nabla h + 2\omega \times \mathbf{u} = 0$

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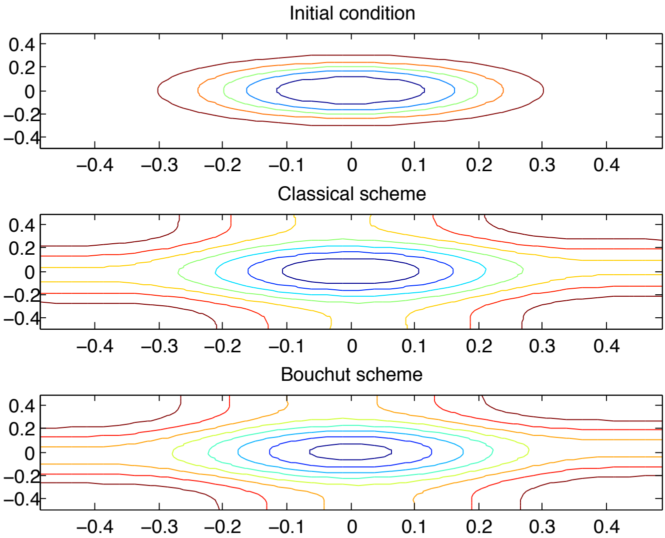
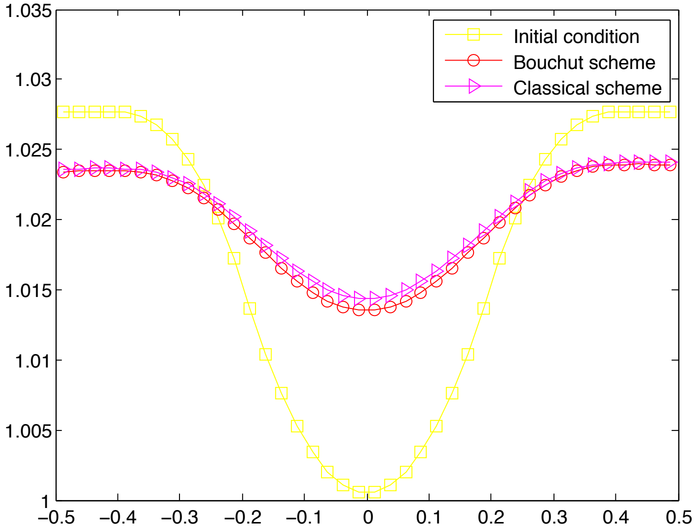
Stationary vortex : Numerical Accuracy



Stationary vortex : Numerical Accuracy



Finite Volume Framework

- Accuracy at low Mach number : $\nabla \cdot \mathbf{u} = 0$

FV Compres. Flow Solvers for Multi-D, Var. Dens. Zero Mach Number Flows,
Schneider et al., JCP, 1999.

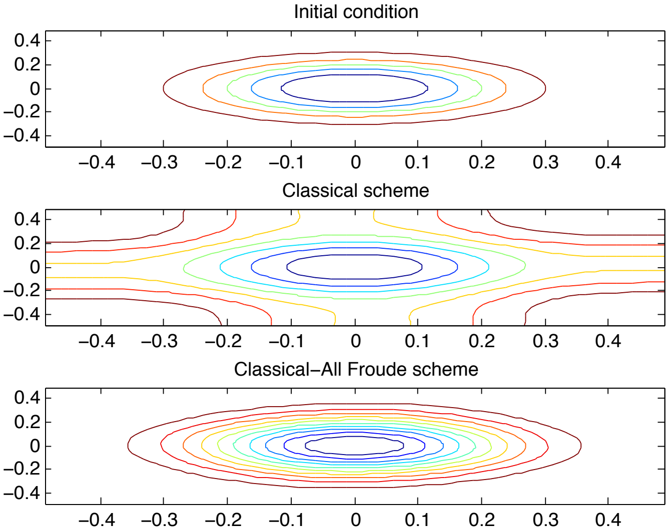
Dissipation mechanism of upwind-schemes in the low Mach number regime,
Rieper, JCP, 2009.

Stability of a Cartesian Grid Projection Meth. for Zero Froude SW Flows,
Klein & Vater, Num. Math., 2009.

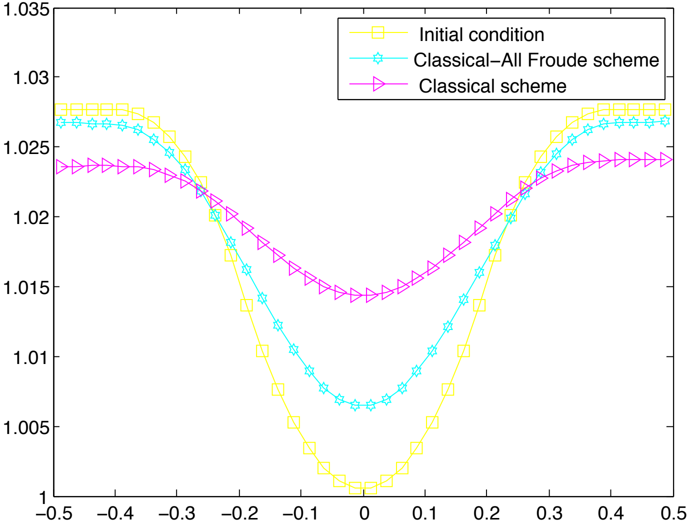
Godunov type schemes for compressible Euler system at Low Mach Number,
Dellacherie, JCP, 2010.

A Weakly AP Low Mach Number Scheme for the Euler Equations
Noelle et al., SIAM JSC, 2014.

Stationary vortex : Numerical Accuracy



Stationary vortex : Numerical Accuracy



Wave Equation with Coriolis Force

- Linearization of the SW model
  - Deep ocean study $F_r = R_o = \epsilon$
  - Flat topography

$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \nabla h &= \mathbf{u}^\perp \end{aligned}$$

- Collocated Finite Volume Study
  - Hability to capture equilibrium states
  - Accuracy at Low Froude / Low Rossby
  - Stability

Wave Equation with Coriolis Force

- Linearization of the SW model
  - Deep ocean study $F_r = R_o = \epsilon$
  - Flat topography

$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \nabla h &= \mathbf{u}^\perp \end{aligned}$$

- Collocated Finite Volume Study

$$h_i^{n+1} = h_i^n + \frac{\Delta t}{|C_i|} \sum_j F^h(h_i^n, \mathbf{u}_i^n, h_j^n, \mathbf{u}_j^n)\mathbf{n}_{ij}$$

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \frac{\Delta t}{|C_i|} \sum_j F^{\mathbf{u}}(h_i^n, \mathbf{u}_i^n, h_j^n, \mathbf{u}_j^n)\mathbf{n}_{ij} + \left(\mathbf{u}^\perp\right)_i^{n,n+1}$$

Wave Equation with Coriolis Force

- Linearization of the SW model
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  - Flat topography

$$
\begin{aligned}
\partial_t h + \nabla \cdot \mathbf{u} &= 0 \\
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\end{aligned}
$$

- Collocated Finite Volume Study

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$$F^h(h_i^n, \mathbf{u}_i^n, h_j^n, \mathbf{u}_j^n)\mathbf{n}_{ij} = \frac{\mathbf{u}_i^n + \mathbf{u}_j^n}{2}\mathbf{n}_{ij} - |D| \left(h_i^n - h_j^n\right)$$

Wave Equation with Coriolis Force

- Study of the Modified Equation
  - Semi discrete Godunov type scheme
  - Numerical viscosity of size Δx

$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{u} - \mu_h \Delta h &= 0 \\ \partial_t \mathbf{u} + \nabla h - \mu_{\mathbf{u}} \Delta \mathbf{u} &= \mathbf{u}^\perp \end{aligned}$$

- Collocated Finite Volume Study

$$h_i^{n+1} = h_i^n + \frac{\Delta t}{|C_i|} \sum_j F^h(h_i^n, \mathbf{u}_i^n, h_j^n, \mathbf{u}_j^n)\mathbf{n}_{ij}$$

$$F^h(h_i^n, \mathbf{u}_i^n, h_j^n, \mathbf{u}_j^n)\mathbf{n}_{ij} = \frac{\mathbf{u}_i^n + \mathbf{u}_j^n}{2}\mathbf{n}_{ij} - |D| \left(h_i^n - h_j^n\right)$$

Wave Equation with Coriolis Force

- Case with no Coriolis force
  - Equilibrium states associated to constant h

$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{u} - \mu_h \Delta h &= 0 \\ \partial_t \mathbf{u} + \nabla h - \mu_{\mathbf{u}} \Delta \mathbf{u} &= 0 \end{aligned}$$

Wave Equation with Coriolis Force

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Wave Equation with Coriolis Force

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- Modification of the numerical viscosity
  - Low (All) Froude scheme : $\mu_{\mathbf{u}} = 0$ $(\mathcal{O}(\epsilon))$
  - Dellacherie scheme : $\Delta u \rightsquigarrow \nabla (\nabla \cdot \mathbf{u})$

Godunov type schemes for compressible Euler system at Low Mach Number, Dellacherie, JCP, 2010.

Wave Equation with Coriolis Force

- Quasi 1d case
  - Flow variables independent of y direction
  - Equilibrium states associated to null $\mathbf{u}_x$ component

$$\begin{aligned}
\partial_t h + \partial_x u - \mu_h \Delta h &= 0 \\
\partial_t u + \partial_x h - \mu_u \Delta u &= v \\
\partial_t v \qquad\quad - \mu_v \Delta v &= -u
\end{aligned}$$

Wave Equation with Coriolis Force

- Quasi 1d case
  - Flow variables independent of y direction
  - Equilibrium states associated to null $\mathbf{u}_x$ component

$$
\begin{aligned}
\partial_t h + \partial_x u - \mu_h \Delta h &= 0 \\
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Wave Equation with Coriolis Force

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  - Flow variables independent of y direction
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- Modification of the numerical viscosity
  - Low (All) Rossby scheme : $\mu_{\mathbf{h}} = 0 \ (\mathcal{O}(\epsilon))$
  - Bouchut scheme : $\Delta h \rightsquigarrow \Delta(h + \tilde{b})$ with $\partial_x \tilde{b} = -v$

Godunov type schemes for Quasi 1d Wave Eq. with Coriolis Term,
ADHOP, to appear in ESAIM Proc., 2016.

Wave Equation with Coriolis Force

- Kernel of the schemes

Classical scheme

$$\ker L_{\kappa_r \neq 0,h} = \left\{ q = (r,u,v) \in \mathbb{R}^{3N} \;\middle|\; \exists\, C \in \mathbb{R} : r_i = C, u_i = 0, v_i = 0 \right\}$$

Low Froude scheme

$$\ker L_{\kappa_r = 0,h} = \left\{ q = (r,u,v) \in \mathbb{R}^{3N} \;\middle|\; u_i = 0,\ \frac{a_\star}{2\Delta x}(r_{i+1} - r_{i-1}) = \omega v_i \right\}$$

Bouchut scheme

$$\ker L_{Bc,h} = \left\{ q = (r,u,v) \in \mathbb{R}^{3N} \;\middle|\; u_i = 0,\ \frac{a_\star}{2\Delta x}(r_{i+1} - r_i) = \omega \frac{v_i + v_{i+1}}{2} \right\}$$

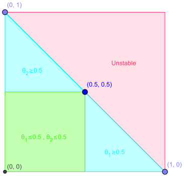
Godunov type schemes for Quasi 1d Wave Eq. with Coriolis Term,
ADHOP, to appear in ESAIM Proc., 2016.

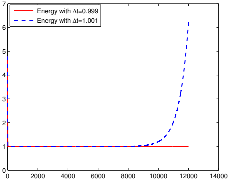
Wave Equation with Coriolis Force

- Stability properties

$$\Delta t_a := \frac{\kappa_u \Delta x}{2|a_\star|} \frac{1}{\left(1 - \frac{\omega \Delta x}{|a_\star|}\sqrt{\Theta_1}\right)_+}$$

$$\Delta t_b := \frac{\Delta x}{\kappa_u |a_\star|} \times \frac{2\kappa_u^2 a_\star^2}{\omega^2 \Delta x^2 \Theta_3} \left[1 - \sqrt{1 - \frac{\omega^2 \Delta x^2}{\kappa_u^2 a_\star^2}\Theta_3}\right]$$





Godunov type schemes for Quasi 1d Wave Eq. with Coriolis Term, ADHOP, to appear in ESAIM Proc., 2016.

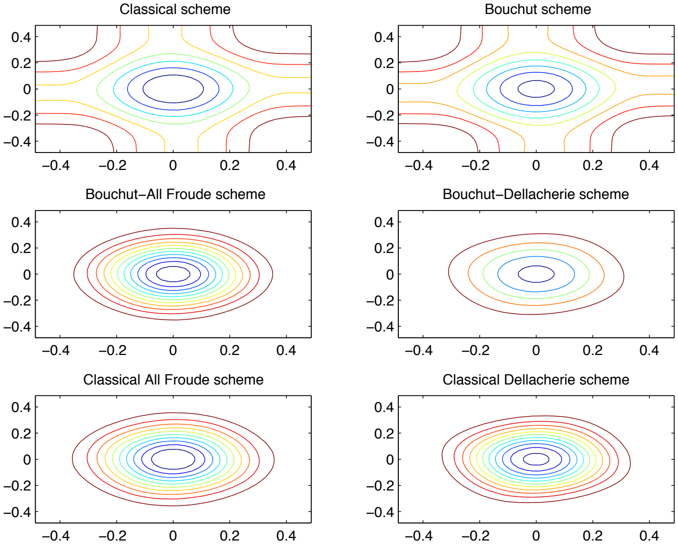
Wave Equation with Coriolis Force

- Fully 2d case
  - Equilibrium states associated to variable h and $\mathbf{u}$
  - Necessary to modify all components of the numerical viscosity

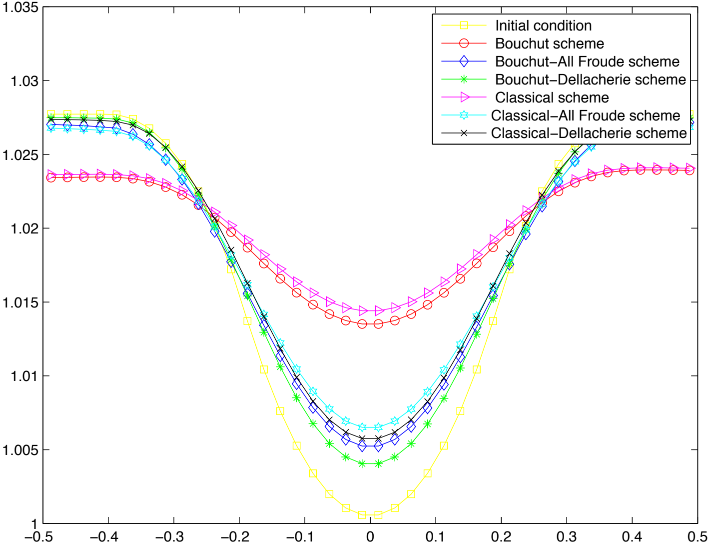
$$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{u} - \mu_h \Delta h &= 0 \\ \partial_t \mathbf{u} + \nabla h - \mu_{\mathbf{u}} \Delta \mathbf{u} &= \mathbf{u}^{\perp} \end{aligned}$$

- Modification of the numerical viscosity
  - Many possible choices
  - Accuracy and Stability issues
    - Classical - Classical : Stable
    - Low Rossby - Low Froude : Unstable
    - Other choices...

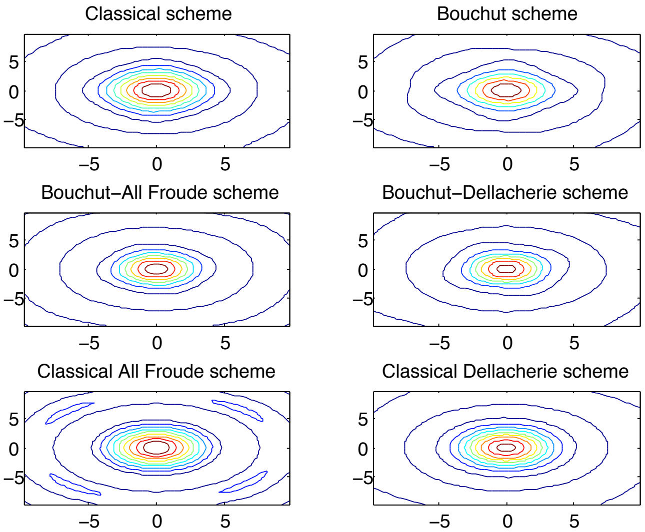
Stationary vortex : Numerical Accuracy



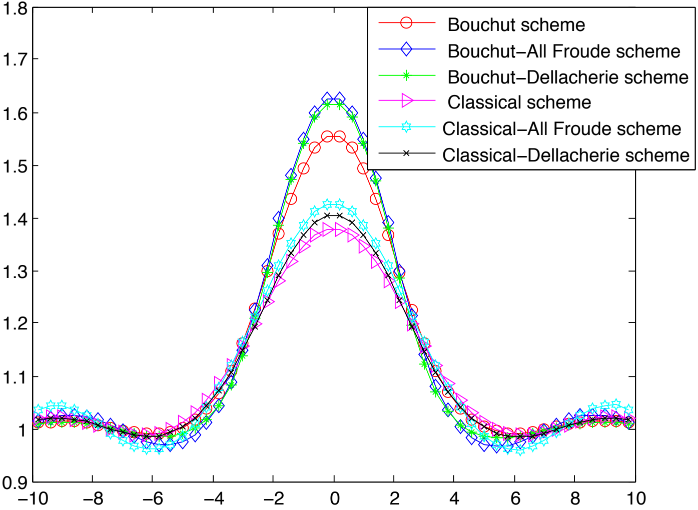
Stationary vortex : Numerical Accuracy



Water column : Stability and Accuracy



Water column : Stability and Accuracy



Perspectives

- 2d Linear & Non linear studies
- Staggered FV schemes
- Extension to 3d (through multilayer models)

$$\partial_t H + \nabla \cdot \left(H \sum \mathbf{U}_\alpha \right) = 0$$

$$\partial_t \left(H\mathbf{U}_\alpha \right) + \nabla \cdot \left(H\mathbf{U}_\alpha \otimes \mathbf{U}_\alpha \right) + \qquad \nabla \frac{gH^2}{2}$$

$$= \quad -gH \nabla B \qquad - 2\Omega \times \left(H\mathbf{U}_\alpha \right)$$

$$+\mathbf{U}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{U}_{\alpha-1/2} G_{\alpha-1/2}$$

A multilayer Saint-Venant system with mass exchanges for SW flows, ABPS, M2AN, 2011.
A hierarchy of non-hydrostatic models for free surface flows, Fernandez Nieto et al., 2016.

Perspectives

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- Staggered FV schemes
- Extension to 3d (through multilayer models)

Finite Volume for Complex Application, FVCA 8
Lille Learning Center, France, 12-16 June 2017