Parametrized Model Order Reduction for Component-to-System Synthesis

or

PDE Apps for Acoustic Ducts

(and Elastic Shafts, Historic Structures, Thermal Fins,. . .)

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[1] Akselos, S.A., licenses technology developed in the MIT research group of ATP. ATP has *no financial interest* in Akselos, S.A.

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[2]None of the academic collaborators has any financial interest in Akselos, S.A.

Sponsors

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Parametrized Partial Differential Equations (PDEs)

- General Setting
- PDE Apps

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Acoustics Preliminaries

Time domain and frequency domain: Laplace transform

$$\text{pressure}(\mathbf{x}, t) = \Re\{u(\mathbf{x}, f)\exp(2\pi \mathrm{i} f t)\}.$$

Frequency-domain pressure satisfies Helmholtz equation:

$$-\underbrace{(1 + \mathrm{i}\epsilon k)}_{\kappa} \nabla^2 u - \underbrace{\frac{(2\pi f)^2}{c_{\text{sound}}^2}}_{k^2} u = 0 \ ,$$

for ϵ a dissipation coefficient and c_{sound} speed of sound.

Particle velocity is related to pressure by

$$\text{velocity}(\mathbf{x}, f) = \frac{\mathrm{i}}{2\pi f \rho} \nabla u(\mathbf{x}, f) \quad ,$$

for ρ the (uniform) density.

Examples of Parametrized PDEs

Heat Transfer (Conduction):

$$-\nabla(\sigma \nabla u) = q \text{ in } \Omega_\lambda, \quad s \equiv \bar{u}_{\text{root}} .$$

Linear Elasticity:

$$-\frac{\partial}{\partial x_j} E_{ij\ell m} \frac{\partial u_\ell}{\partial x_m} = F_i \text{ in } \Omega_\lambda, \quad s \equiv \text{SCF} .$$

Helmholtz Acoustics:

$$-(1 + \mathrm{i}\epsilon k)\nabla^2 u - k^2 u = F \text{ in } \Omega_\lambda, \quad s \equiv Z^{\text{inlet}} .$$

INPUT PARAMETER $\mu \equiv (k, \lambda) \in \mathbb{R}^P$

$\rightarrow$ FIELD $u_\mu(x)$ and OUTPUT (QoI) s_μ

Given $\mu \in \mathcal{P}$ (compact) $\subset \mathbb{R}^P$, find

field $u_\mu \in X(\Omega_\mu)$ (say) scalar, real

$$\begin{aligned} A_\mu u_\mu &= F_\mu \text{ in } \Omega_\mu \ , \ \text{or} \\ \langle A_\mu u_\mu, v\rangle &= \langle F_\mu, v\rangle, \forall v \in X \ , \ \text{or} \\ a_\mu(u_\mu, v) &= F_\mu(v), \forall v \in X \ , \end{aligned}$$

output(s) $s_\mu \in \mathbb{R}$

$$s_\mu = \langle L_\mu, u_\mu\rangle, \ \text{or } s_\mu = \ell_\mu(u_\mu) \ ,$$

where $\Omega_\mu \subset \mathbb{R}^3$, $X = H^1_{(0)}(\Omega_\mu)$, and $F_\mu, L_\mu \in X'$.

Note boundary conditions are included in a_μ and F_μ.

Model and Family

A *Model* is a particular problem definition:

parametrization: $\mu \in \mathcal{P} \subset \mathbb{R}^P$;

spatial domain: $x \in \Omega_\mu \subset \mathbb{R}^3$;

physical discipline: a_μ, F_μ;

engineering outputs (QoI): ℓ_μ.

A Model maps parameter $\mu \in \mathcal{P}$ to

field $u_\mu(x)$ and output(s) s_μ.

A *Family* is a set of Models which share

a physical discipline and engineering context.

Acoustic Ducts, Elastic Shafts, Historic Structures,. . .

Parametrized Partial Differential Equations (PDEs)

- General Setting
- PDE Apps

A PDE App is

software associated to a Model

which maps any $\mu \in \mathcal{P}$ to an

$$\text{approximate} \begin{cases} \text{field } \tilde{u}_\mu(x) & \approx u_\mu(x) \\ \text{output } \tilde{s}_\mu = \ell_\mu(\tilde{u}_\mu) & \approx s_\mu \end{cases}$$

subject to performance requirements:

response time and accuracy.

PDE App: Performance Requirements

A *deployed* PDE App should satisfy:

- $\lessapprox$ 5-second problem set-up time; "app-ification"
- $\lessapprox$ 5-second problem solution time, field and outputs;
- $\lessapprox$ 5% solution error, specified metrics;
- $\lessapprox$ 5-second field visualization time.

The choice of 5 seconds is informed by
the human attention span: *interaction*.

Offline I: Very Slow — Days

Given Family, form associated Online Dataset $\mathbb{D}$.

Offline II: Slow — Hours

Given Model $\in$ Family, script PDE App.

Online: Fast — Seconds

Given PDE App, evaluate $\mu \in \mathcal{P} \xrightarrow{\mathbb{D}} \tilde{u}_\mu(x), \tilde{s}_\mu$.

The PDE App Offline-Online approach
is computationally competitive in
the many-query context — Offline amortized, and
the interactive context — Offline "irrelevant."

Computational Methodology

- Perspective
- Components and System Synthesis
- Finite Element (FE) Approximation
- Static Condensation Reformulation of FE
- Model Order Reduction
- Remarks
- Computational Procedure: PDE App Workflow

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Component Mode Synthesis, 1960s PR

Hurty, Craig-Bampton, Bourquin, Hetmaniuk,. . .

Static Condensation 1970s SC

Reduced Basis Methods, 1980s RB

Almroth, Noor, Porsching, Gunzburger,. . .

Post-Modern Reduced Basis Methods, 2000s

MoRePaS I-III: *a priori/posteriori* error estimation,
Weak Greedy sampling,
(approximate) affine expansions,
strict Offline-Online decomposition,. . .

Reduced Basis Element Method, 2000s E

Maday-Rønquist

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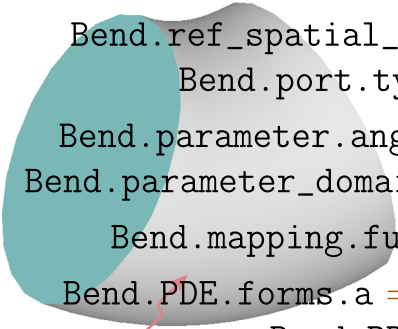
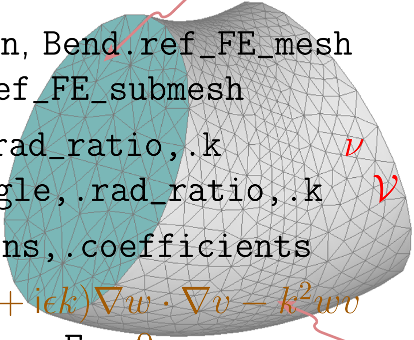
Parametrized Archetype Component





`Bend`

`Bend.ref_spatial_domain`, `Bend.ref_FE_mesh`

`Bend.port.type.ref_FE_submesh`

`Bend.parameter.angle,.rad_ratio,.k` ν

`Bend.parameter_domain.angle,.rad_ratio,.k` $\mathcal{V}$

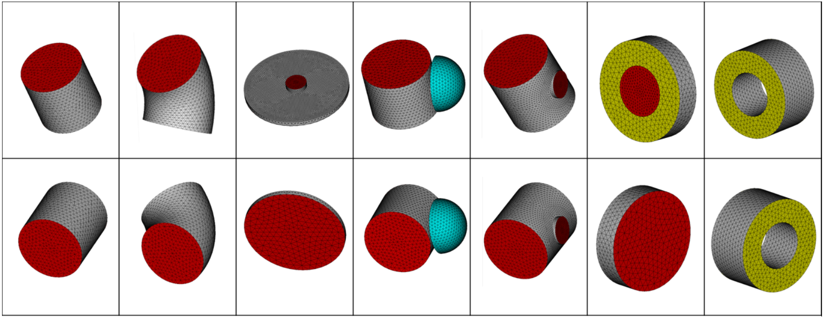
`Bend.mapping.functions,.coefficients`

`Bend.PDE.forms.a` $= \int (1 + \mathrm{i}\epsilon k) \nabla w \cdot \nabla v - k^2 w v$

`Bend.PDE.forms.F` $= 0$

Acoustic Ducts

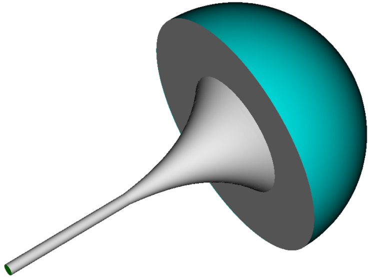
(selected archetype components)



Admissible connections:

ports of common color $\leftrightarrow$ common port type.

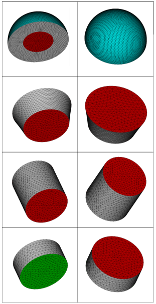
System Synthesis: A Model

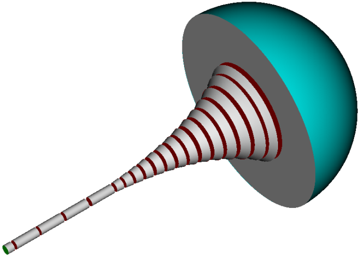


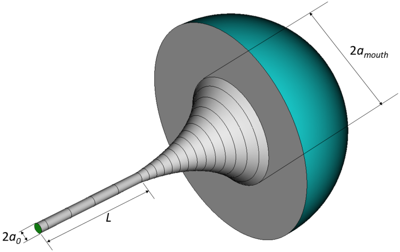
`Model_Exponential_Horn` (Flanged)

$$\mu \equiv (L/a_0, m^{\mathrm{horn}}, a_{\mathrm{mouth}}/a_0, ka_0)$$
$$\in \mathcal{P} \equiv [2, 20] \times [0.0334, 0.1666] \times [4, 12] \times [0, 1]$$

System Synthesis: Instantiation and Connection





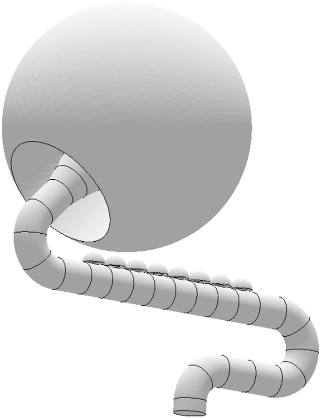


Instantiation

$\mu_{\text{model}} \in \mathcal{P} \rightarrow$
$\{\nu_{\text{local}} \in \mathcal{V}\}_{\text{instantiated components}}$

Connection

local port pairs $\rightarrow$
global ports $\Gamma \in G$



`Model_Nguyenophone`

$$\mu \equiv (\texttt{Hole\_Location}, \texttt{Hole\_Open}, k)$$
$$\in \mathcal{P} \equiv \texttt{Wedge} \subset \mathbb{R}^8 \times \{0,1\}^8 \times [0,2]$$

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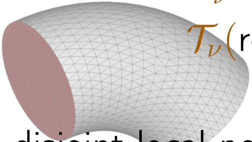
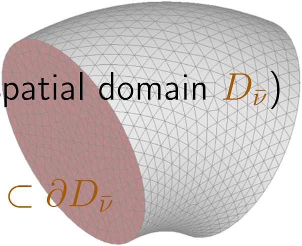
Geometry Mappings

An archetype component is characterized by

spatial domain $D_\nu = \mathcal{T}_\nu(\text{reference spatial domain } D_{\bar{\nu}})$



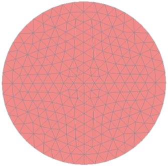


and

two disjoint local ports $\gamma_1, \gamma_2 \subset \partial D_{\bar{\nu}}$

such that

$$\gamma_i = \tau_i \gamma_0, \; i = 1, 2 \; ,$$



for γ_0 a fiducial port (type).

We may easily consider more than two local ports.

FE Approximation Spaces

Associate to

each archetype component

a reference FE mesh,

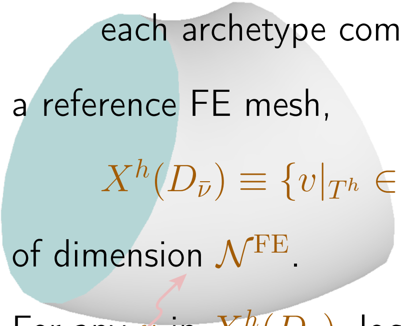
$$X^h(D_{\bar{\nu}}) \equiv \{v|_{T^h} \in \mathbb{P}_p(T^h), \forall T^h \in \mathbb{T}^h\}$$

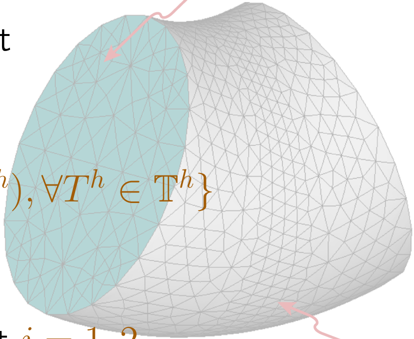
of dimension $\mathcal{N}^{\text{FE}}$.

For any v in $X^h(D_{\bar{\nu}})$, local port $i = 1, 2$,

$$v|_{\gamma_i} \in \{ \underbrace{\chi_j^{h\ \text{fid}}}_{\text{fiducial port modes}} \circ \tau_i^{-1}, 1 \leq j \leq J^{\text{FE}}\};$$

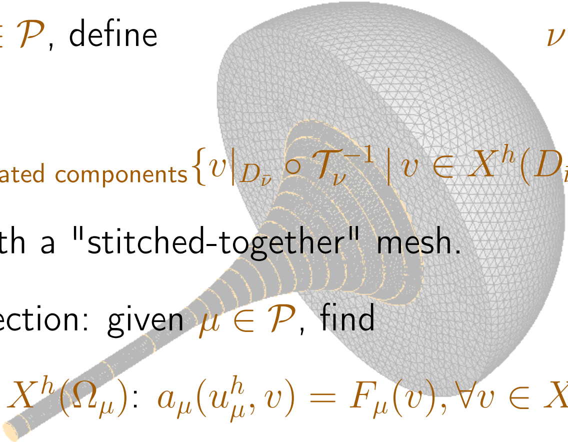
implicit conforming condition on ports of common type.





Finite Element (FE) Approximation of Model

For given $\mu \in \mathcal{P}$, define $\nu = \nu(\mu)$



$X^h(\Omega_\mu) \equiv$

$$\bigoplus_{\text{instantiated components}} \{ v|_{D_{\bar{\nu}}} \circ \mathcal{T}_\nu^{-1} \,|\, v \in X^h(D_{\bar{\nu}}) \} \cap X$$

associated with a "stitched-together" mesh.

Galerkin projection: given $\mu \in \mathcal{P}$, find

field $u^h_\mu \in X^h(\Omega_\mu)$: $a_\mu(u^h_\mu, v) = F_\mu(v), \forall v \in X^h(\Omega_\mu)$,

and subsequently

output $s^h_\mu = \ell_\mu(u^h_\mu)$.

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In *given instantiated component*, $\nu = \nu(\mu)$

for local port $i = 1, 2$

for port mode $j = 1, \ldots, J^{\mathrm{FE}}$:

$\psi^h_{i,j} = \mathcal{L}_i(\chi^h_j \circ \tau_i^{-1})$ is lifting to reference domain of port mode j on port i, and

$\varphi^h_{i,j;\nu} = \psi^h_{i,j} + \eta^h_{i,j;\nu} \in X^h_{[\gamma_1,\gamma_2:0]}(D_{\bar{\nu}})$ satisfies

$a^{D_{\bar{\nu}}}_{\nu}(\varphi^h_{i,j;\nu}, v) = 0, \forall v \in X^h_{[\gamma_1,\gamma_2:0]}(D_{\bar{\nu}})$, subject to

$\varphi^h_{i,j;\nu}|_{\gamma_{i'}} = \chi^h_j\, \delta_{ii'}$, $\mathcal{N}^{\mathrm{FE}} \times \mathcal{N}^{\mathrm{FE}}$

where for simplicity all sources reside on ports.

Express $\eta^h_{i,j;\nu}$ as $\qquad 1 \le j \le J^{\mathrm{FE}}\,,\; 1 \le i \le 2$

$$\eta^h_{i,j;\nu}(\bar{x}) = \sum_{k=1}^{\mathcal{N}^{\mathrm{FE}}} \alpha^h_{i,j,k;\nu}\,\xi^h_k(\bar{x}) \text{ for } \bar{x} \in D_{\bar{\nu}} \quad ;$$

then $\alpha^h_{i,j,k;\nu}$ satisfy $\qquad \mathbb{G}_\nu \equiv \mathbb{J}_\nu^{-1}\mathbb{J}_\nu^{-\mathrm{T}}$

$$\sum_{k=1}^{\mathcal{N}^{\mathrm{FE}}} \left(\int_{D_{\bar{\nu}}} \left(\kappa(\nabla \xi^{h\,*}_{k'})^{\mathrm{T}} \mathbb{G}_\nu \nabla \xi^h_k - k^2 \xi^{h\,*}_{k'} \xi^h_k \right) |\mathbb{J}_\nu| \, d\bar{x} \right) \alpha^h_{i,j,k;\nu}$$

$$= -\int_{D_{\bar{\nu}}} \left(\kappa(\nabla \xi^{h\,*}_{k'})^{\mathrm{T}} \mathbb{G}_\nu \nabla \psi^h_{i,j} - k^2 \xi^{h\,*}_{k'} \psi^h_{i,j} \right) |\mathbb{J}_\nu| \, d\bar{x}$$

for $1 \le k' \le \mathcal{N}^{\mathrm{FE}}$.

In *given instantiated component*, (local) LINEARITY

$$u^h_\mu|_{D_{\nu(\mu)}} = \sum_{i=1}^{2} \sum_{j=1}^{J^{\mathrm{FE}}} u^h_{i,j;\nu} \, (\varphi^h_{i,j;\nu} \circ \mathcal{T}_\nu^{-1})$$

for appropriate coefficients $u^h_{i,j;\nu}, 1 \leq j \leq J^{\mathrm{FE}}, i = 1, 2$.

Form $2J^{\mathrm{FE}} \times 2J^{\mathrm{FE}}$ stiffness matrix $A^{h\,\text{Galerkin}}_{[i,j],[k,\ell];\nu}$:

normal velocity moment on local port i flux

with respect to test port mode j

expressed in terms of

pressure coefficient on local port k

associated with trial port mode ℓ.

We may write stiffness matrix as

$$A^h_{[i,j],[k,l];\nu} \equiv$$

$$\int_{D_{\bar{\nu}}} \big(\, \kappa(\nabla \psi^{h\,*}_{i,j})^{\mathrm{T}} \mathbb{G}_\nu \nabla(\psi^h_{i,j} + \eta^h_{k,\ell;\nu})$$

$$-k^2(\psi^{h\,*}_{i,j})(\psi^h_{i,j} + \eta^h_{k,\ell;\nu})\,\big)\, |\mathbb{J}_\nu|\, d\bar{x} \;,$$

for $1 \le i,k \le 2,\ 1 \le j,\ell \le J^{\mathrm{FE}}$.

SC: System Level

Require on global ports $\Gamma \in G$

continuity of pressure, and

weak continuity of normal velocity

implemented as direct stiffness assembly:

$$\{A^h_{\nu(\mu)}\}_{\text{instantiated components}} \to \mathcal{A}^h_\mu \quad ; \qquad \mathcal{F}^h_\mu$$

here $\mathcal{A}^h_\mu$ is $|G|J^{\mathsf{FE}} \times |G|J^{\mathsf{FE}}$ block-sparse Schur complement.

Issues: J^{FE} will be large, and

$\mathcal{N}^{\mathrm{FE}}$ will be large,

such that $\mathcal{A}^h_\mu$ costly to form and to "invert."

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for local port $i = 1, 2$

for port mode $j = 1, \ldots, M$:

$\psi^h_{i,j} = \mathcal{L}_i(\chi^{h\,\mathrm{fid}}_j \circ \tau_i^{-1})$ is lifting to reference domain of port mode j on port i, and

$\varphi^{h,N}_{i,j;\nu} = \psi^h_{i,j} + \eta^{h,N}_{i,j;\nu} \in Z^{h,N\,\mathrm{arch}}_{i,j\,[\gamma_1,\gamma_2:0]}(D_{\bar{\nu}})$ satisfies

$a^{D_{\bar{\nu}}}_{\nu}(\varphi^h_{i,j;\nu}, v) = 0, \forall v \in Z^{h,N}_{i,j\,[\gamma_1,\gamma_2:0]}(D_{\bar{\nu}})$, subject to

$\varphi^h_{i,j;\nu}|_{\gamma_{i'}} = \chi^h_j\, \delta_{ii'}$; $N \times N$

where for simplicity all sources reside on ports.

Express $\eta_{i,j;\nu}^{h,N}$ as $\quad 1 \le j \le M \,,\; 1 \le i \le 2$

$$\eta_{i,j;\nu}^{h,N}(\bar{x}) = \sum_{k=1}^{N} \eta_{i,j,k;\nu}^{h,N}\, \zeta_{i,j,k}(\bar{x}) \text{ for } \bar{x} \in D_{\bar{\nu}} \quad ;$$

then $\eta_{i,j,k;\nu}^{h,N}$ satisfy

$$\sum_{k=1}^{N}\Big(\int_{D_{\bar{\nu}}}\big(\,\kappa(\nabla\zeta_{i,j,k'}^{*})^{\mathrm{T}}\mathbb{G}_{\nu}\nabla\zeta_{i,j,k} - k^2\zeta_{i,j,k}^{*}\zeta_{i,j,k}\,\big)|\mathbb{J}_{\nu}|d\bar{x}\,\Big)\,\eta_{i,j,k;\nu}^{h,N}$$
$$= -\int_{D_{\bar{\nu}}}\big(\,\kappa(\nabla\zeta_{i,j,k'}^{*})^{\mathrm{T}}\mathbb{G}_{\nu}\nabla\psi_{i,j}^{h} - k^2\zeta_{i,j,k'}^{*}\psi_{i,j}^{h}\,\big)|\mathbb{J}_{\nu}|d\bar{x}$$

for $1 \le k' \le N$.

In *given instantiated component*,

$$u_\mu^{h,M,N}|_{D_{\nu(\mu)}} = \sum_{i=1}^{2}\sum_{j=1}^{M} u_{i,j;\nu}^{h,M,N}\,(\varphi_{i,j;\nu}^{h,N} \circ \mathcal{T}_\nu^{-1})$$

for appropriate coefficients $u_{i,j;\nu}^{h,M,N}, 1 \le j \le M, i = 1,2$.

Form $2M \times 2M$ stiffness matrix $A_{[i,j],[k,\ell];\nu}^{h,M,N\ \text{(Petrov)-Galerkin}}$:

normal velocity moment on local port i flux

with respect to test port mode j

expressed in terms of

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associated with trial port mode ℓ.

We may write stiffness matrix as

$$A^{h,M,N}_{[i,j],[k,l];\nu} \equiv$$

$$\int_{D_{\bar{\nu}}} \left(\kappa (\nabla \psi^{h\,*}_{i,j})^{\mathrm{T}} \mathbb{G}_\nu \nabla (\psi^h_{i,j} + \eta^{h,N}_{k,\ell;\nu}) \right.$$

$$\left. - k^2 (\psi^{h\,*}_{i,j})(\psi^h_{i,j} + \eta^{h,N}_{k,\ell;\nu}) \right) |\mathbb{J}_\nu| \, d\bar{x} \ ,$$

for $1 \leq i, k \leq 2, \ 1 \leq j, \ell \leq M$.

Require on global ports $\Gamma \in G$

continuity of pressure, and

weak continuity of normal velocity

implemented as direct stiffness assembly:

$$\{A^{h,M,N}_{\nu(\mu)}\}_{\text{instantiated components}} \to \mathcal{A}^{h,M,N}_{\mu} \qquad\qquad \mathcal{F}^{h,M,N}_{\mu}$$

where $\mathcal{A}^{h,M,N}_{\mu}$ is $|G|M \times |G|M$ block-sparse.

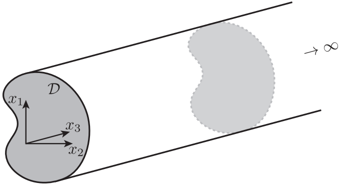
Issues "resolved": $M \ll J^{\mathrm{FE}}$, and

$N \ll \mathcal{N}^{\mathsf{FE}}$,

such that $\mathcal{A}^{h,M,N}_{\mu}$ is inexpensive to form and to "invert."

Port Reduction, $M \ll J^{\mathrm{FE}}$: Rationale...

Consider a waveguide $\mathcal{D} \times (0, \infty)$,



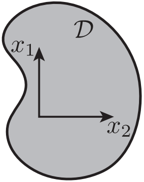
and find $p(x_1, x_2, x_3)$ such that

$$-\nabla^2 p - k^2 p = 0 \text{ in } \mathcal{D} \times (0, \infty) \quad ,$$

and

$$p = g \text{ on } (x_1, x_2) \in \mathcal{D}, x_3 = 0,$$
$$\frac{\partial p}{\partial n} = 0 \text{ on } (x_1, x_2) \in \partial\mathcal{D}, 0 < x_3 < \infty,$$
$$p \text{ (say) outgoing bounded wave as } x_3 \to \infty.$$

Restrict attention to the transverse domain $\mathcal{D}$,



and find $(\Upsilon_i(x_1, x_2), \lambda_i)_{i=1,\ldots}$ solution of eigenproblem

$$-\nabla^2_{x_1,x_2}\Upsilon = \lambda\Upsilon, \text{ in } \mathcal{D} \quad ,$$

$$\frac{\partial\Upsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D};$$

order (real) eigenvalues $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \ldots$

. . . Port Reduction, $M \ll J^{\mathrm{FE}}$: Rationale — Evanescence

Consider $k \in [\sqrt{\lambda_n}, \sqrt{\lambda_{n+1}})$: then $\Re\{\cdot\, e^{\mathrm{i}\omega t}\}$

$$p = \sum_{j=1}^{n} \alpha_j^g \, \Upsilon_j(x_1, x_2) \, e^{-\mathrm{i}\sqrt{k^2 - \lambda_j}\, x_3} + \sum_{j=n+1}^{\infty \text{ (or } J^{\mathrm{FE}})} \alpha_j^g \, \Upsilon_j(x_1, x_2) \, e^{-\sqrt{\lambda_j - k^2}\, x_3}$$

for coefficients α_j^g chosen to realize $p(\cdot, \cdot, x_3 = 0) = g$.

For any global port $\Gamma \in G$*, higher modes introduced*

in neighboring components, and

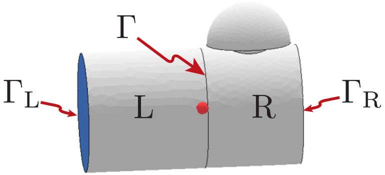
at neighboring global ports,

will be filtered prior to "arrival" at Γ*.*

Port Reduction, $M \ll J^{\mathrm{FE}}$: A *Library* Training Procedure

For all compatible archetype component pairs in Library,

form *test subsystem*



and find $z \in X^h(\mathtt{L},\mathtt{R})$ such that

$$a^{(\mathtt{L},\mathtt{R})}_{\nu_{\mathtt{L}},\nu_{\mathtt{R}}}(z,v) = 0, \ \forall v \in X^h(\mathtt{L},\mathtt{R}) \ ,$$

for a rich set of Dirichlet conditions on $\Gamma_{\mathtt{L}}$, $\Gamma_{\mathtt{R}}$, and admissible parameters $\nu_{\mathtt{L}}$ and $\nu_{\mathtt{R}}$.

Collect $z|_{\Gamma} \circ \mathcal{T}_{\nu} \circ \tau.$ *from all test subsystems* in a set S.

Apply POD to S: fiducial port modes $\{\chi_j\}_{j=1,\dots,M}$.

Bubble Reduction, $N \ll \mathcal{N}^{\mathrm{FE}}$: Rationale

For any archetype component in Library,

for local port $i = 1, 2$,

for port mode $j = 1, \ldots, M$,

$$\eta^h_{i,j;\nu} \quad \in \quad \underbrace{\{\eta^h_{i,j;\nu} \text{ on } D_{\bar{\nu}} \,|\, \nu \in \mathcal{V}\}}_{\text{low-dimensional smooth manifold}} \quad \subset \quad \underbrace{X^h_{[\gamma_1,\gamma_2;0]}(D_{\bar{\nu}})}_{\text{high-dimensional space}} \quad ;$$

note that

$$\nu_{\text{local}} \in \mathcal{V} \subset \mathbb{R}^V, \; \mu_{\text{model}} \in \mathcal{P} \subset \mathbb{R}^P$$

for (typically) $V \ll P$ — *components divide and conquer*.

Bubble Reduction, $N \ll \mathcal{N}^{\mathrm{FE}}$: A *Library* Training Procedure

For each archetype component in Library,

for local port $i = 1, 2$,

for port mode $j = 1, \ldots, M$,

form $Z^{h,N}_{i,j[\gamma_1,\gamma_2:0]}$ as RB Lagrangian snapshot space $\perp$-ized

$$Z^{h,N}_{i,j[\gamma_1,\gamma_2:0]} \equiv \text{span}\{\eta^h_{i,j;\nu^n_{i,j}}, 1 \le n \le N_{i,j}\}$$

for quasi-optimal parameter values

$$\{\nu^1_{i,j} \in \mathcal{V}, \ldots, \nu^N_{i,j} \in \mathcal{V}\}$$

selected by the RB Weak-Greedy procedure.

Computational Methodology

- Perspective
- Components and System Synthesis
- Finite Element (FE) Approximation
- Static Condensation Reformulation of FE
- Model Order Reduction
- Remarks
- Computational Procedure: PDE App Workflow

Optimality

Under certain hypotheses, the best fits associated with

the port reduced spaces, span$\{\chi_j, 1 \leq j \leq M\}$,

and

the bubble reduced spaces, $Z_{i,j}^{h,N}$,

converge at rates similar to the corresponding

Kolmogorov M (respectively, N) width.

The (Petrov-)Galerkin projections are optimal to within

a (Model,μ *and* M, N)-dependent stability constant.

Verification (and Validation)

A posteriori error indicators play a role in

optimal choice of snapshots $\rightarrow Z_{i,j}^{h,N}$

and

optimal choice of M and N.

Each Model is verified over $\Xi_{\text{verification}} \subset \mathcal{P}$:

refinement in $h \downarrow, M \uparrow$, and $N \uparrow$;

reference to appropriate closed-form approximations;

comparison to 3rd-party computations and experiments.

Verification of each Model improves

archetype components: convergence of *Library*.

Computational Methodology

- Perspective
- Components and System Synthesis
- Finite Element (FE) Approximation
- Static Condensation Reformulation of FE
- Model Order Reduction
- Remarks
- Computational Procedure: PDE App Workflow

Offline I: Library → Online Dataset $\mathbb{D}$ Expensive

Prepare Online Dataset $\mathbb{D}$ for Library:

archetype component reference FE meshes;

archetype *affine* component mappings $\mathcal{T}_\nu$;

port modes $\chi_j^h, 1 \leq j \leq M$ (for each port type);

RB spaces $Z_{i,j}^{h,N}$ for each archetype component,
local port i, and port mode j;

(Petrov-)Galerkin parameter-independent inner products.

Role of components:

no Models formed or evaluated in Offline I stage;

all Models in Online stage amortize Offline I effort.

A typical term in

$$A^{h,M,N}_{[i,j],[k,l];\nu} \equiv$$

$$\int_{D_{\bar{\nu}}} \left(\kappa(\nabla\psi^{h\,*}_{i,j})^{\mathrm{T}}\mathbb{G}_\nu|\mathbb{J}_\nu|\nabla(\psi^h_{i,j} + \eta^{h,N}_{k,\ell;\nu}\right) + \ldots$$

leads to inner product

$$\sum_{n=1}^{N} \eta^{h,N}_{k,\ell,n;\nu} \int_{D_{\bar{\nu}}} \left(\kappa(\nabla\psi^{h\,*}_{i,j})_1 \, \underbrace{(\mathbb{G}_\nu)_1|\mathbb{J}_\nu|}_{\text{EIM expansion}} \, (\nabla(\psi^h_{i,j} + \zeta_{k,\ell,n}))_1 \right. .$$

Online: $\mathbb{D}$; Model; $\mu \in \mathcal{P} \to u_\mu^{h,M,N}, s_\mu^{h,M,N}$ Fast

Web-User-Interface (WUI) Cloud Implementation

Query the PDE App:

input $\mu \in \mathcal{P}$,	User
synthesize Model from (say) script,	Model Server
invoke Online Dataset $\mathbb{D}$	Compute Server
form and solve Schur complement,	Compute Server
calculate field and output,	Compute Server
download and display solution.	User, Servers

(Offline II — prepare Model Server for each Model: parametrization, instantiation, connections, and outputs.)

Acoustics Ducts: PDE App Examples

- A Flanged Exponential Horn
- An Expansion Chamber
- A Circular Duct with Toroidal Bend
- An Extended-Tube Expansion Chamber (ETEC)

Acoustics Ducts: PDE App Examples

- A Flanged Exponential Horn
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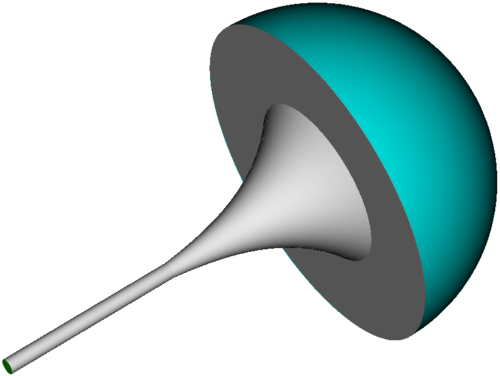
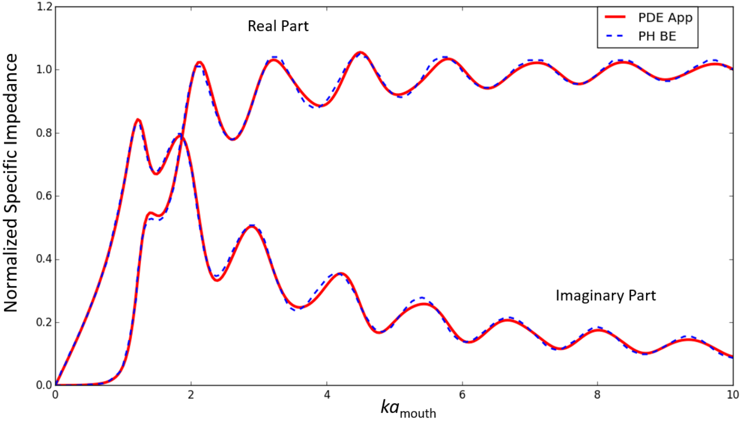
Model: Parametrization and Spatial Domain



$$\mu \equiv (L/a_0, m^{\text{horn}}, a_{\text{mouth}}/a_0, ka_0) \\ \in \mathcal{P} \equiv [2, 20] \times [0.0334, 0.1666] \times [4, 12] \times [0, 1]$$

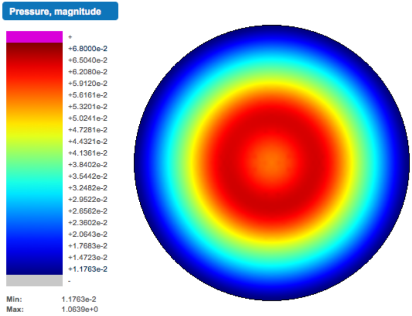
Throat Impedance

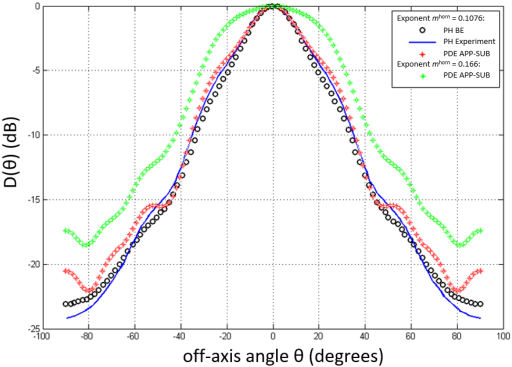
Parameters: $m^{\text{horn}} = 0.1076, a_{\text{mouth}}/a_0 = 10.67$.



PH: Post & Hixson, PhD Thesis, 1974.

Visualization: Radiation Directivity $ka_{\text{mouth}} = 10$





Nearfield

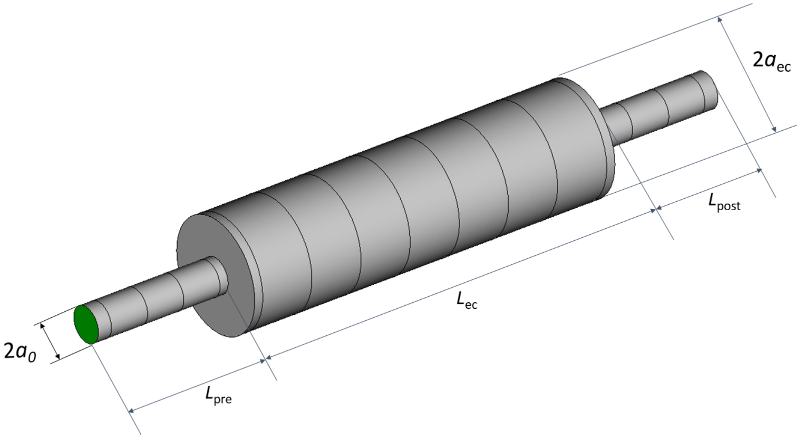
Farfield

Modulus of Pressure

Acoustics Ducts: PDE App Examples

- A Flanged Exponential Horn
- **An Expansion Chamber**
- A Circular Duct with Toroidal Bend
- An Extended-Tube Expansion Chamber (ETEC)

Model: Parametrization and Spatial Domain

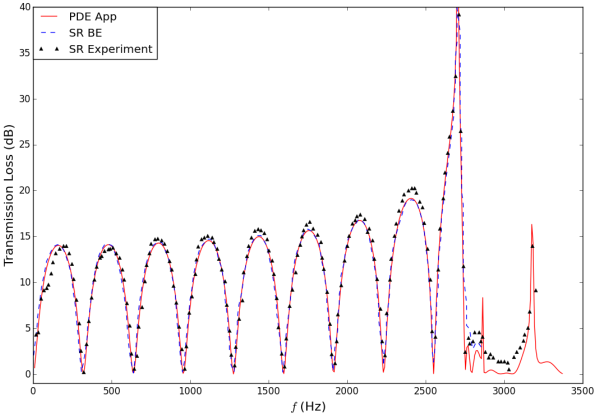


$$\mu \equiv (L_{\text{pre}}/a_0, L_{\text{post}}/a_0, L_{\text{ec}}/a_0, a_{\text{ec}}/a_0, ka_0)$$
$$\in \mathcal{P} \equiv [4, 12]^2 \times [1.5, 25] \times [1.5, 6.5] \times [0, 1.5]$$

Transmission Loss

Parameters: $L_{\mathrm{ec}}/a_0 = 22.26, a_{\mathrm{ec}}/a_0 = 3.152;$

$a_0 = 0.0243$ cm.

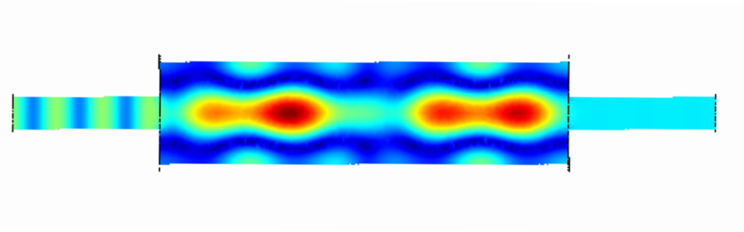


SR: Selamet and Radavich, J Sound Vibration, 1997.

Visualization: Excitation of (Axisymmetric) Higher Modes

Parameters: $L_{\mathrm{ec}}/a_0 = 22.26, a_{\mathrm{ec}}/a_0 = 3.1525;$

$f = 2.8$ kHz, $a_0 = 0.0243$ cm.



Modulus of Pressure

Acoustics Ducts: PDE App Examples

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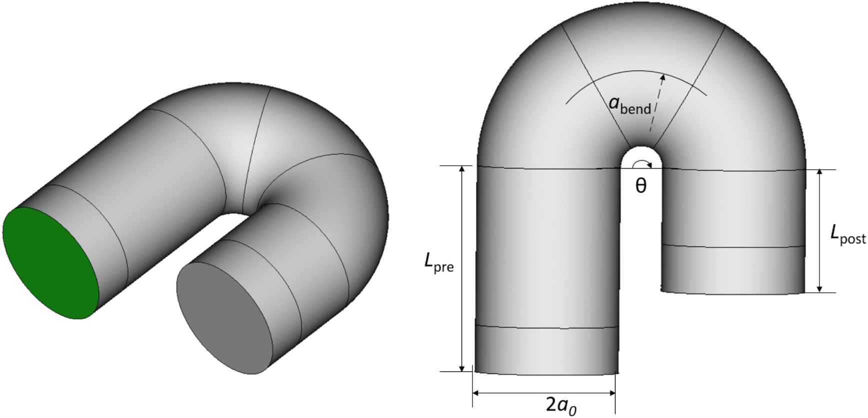
Model: Parametrization and Spatial Domain



$$\mu \equiv (L_{\text{pre}}/a_0, L_{\text{post}}/a_0, a_{\text{bend}}/a_0, \theta_{\text{bend}}, ka_0)$$
$$\in \mathcal{P} \equiv [1.5, 15]^2 \times [1.2, 3] \times [30°, 180°] \times [0, 1.8412]$$

WUI: Model Selection

PDE Apps for Education
The PDE App Project Team

Quick Start

- List of Models
  - Acoustic Ducts
    - Circular Duct with Bend
    - Expansion Chamber
    - ETEC Muffler
  - Acoustic Ducts SUB

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**Akselos S.A. licenses technology developed in the MIT research group of AT Patera. AT Patera has no financial interest in Akselos S.A.*

WUI: Parameter Specification

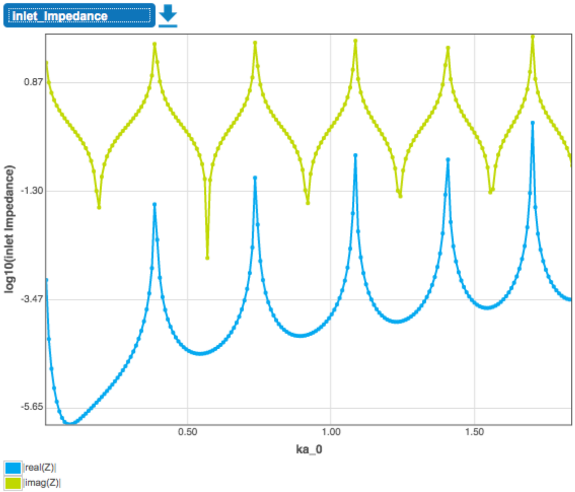
ACOUSTIC DUCTS - CIRCULAR DUCT WITH TOROIDAL BEND

SOLVE | MODEL INFO | HOW TO USE | LOGS

	Current value	New value	Range
L_pre/a_0	2.8571	2.8571	[1.5000, 15.0000]
L_post/a_0	1.7143	1.7143	[1.5000, 15.0000]
a_bend/a_0	1.2857	1.2857	[1.2000, 3.0000]
Bend angle	180.0000	180	[30.0000, 180.0000]
ka_0	1.8200	1.82	[0.0000, 1.8412]
Inlet BC	V	Velocity	
Outlet BC	V	Velocity	
Number of sweep points	200.0000	200	

Update Model

WUI: Output

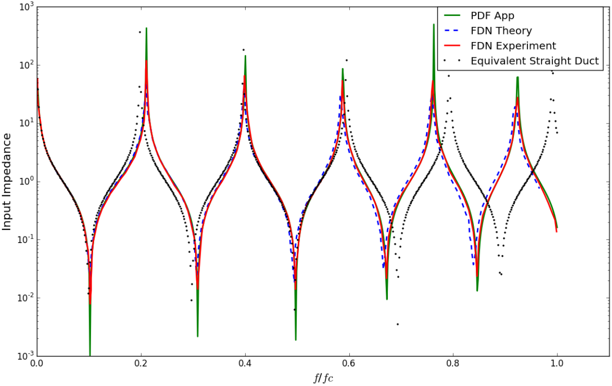


Response Time (all-inclusive) 8.4 seconds:
4-core GCE instance and commodity Internet.

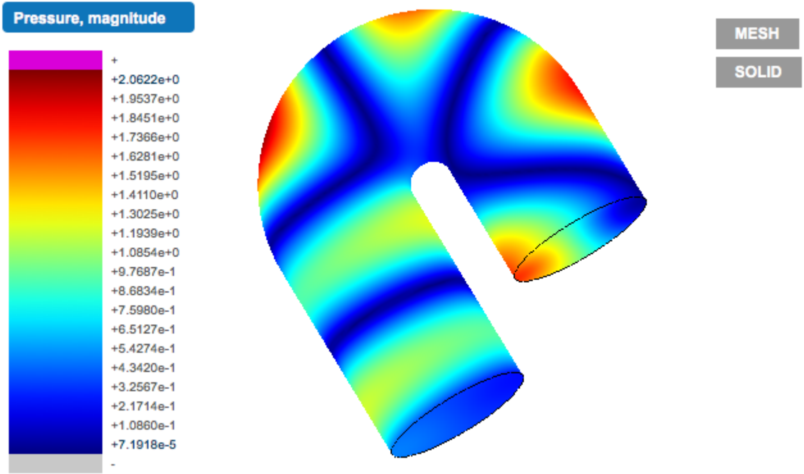
Inlet Impedance (Reactive)

Parameters: $L_{\text{pre}}/a_0 = 2.8571, L_{\text{post}}/a_0 = 1.7143,$
$a_{\text{bend}}/a_0 = 1.2857, \theta_{\text{bend}} = 180°.$

Boundary Conditions: velocity-velocity.



FDN: Félix, Dalmont, and Nederveen, JASA, 2012.



Response Time (all-inclusive) 8.4 seconds:
4-core GCE instance and commodity Internet.

Acoustics Ducts: PDE App Examples

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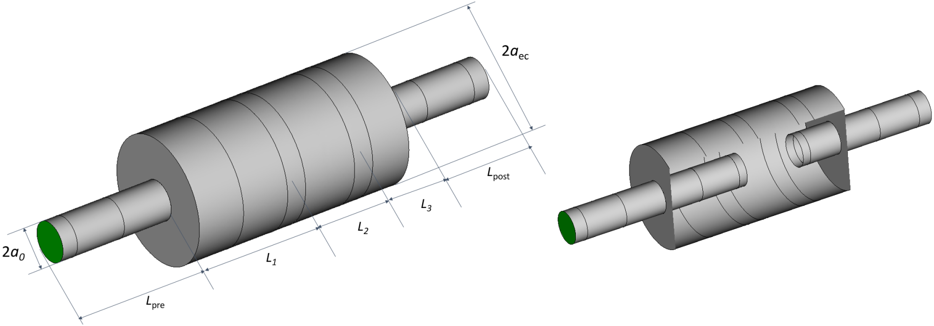
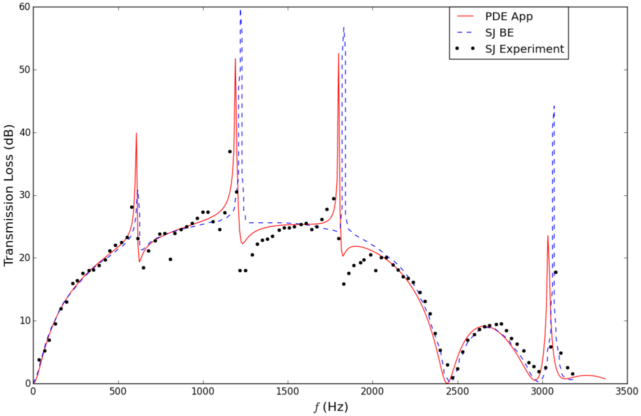
Model: Parametrization and Spatial Domain



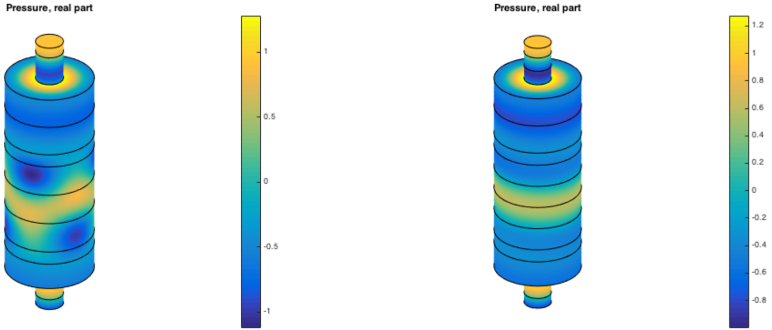
$$\mu \equiv (L_{\mathrm{pre}}/a_0, L_{\mathrm{post}}/a_0, L_1/a_0, L_2/a_0, L_3/a_0, a_{\mathrm{ec}}/a_0, ka_0)$$
$$\in \mathcal{P} \equiv [2,6]^2 \times [2,16]^3 \times [1.5,4.0] \times [0,1.5]$$

Parameters: $L_1/a_0 = 5.391, L_2/a_0 = 3.716,$
$L_3/a_0 = 2.510, a_{\mathrm{ec}}/a_0 = 3.152; a_0 = 0.0243$ cm.



SJ: Selamet and Ji, J Sound Vibration, 1999.

Achtung!

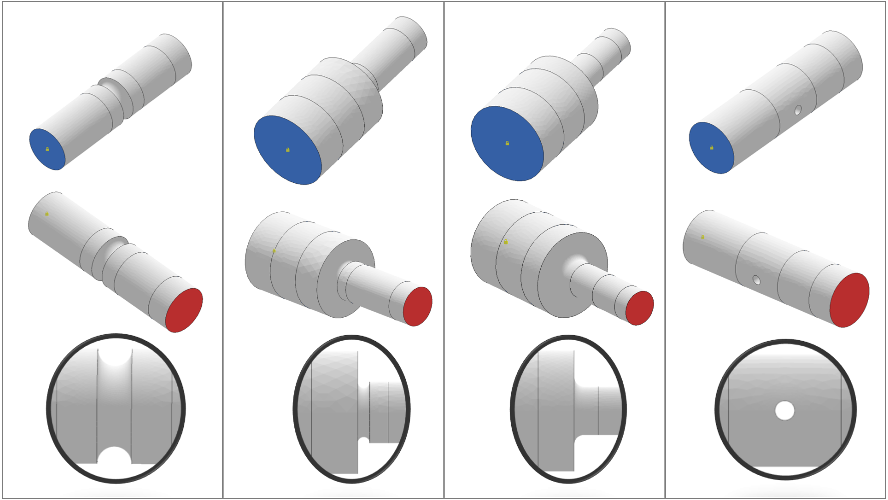


Issues: physical stability, numerical stability,. . .

. . . components, training, projection.

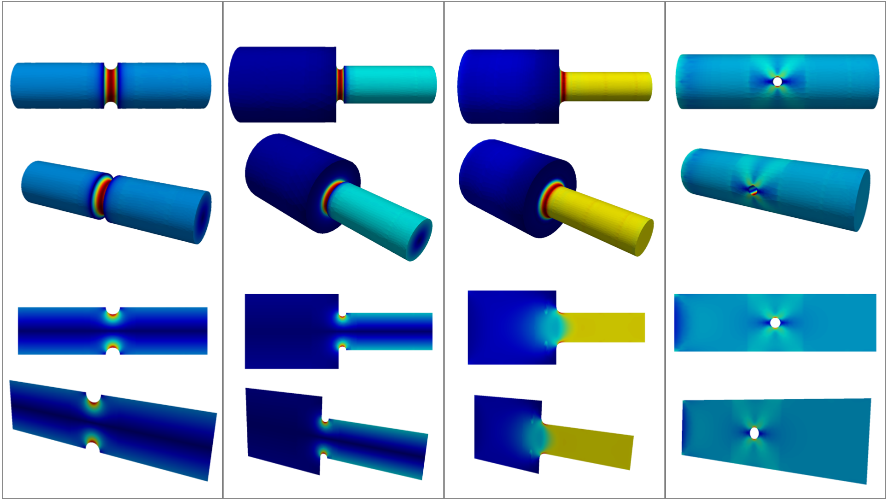
Elastic Shafts: PDE App Examples — briefly

A Family $\mathcal{F}$ of Models



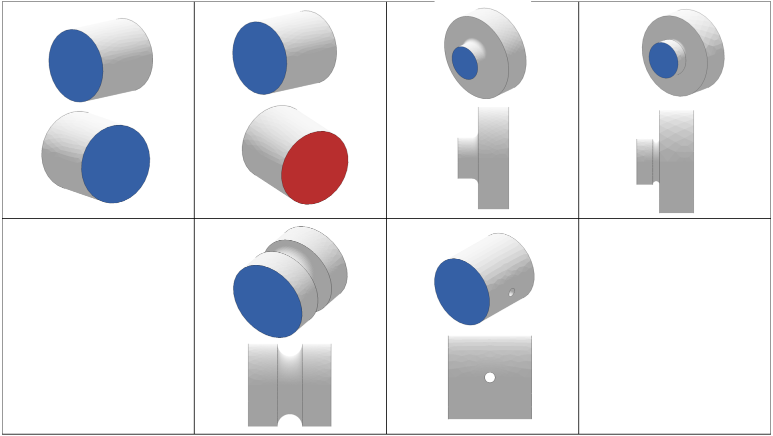
$\mathcal{F}$: Elasticity, Shafts; Stress Concentration Factors (SCFs)

A Family of Models, $\mathcal{F}$

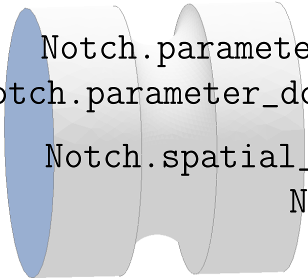


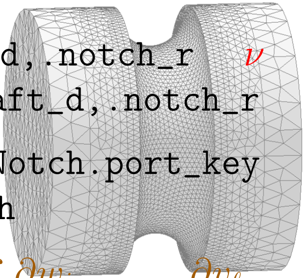
$\mathcal{F}$: Elasticity, Shafts; SCFs

A *Library* $\mathcal{L}$ of Archetype Components for $\mathcal{F}$ Port Key



A Parametrized *Archetype Component in* $\mathcal{L}$





`Notch`

`Notch.parameter.shaft_d`, `.notch_r` ν

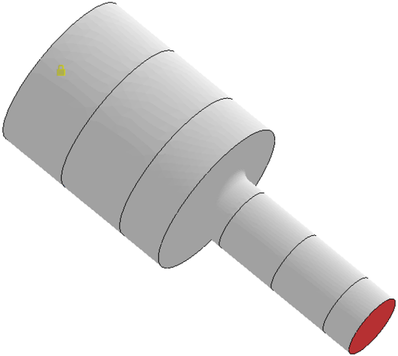
`Notch.parameter_domain.shaft_d`, `.notch_r` $\mathcal{V}$

`Notch.spatial_domain`, `Notch.port_key`

`Notch.mesh`

`Notch.PDE.forms.a` $= \int \frac{\partial w_i}{\partial x_j} E_{ij\ell m} \frac{\partial v_\ell}{\partial x_m}$

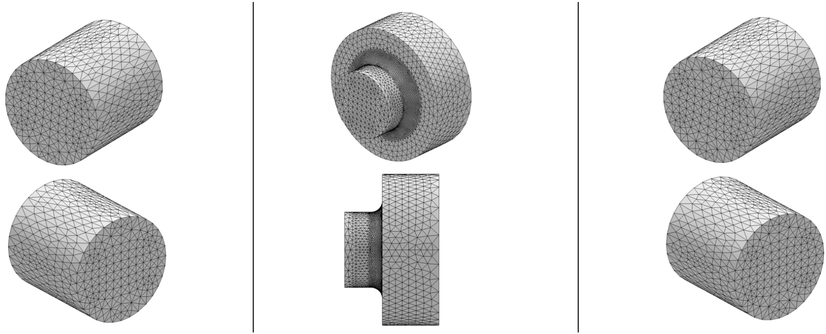
`Notch.PDE.forms.f` $= 0$



Model_Shoulder_Fillet

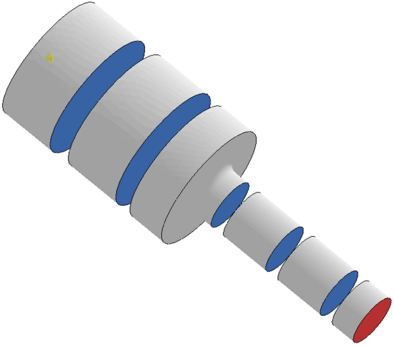
$\mu \equiv (L, D_1/D_2, r_{\text{fillet}}/D_2, E)$

Synthesis: Model_Shoulder_Fillet in $\mathcal{F}$



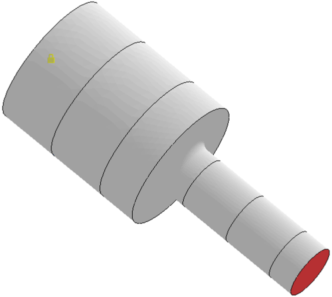
(Relevant) Parametrized Archetype Components:
Tension_Torsion_Load,
Shoulder_Fillet, Circular_Shaft

Synthesis: `Model_Shoulder_Fillet` in $\mathcal{F}$



$$\mu \equiv (L, D_1/D_2, r_{\mathrm{fillet}}/D_2, E)$$
$\rightarrow$ (ν) Parameter-Instantiated (Archetype) Components

Synthesis: `Model_Shoulder_Fillet` in $\mathcal{F}$



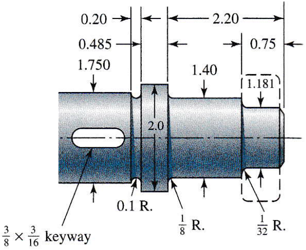
Geometry

Mesh

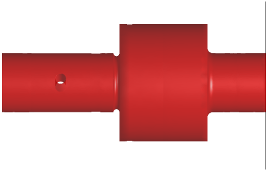
Connection of Local Ports $(\mu) \rightarrow$

Global Ports and Assembled *System*

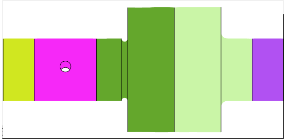
Synthesis: `Model_Shaft` in $\mathcal{F}_{\mathcal{L}}$ — Formulation



Shigley *et al.* Problem 18-81

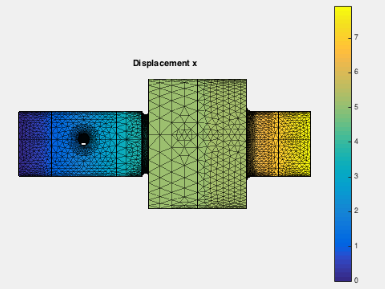


Model: $\mu \in \mathcal{P} \to \Omega_\mu$

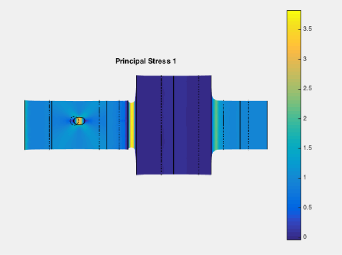


Assembled System $\leftarrow \mathcal{L}$

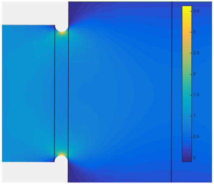
Synthesis: `Model_Shaft` in $\mathcal{F}_{\mathcal{L}}$ — Solution (accuracy?)



Axial Displacement



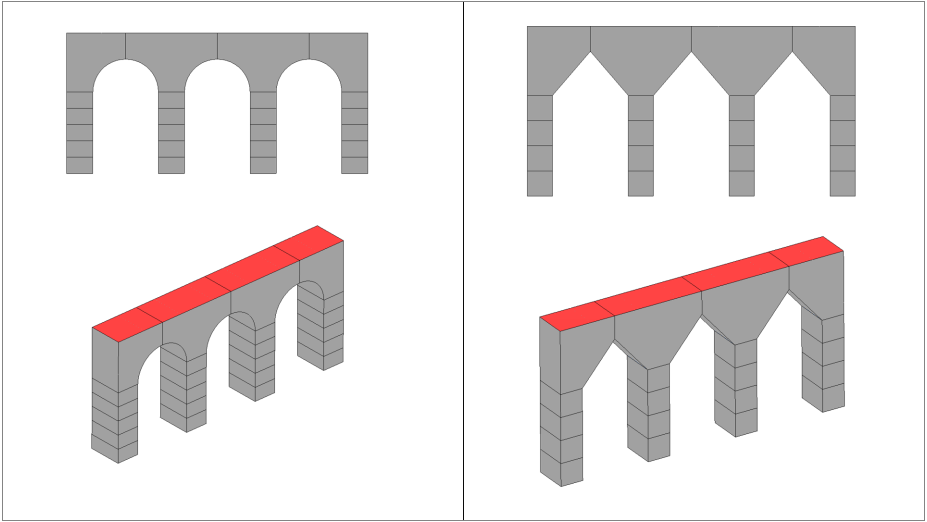
Principal Stress: σ_1



Axial Stress: σ_{11}

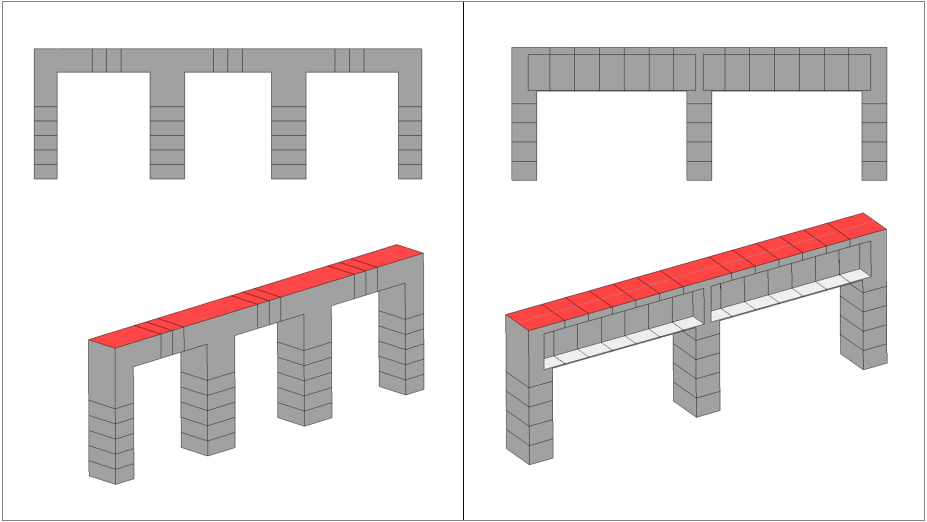
Lintels and Arches: PDE App Examples — briefly

A Family $\mathcal{F}$: Models. . .



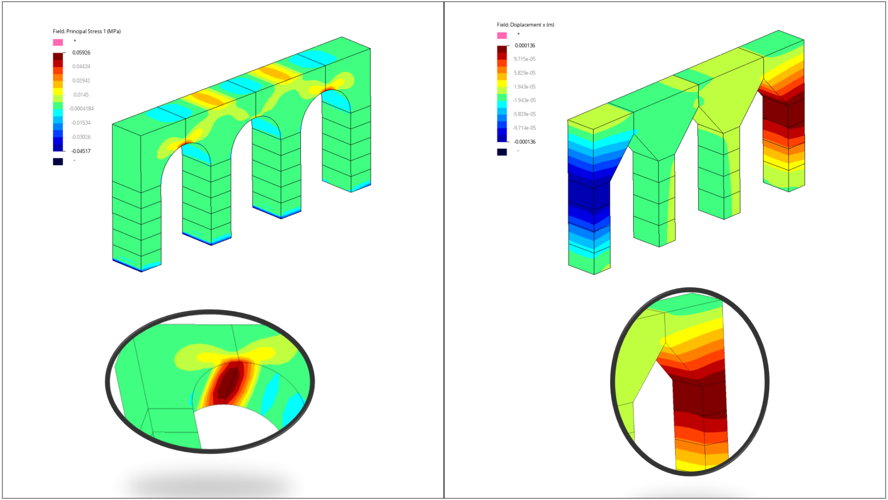
$\mathcal{F}$: Elasticity, Lintels and Colonnades; Max $|u|$, Max σ_1

A Family $\mathcal{F}$: Models. . .



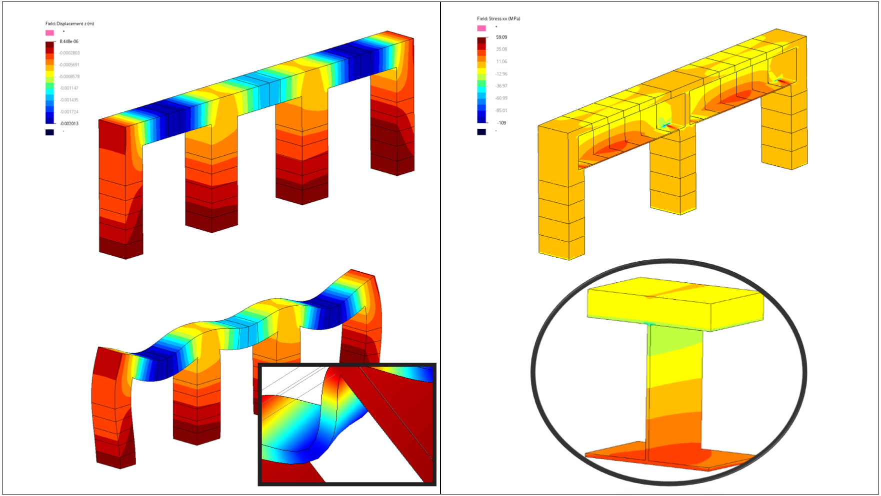
$\mathcal{F}$: Elasticity, Lintels and Colonnades; Max $|u|$, Max σ_1

A Family $\mathcal{F}$: ...and Representative Solutions

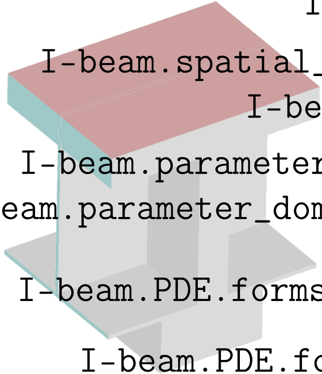
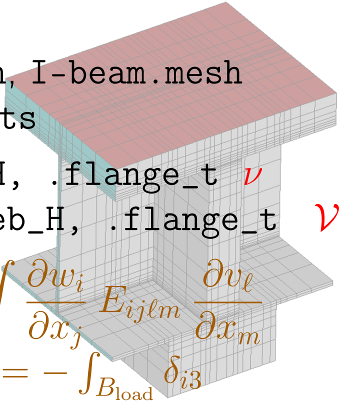


$\mathcal{F}$: Elasticity, Lintels and Colonnades; Max $|u|$, Max σ_1

A Family $\mathcal{F}$: ...and Representative Solutions



$\mathcal{F}$: Elasticity, Lintels and Colonnades; Max $|u|$, Max σ_1

`I-beam`

`I-beam.spatial_domain`, `I-beam.mesh`

`I-beam.ports`

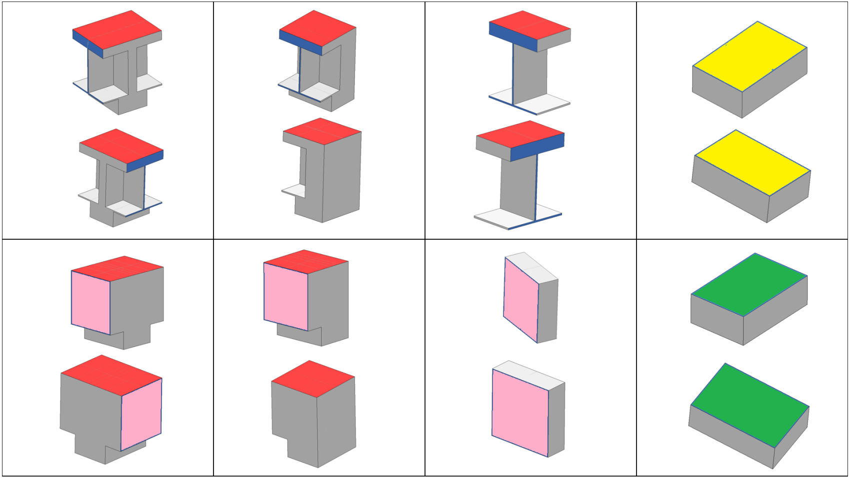
`I-beam.parameter.web_H, .flange_t` ν

`I-beam.parameter_domain.web_H, .flange_t` $\mathcal{V}$

`I-beam.PDE.forms.a` $= \int \frac{\partial w_i}{\partial x_j} E_{ij\ell m} \frac{\partial v_\ell}{\partial x_m}$

`I-beam.PDE.forms.f` $= -\int_{B_{\mathrm{load}}} \delta_{i3}$

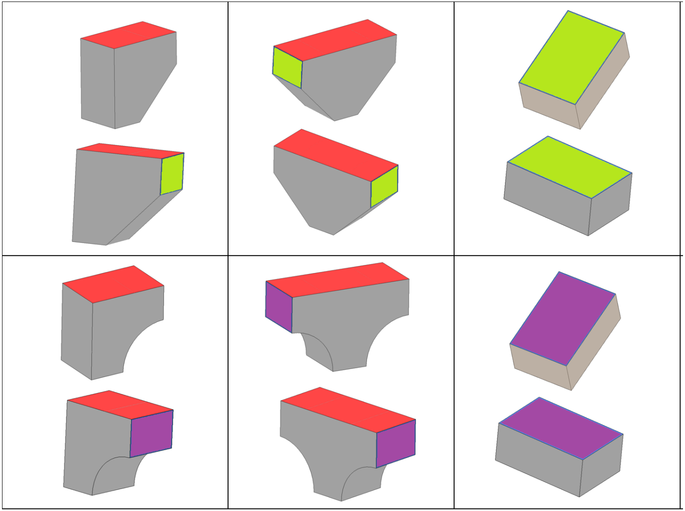
A *Library* $\mathcal{L}$ of Parametrized Archetype Components for $\mathcal{F}$



Admissible connections:

ports of common color $\leftarrow$ common port key.

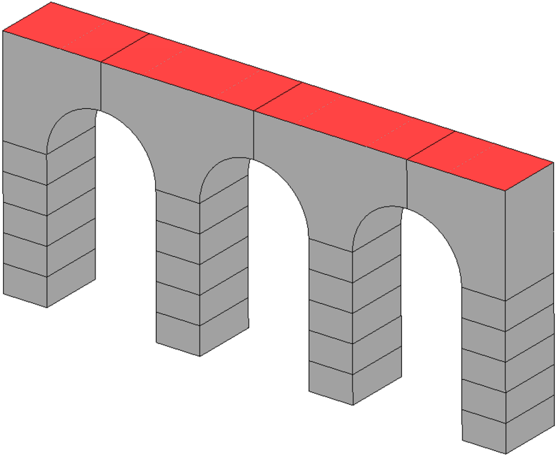
A *Library* $\mathcal{L}$ of Parametrized Archetype Components for $\mathcal{F}$



Admissible connections:

ports of common color $\leftarrow$ common port key.

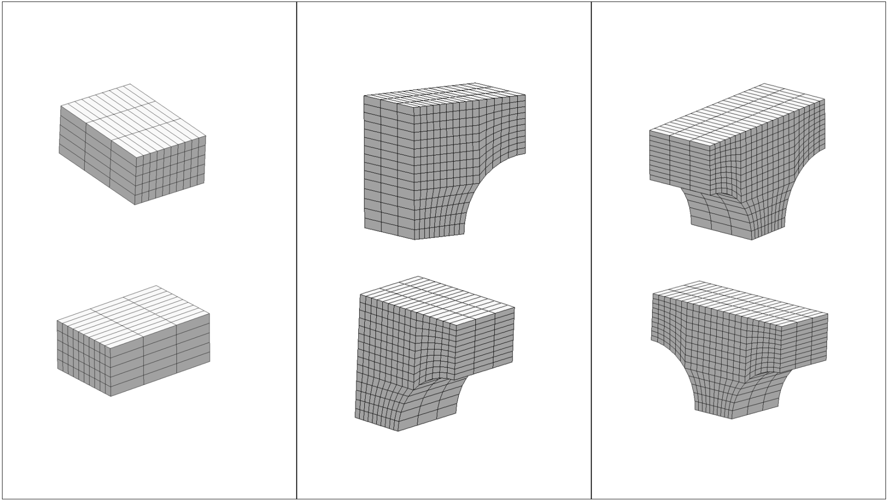
Synthesis: `Model_Roman_Arch` in $\mathcal{F}$



`Model_Roman_Arch`

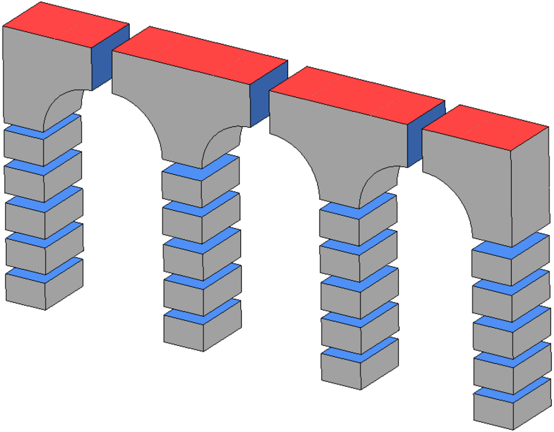
$$\mu \equiv (H_{\text{col}}/D, W_{\text{col}}/D, r_{\text{arch}}/D, n_{\text{arches}}, E) \in \mathcal{P}$$

Synthesis: `Model_Roman_Arch` in $\mathcal{F}$



(Relevant) Parametrized Archetype Components:
`Column`, `Arch_End`, `Arch_Middle`

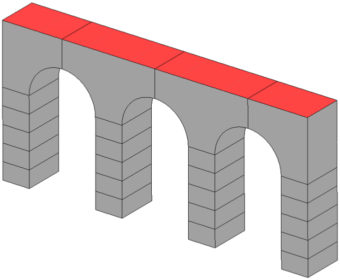
Synthesis: `Model_Roman_Arch` in $\mathcal{F}$



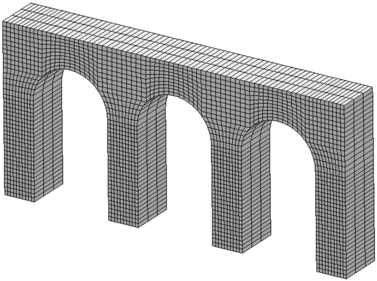
$$\mu \equiv (H_{\mathrm{col}}/D, W_{\mathrm{col}}/D, r_{\mathrm{arch}}/D, n_{\mathrm{arches}}, E) \in \mathcal{P}$$

$\rightarrow \nu$ Parameter-Instantiated (Archetype) Components

Synthesis: `Model_Roman_Arch` in $\mathcal{F}$



Geometry



Mesh

Connection of Local Ports $(\mu) \rightarrow$

Global Ports and Assembled *System*

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