# *High-performance climate modelling : mimetic finite differences, and beyond ?*


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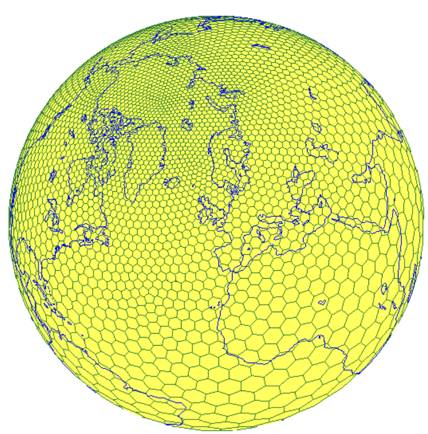
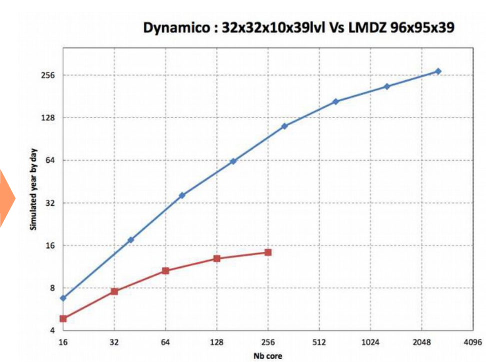
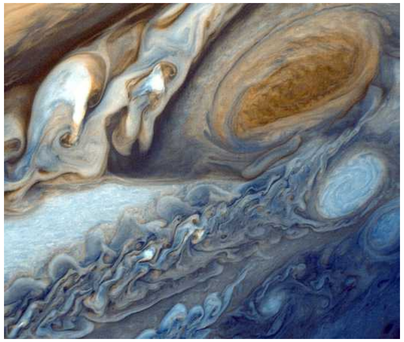
*Yann Meurdesoif (LSCE/IPSL), Evaggelos Kritsikis (LAGA/Paris XIII), ...*


$$\delta \int \mathcal{L} \mathrm{d}t = 0$$





## Background on climate modelling

- Weather vs climate
- Characteristic scales
- Equations for atmospheric flow motion

## DYNAMICO, an energy-conserving finite difference/finite volume atmospheric solver

- Conservation of energy : why ?
- Hamiltonian formulation
- Preliminary results

## Finite elements, the path to higher-order accuracy ?

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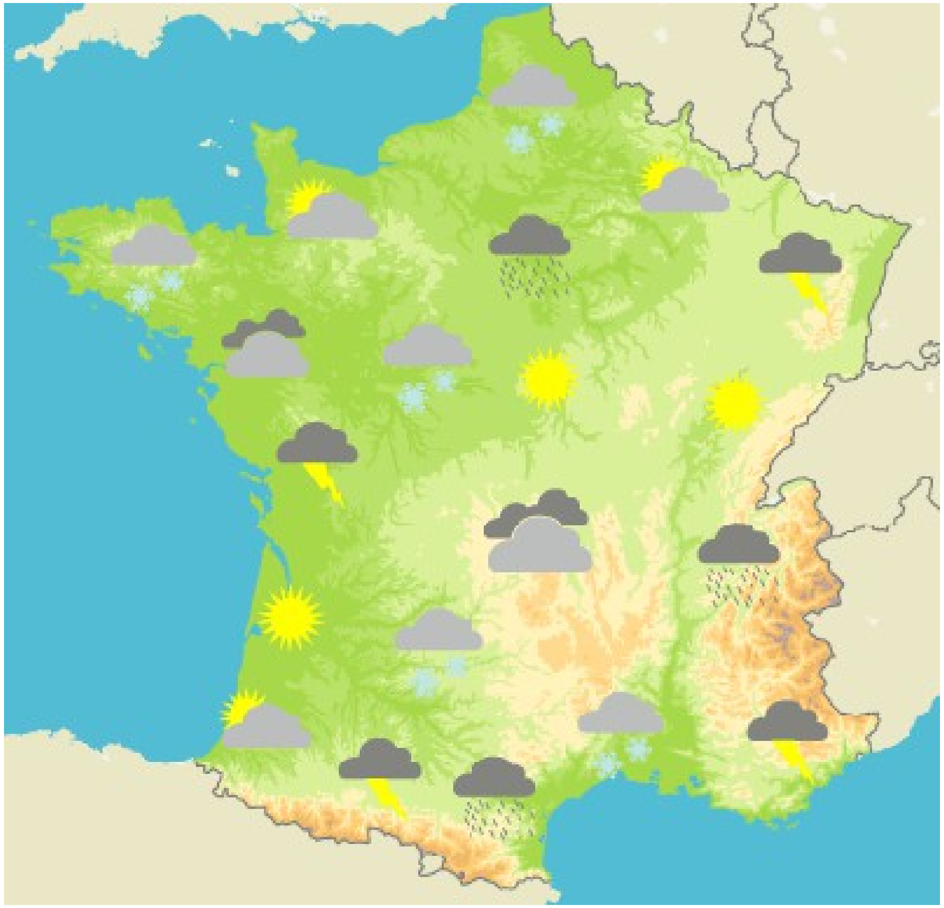
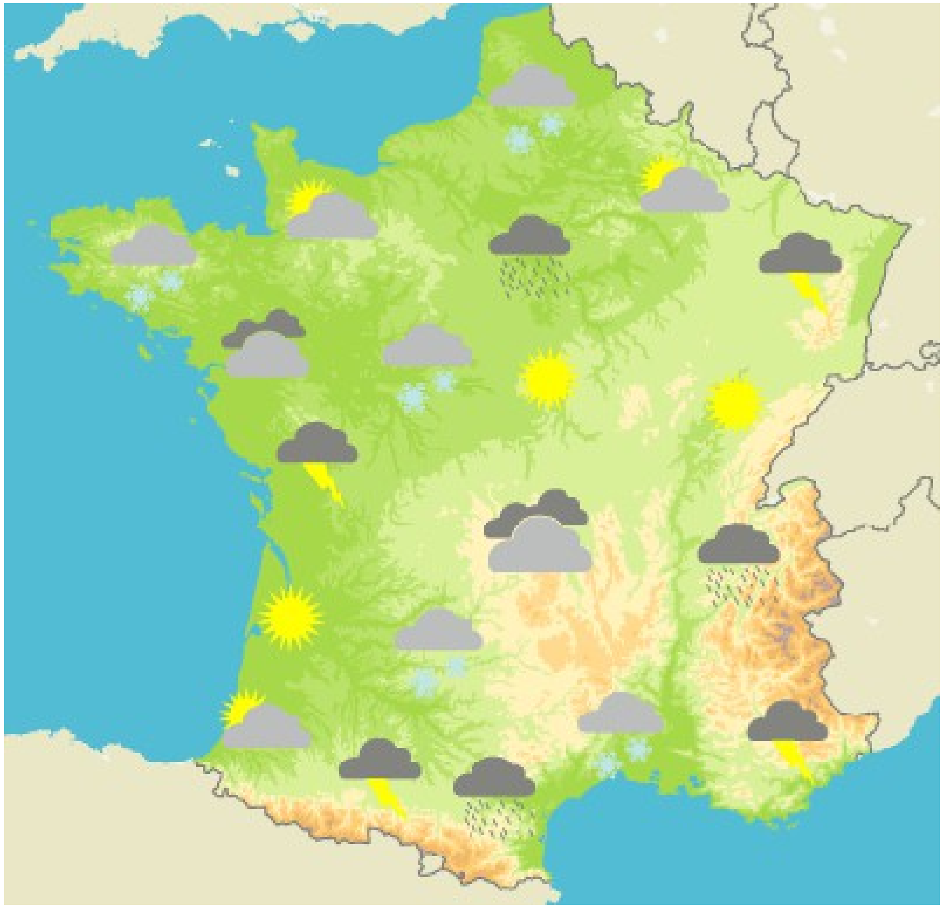
# Weather forecasting

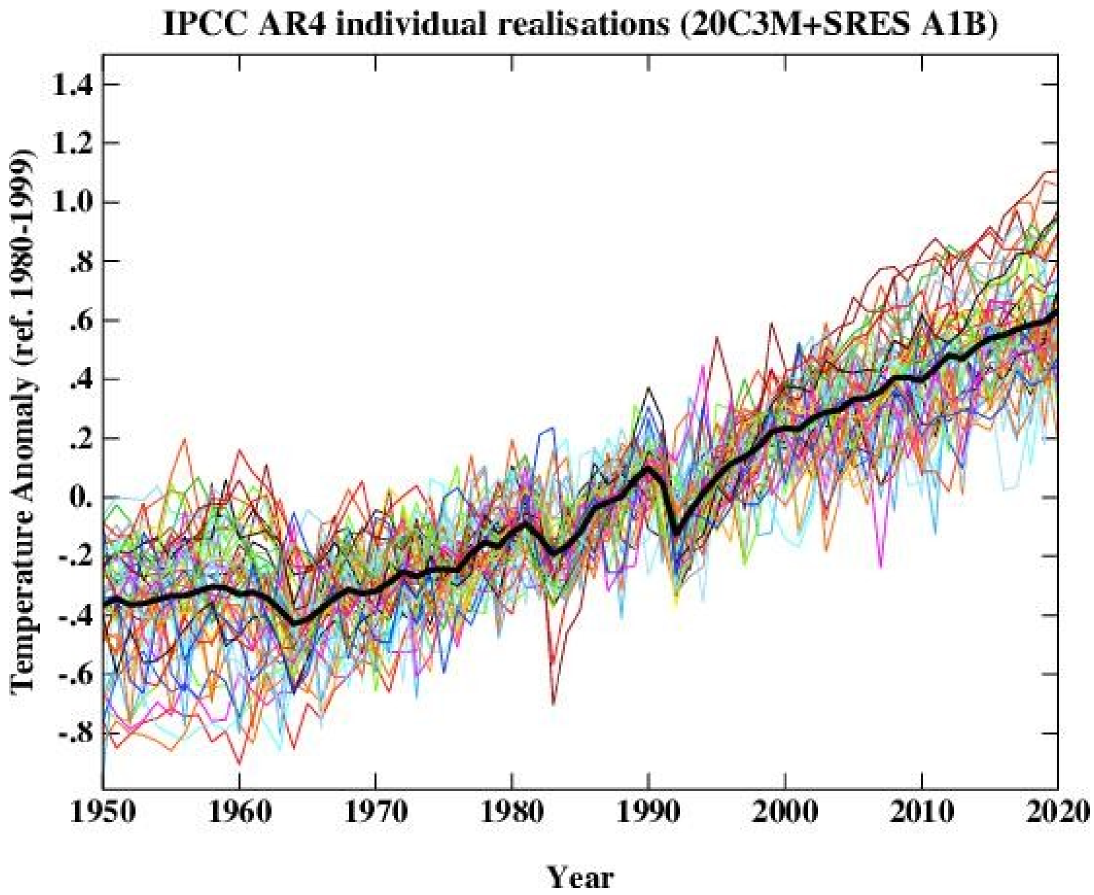


# Weather forecasting



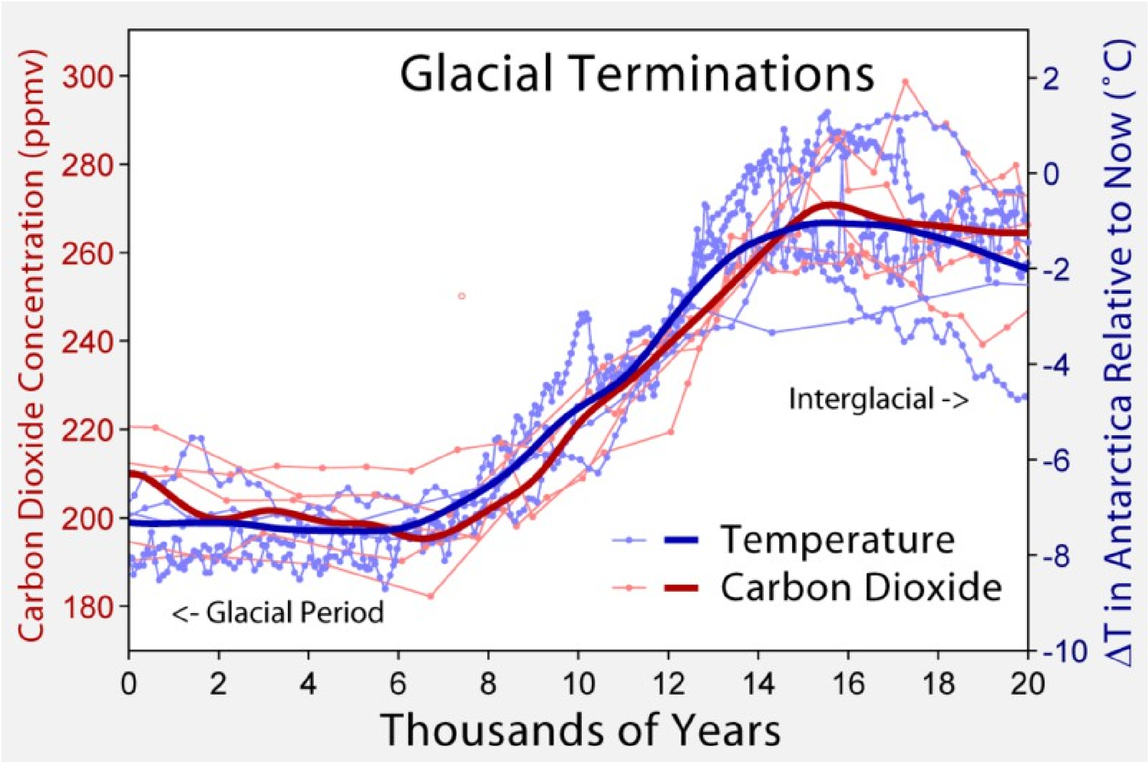
$$\frac{\text{4 simulated days}}{\text{1 h of walltime}} = \mathbf{x\ 100}$$

# Modern climate

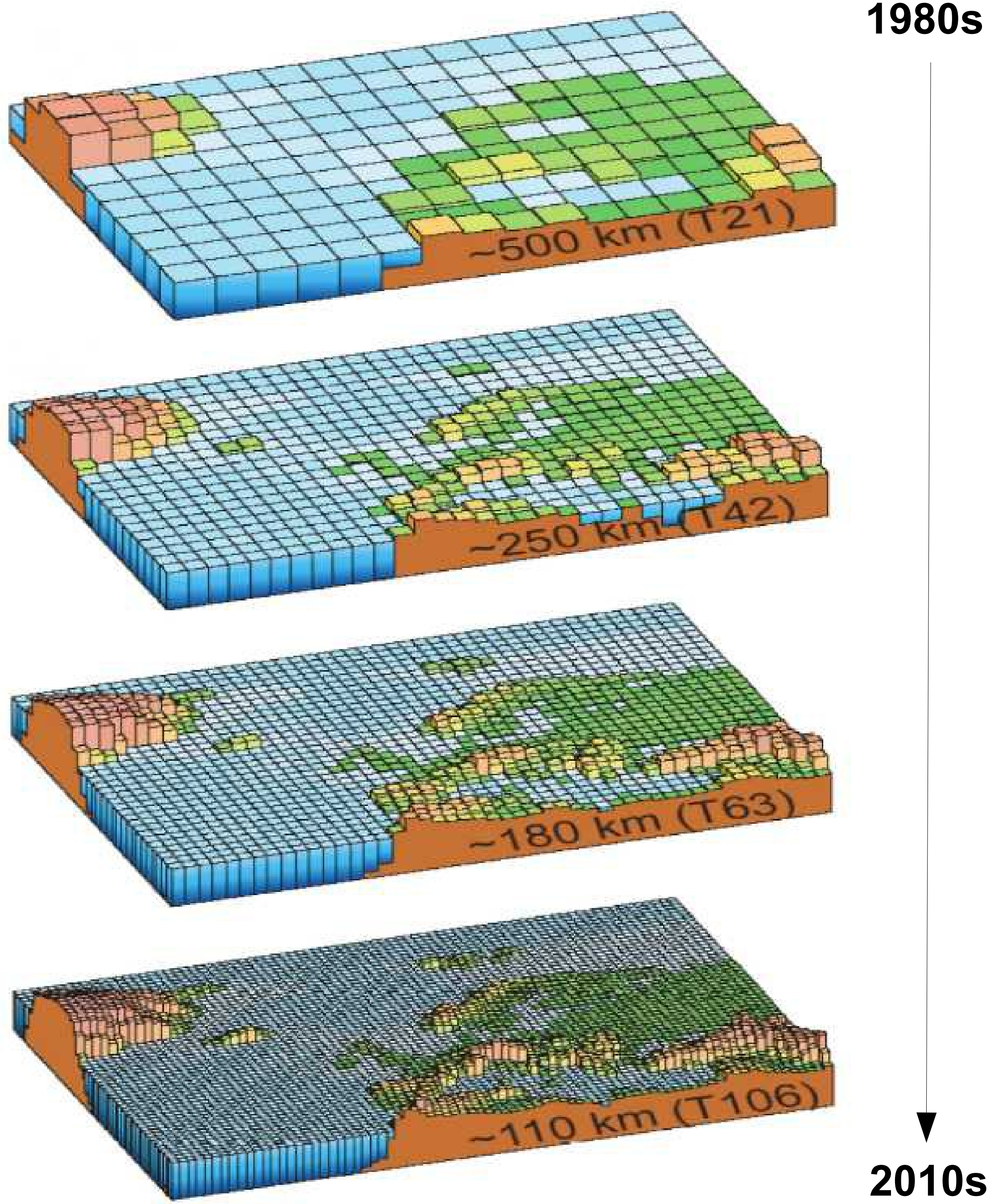



$$\frac{\text{100 simulated years}}{\text{1 month of walltime}} = \textbf{x 1000}$$

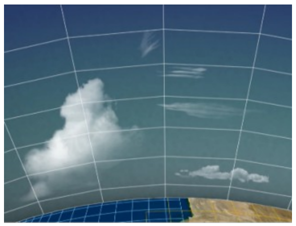
# Paleoclimate

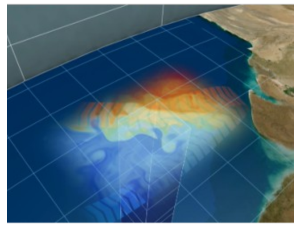


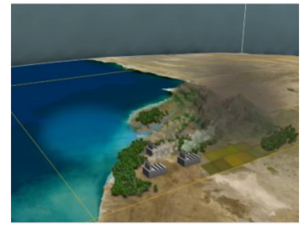

$$\frac{\text{10000 simulated years}}{\text{1 year of walltime}} = \times \; \mathbf{10000}$$

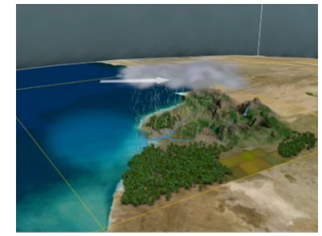

~500 km (T21)
~250 km (T42)
~180 km (T63)
~110 km (T106)
1980s
2010s

# Earth System Modelling at IPSL









**INCA / REPROBUS**
(chimie atmosphérique)
(aérosol)

**ORCHIDEE**
(surfaces continentales)
(végétation)

**LMDZ**
(atmosphère)

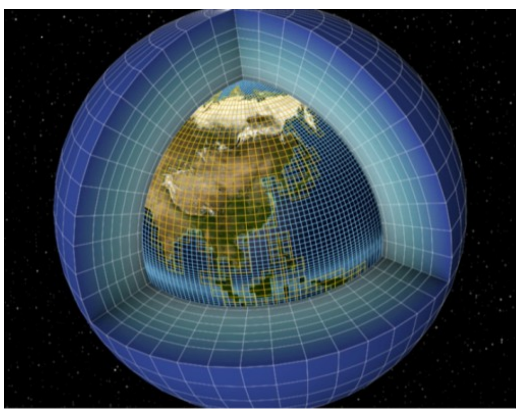
**OASIS**
(coupleur)

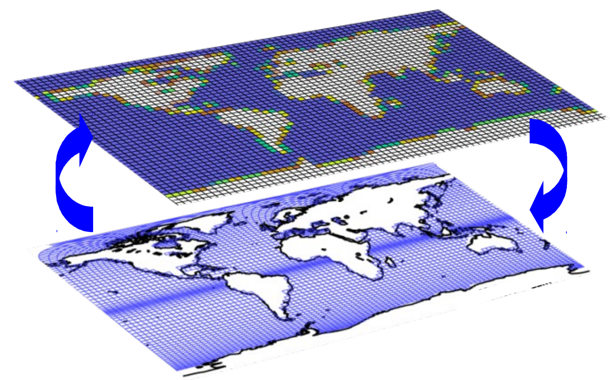
**OPA**
(océan)

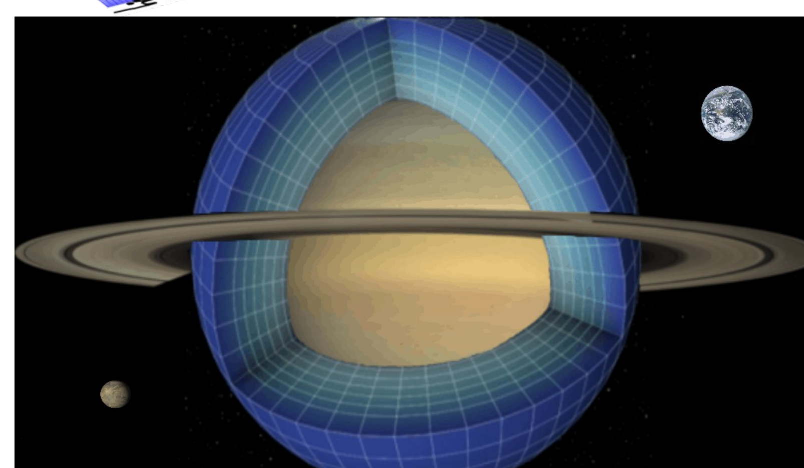
**LIM**
(glace de mer)

**PISCES**
(biogéochimie marine)

**NEMO**





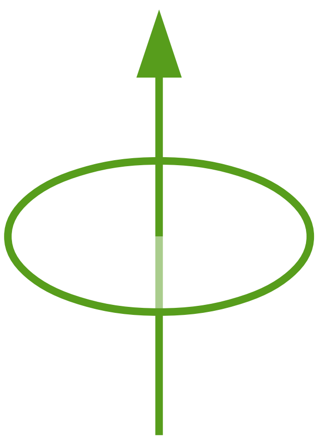


+ planetary atmospheres
(F. Forget, ...)

- Newton's fundamental principle of dynamics
- Forces : pressure and gravity
- Pseudo-forces : Coriolis and centrifugal

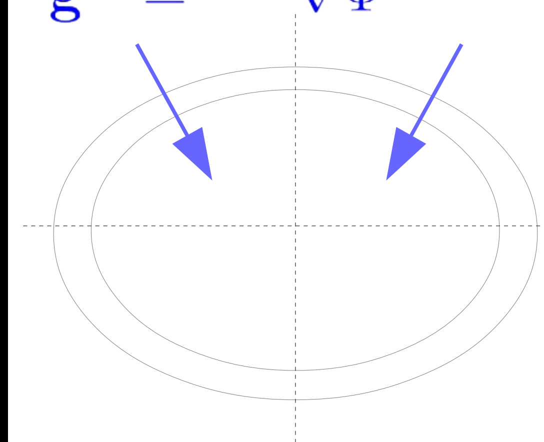
*Planetary velocity*

$$\mathbf{R} = \mathbf{\Omega} \times \mathbf{x}$$



*Geopotential*

$$\Phi = V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R}$$
$$\mathbf{g} = -\nabla\Phi$$



$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\ \rho\dot{\mathbf{x}} = 0 \qquad \frac{Ds}{Dt} = \frac{q}{T}$$

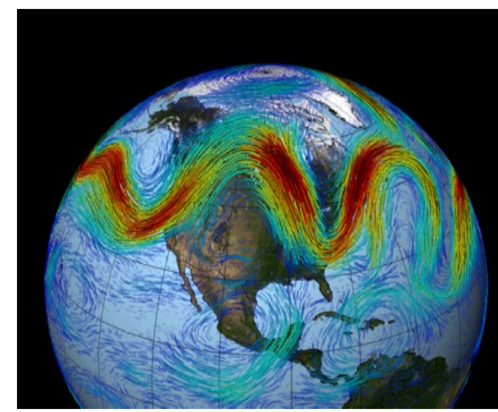
# Characteristic scales

- *Velocity* : Sound $c \sim 340$ m/s — Wind $U \sim 30$ m/s
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2}\,s^{-1}$ — Coriolis $f \sim 10^{-4}\,s^{-1}$
- *Length* : **Scale height $H=c^2/g=10$km** — **Rossby radius : $R=c/f \sim 1000$ km**
  Planetary radius $a=6400$km

*Flatness* $af^2/g \ll 1$ — *Shallowness* $H/a \ll 1$

*Mach number* $M=U/c \ll 1$ — *Scale separation* $f/N \sim H/R \ll 1$



small-scale — scale height — mesoscale — synoptic — planetary

1 km — **10 km** — 100 km — **1000 km** — 10000 km

# Hydrostatic approximation

Vertical momentum budget

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

If vertical acceleration is small … $\frac{\mathrm{d}w}{\mathrm{d}t} \ll g, \frac{1}{\rho}\frac{\partial p}{\partial z}$

… then

$$\frac{\partial p}{\partial z} = -\rho g$$

*Hydrostatic balance*

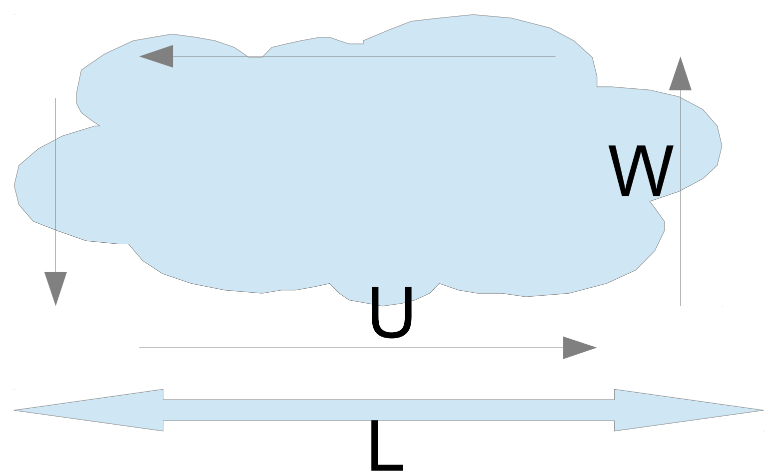
When does this approximation work ? $\frac{\mathrm{d}}{\mathrm{d}t} \sim \frac{W}{H} \sim \frac{U}{L}$

$$\frac{\partial p}{\partial x} \sim \rho\frac{\mathrm{d}u}{\mathrm{d}t} \sim \rho\frac{U}{L}U$$

$$\frac{\partial p}{\partial z} \sim \frac{L}{H}\frac{\partial p}{\partial x} \sim \frac{\rho U^2}{H}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} \sim \frac{W^2}{H}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}/\rho\frac{\partial p}{\partial z} \sim \left(\frac{W}{U}\right)^2 \sim \left(\frac{H}{L}\right)^2$$

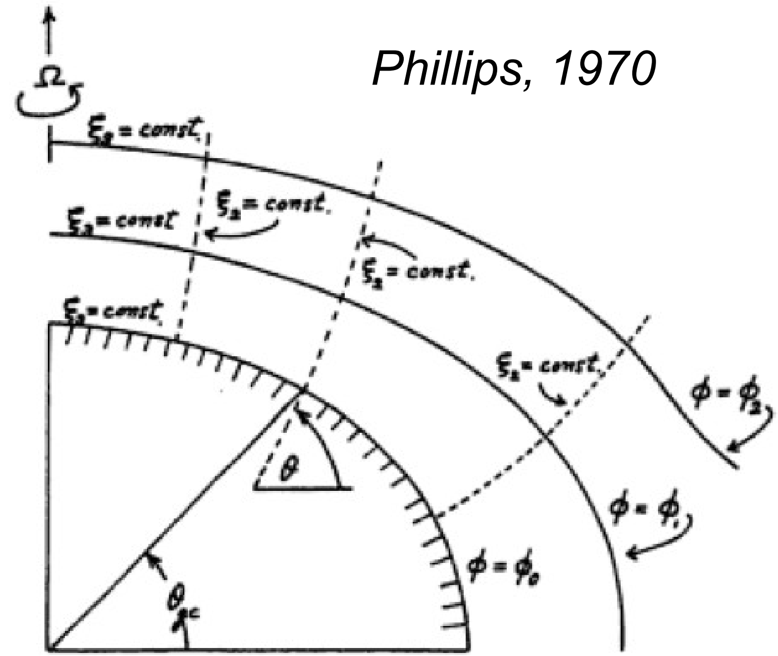



Works for circulations with a large horizontal scale (L >> H ~ 10km)

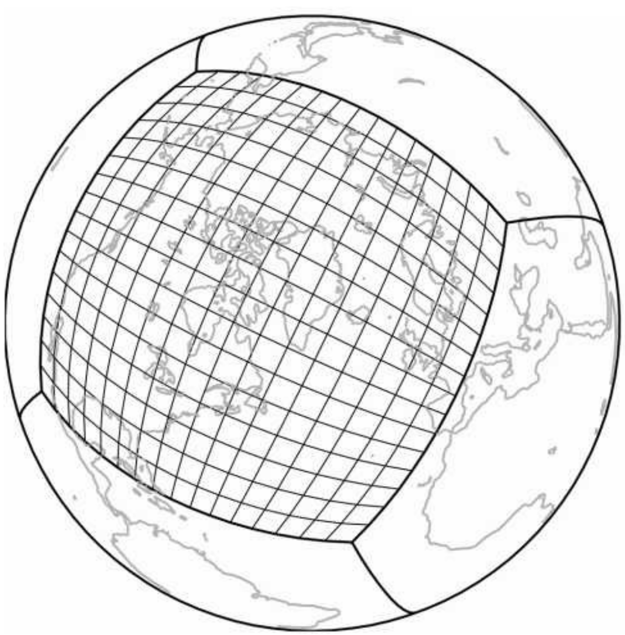
Incorrect for convection (storms), mountain flow, ...

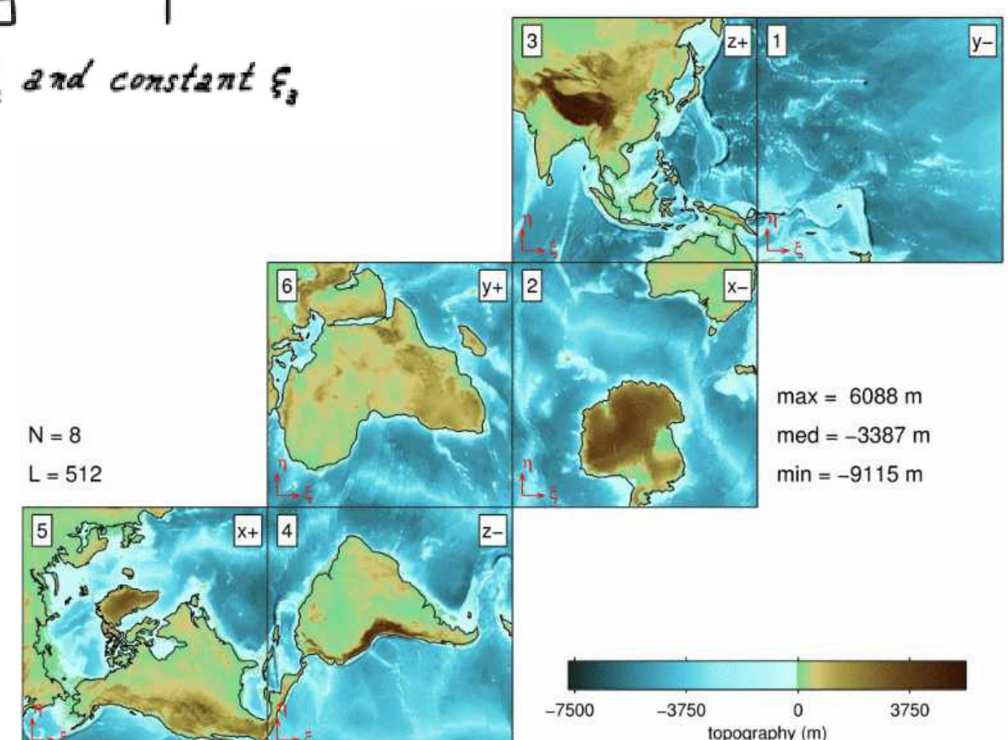
# Geopotential (curvilinear) coordinates

*Phillips, 1970*




Surfaces of constant $\xi_2$ and constant $\xi_3$





# Transport in curvilinear coordinates

$$(x,\, y,\, z) \rightarrow \left(\xi^1,\, \xi^2,\, \Phi\right)$$

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$$\dot{x} \Rightarrow u^i = \frac{D\xi^i}{Dt}$$

*contravariant* velocity components

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*contravariant* velocity components

$$\begin{aligned} J(\xi^i)\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 &= \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ \mathrm{d}m = \mu\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 &= \rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ \mu &= \rho J \end{aligned}$$

*pseudo-density*

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*pseudo-density*

$$\begin{aligned} \frac{\partial s}{\partial t} + \dot{\lambda}\partial_\lambda s + \dot{\phi}\partial_\phi s + \dot{r}\partial_r s &= \frac{q}{T} \\ \rho r^2 \cos\phi &= \mu \\ \frac{\partial \mu}{\partial t} + \partial_\lambda\left(\mu\dot{\lambda}\right) + \partial_\phi\left(\mu\dot{\phi}\right) + \partial_r\left(\mu\dot{r}\right) &= 0 \end{aligned}$$

- Contravariant formulation independent from the coordinate system
- No information about the geometry needed
- Easily in conservative form (flux-form)

# Dynamics in curvilinear coordinates

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

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$$\updownarrow$$

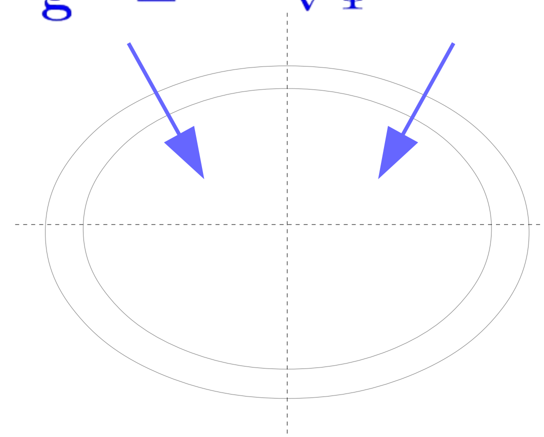
$$\boxed{\begin{aligned} G_{ij}\frac{Du^j}{Dt} + \frac{1}{2}\left(\partial_j G_{ik} + \partial_k G_{ij} - \partial_i G_{jk}\right) u^j u^k & \\ + \left[\partial_j R_i - \partial_i R_j\right] u^j + \frac{J}{\mu}\partial_i p + \partial_i \Phi &= 0 \end{aligned}}$$

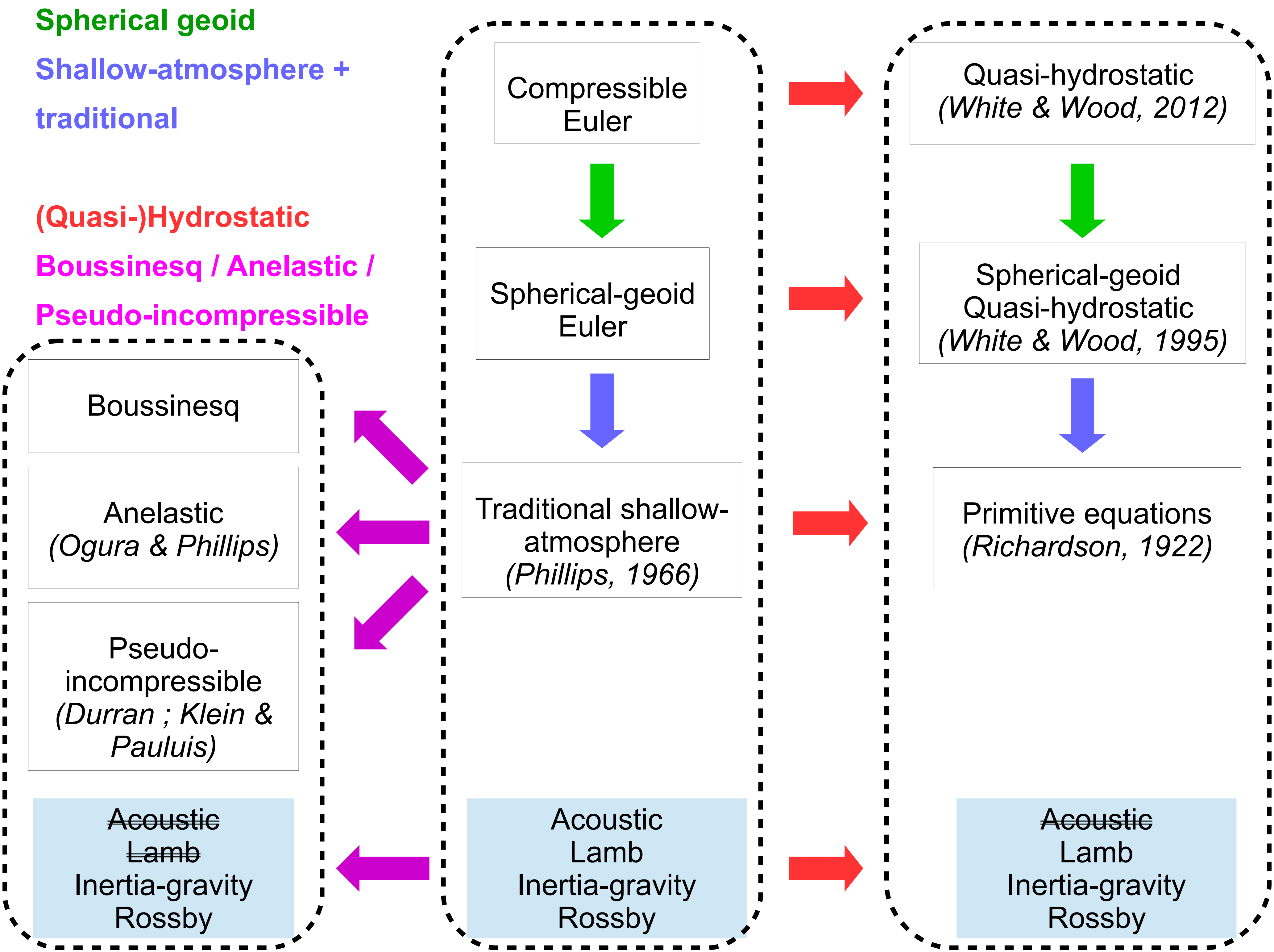
# Spherical geoid approximation

- Ellipticity of geoids ~ centrifugal / gravitational ~ 1/300
- Spherical geoid approximation : pretend that the metric in geopotential coordinates is actually spherical !

*Geopotential*

$$\begin{aligned} \Phi &= V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R} \\ \mathbf{g} &= -\nabla\Phi \end{aligned}$$




Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere (Phillips, 1966)
Quasi-hydrostatic (White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)
Primitive equations (Richardson, 1922)
Boussinesq
Anelastic (Ogura & Phillips)
Pseudo-incompressible (Durran ; Klein & Pauluis)
~~Acoustic~~ ~~Lamb~~ Inertia-gravity Rossby
Acoustic Lamb Inertia-gravity Rossby
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- Hamiltonian formulation
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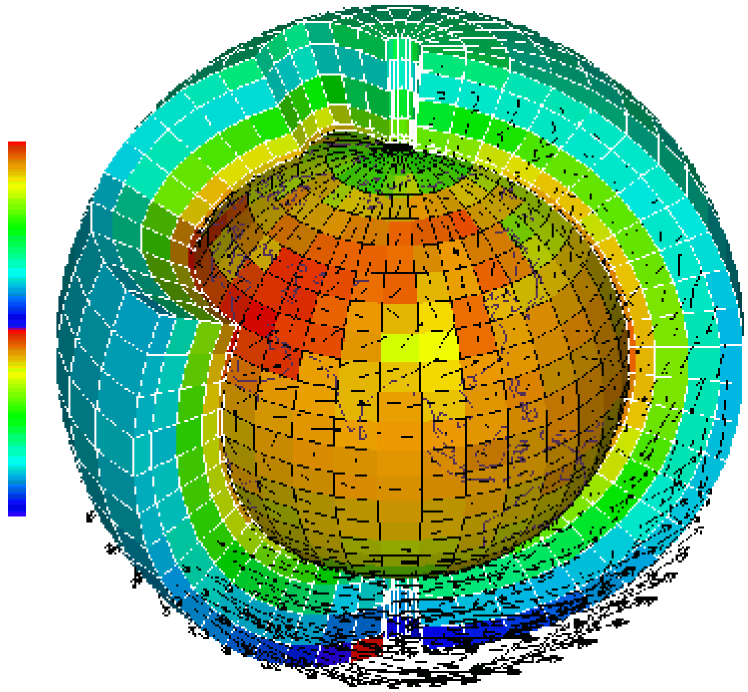
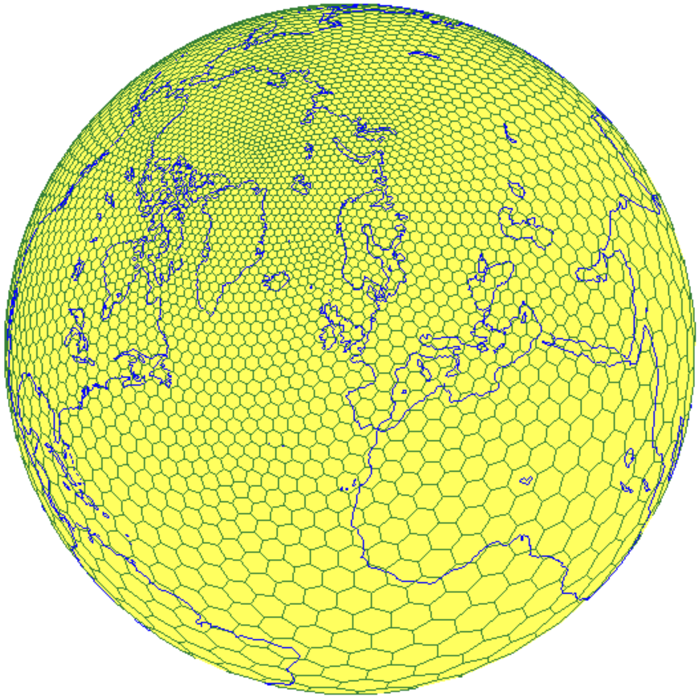
## Finite elements, the path to higher-order accuracy ?



LMD-Z lon-lat core



Enstrophy-conserving finite differences on lon-lat mesh (Sadourny, 1975)

Positive definite finite-volume transport (Hourdin & Armengaud, 1999)

**Scalability**

**Consistency**

***discrete conservation laws***

**Versatility**

***not tied to a unique equation set***

## *Why would a numerical model want to conserve energy ?*

density

specific volume

specific entropy

$$\mathcal{H} = \int \rho \left( \underbrace{\frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}}_{\text{kinetic}} + \underbrace{e\left(\frac{1}{\rho}, s\right)}_{\text{internal}} + \underbrace{gz}_{\text{potential}} \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Energy minimum given total mass and entropy ?

$$\mathcal{M} = \int \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad \mathcal{S} = \int \rho s \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\delta\left(\mathcal{H} - G_0\mathcal{M} - T_0\mathcal{S}\right) = 0$$

$$\Updownarrow$$

$u = v = w = 0$ *Resting*

$T = T_0$ *Isothermal*

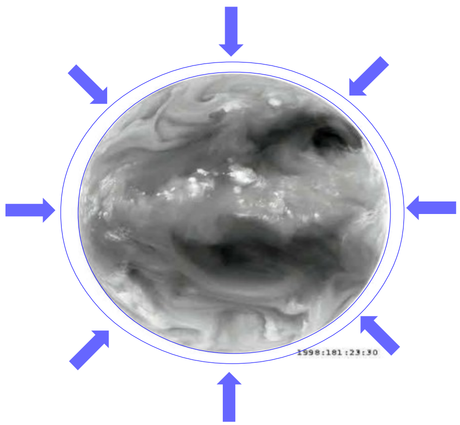
$e + \dfrac{p}{\rho} - Ts + gz = G_0$ *Hydrostatic*

(discrete) conservation limits dynamically accessible states
=> stability

numerics conserve energy, mass , entropy

=> isothermal state of rest is stable

## Adiabatic equations of motion imply **conservation laws** because they derive from a least **action principle**



inertia Coriolis pressure gravity

$$\frac{D\dot{\mathbf{x}}}{Dt} + 2\mathbf{\Omega}\times\dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

$$\boxed{\begin{array}{c} \delta\int\mathcal{L}\mathrm{d}t = 0 \\ \Updownarrow \\ \dfrac{D}{Dt}\dfrac{\partial L}{\partial\dot{\mathbf{x}}} = \dfrac{1}{\rho}\nabla\left(\rho^2\dfrac{\partial L}{\partial\rho}\right) + \dfrac{\partial L}{\partial\mathbf{x}} \end{array}}$$

$$\mathcal{L} = \mathcal{K} + \mathcal{C} - \mathcal{P} = \int L(\mathbf{x},\,\dot{\mathbf{x}},\,\rho,\,s)\mathrm{d}m$$

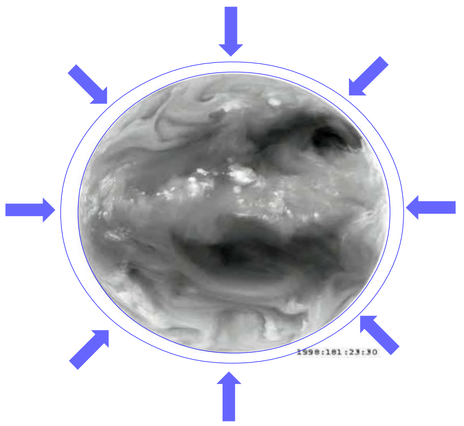
$$\mathcal{K} = \frac{1}{2}\int\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int(\mathbf{\Omega}\times\mathbf{x})\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int\left(e\left(\frac{1}{\rho},s\right) + \Phi(\mathbf{x})\right)\mathrm{d}m$$ Internal energy
Potential energy

Adiabatic equations of motion imply **conservation laws** because they derive from a least **action principle**

$$G_{ij}\frac{Du^j}{Dt}+\frac{1}{2}\left(\partial_j G_{ik}+\partial_k G_{ij}-\partial_i G_{jk}\right)u^j u^k$$
$$+\left[\partial_j R_i-\partial_i R_j\right]u^j+\frac{J}{\mu}\partial_i p+\partial_i \Phi \quad = \quad 0$$



$$\delta\int \mathcal{L}\mathrm{d}t=0$$

$$\Updownarrow$$

$$\frac{D}{Dt}\frac{\partial L}{\partial u^i}=\frac{1}{\hat\rho}\partial_i\left(\hat\rho^2\frac{\partial L}{\partial\hat\rho}\right)+\frac{\partial L}{\partial \xi^i}$$

$$\mathcal{L} = \int L\left(\xi^i, u^i, \hat\rho\right)\mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2}\int G_{ij}u^i u^j \mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int R_j u^j \mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int\left(e\left(\frac{J}{\hat\rho}, s\right)+\Phi(\xi^3)\right)\mathrm{d}m$$ Internal energy Potential energy

# Hamiltonian formulation

- Suppose evolution equations are known for all prognostic variables. Then how an arbitrary functional of those prognostic variables evolves is entirely known :

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = \int \left( \frac{\delta \mathcal{F}}{\delta v_i}\partial_t v_i + \frac{\delta \mathcal{F}}{\delta \mu}\partial_t \mu + \frac{\delta \mathcal{F}}{\delta S}\partial_t S \right) \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$$

- *Conversely, such an evolution equation contains all the equations of motion*
- A Hamiltonian formulation is such an evolution equation of the special form :

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$$

where H is total energy and the Poisson bracket is bilinear, antisymmetric and satisfies the Jacobi identity

- nice, but what is the Hamiltonian formulation for fluid flow ?

# Hamiltonian formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = \{\mathcal{F}, \mathcal{H}\}$$

means that all tendencies can be written as expressions that are *linear in the functional derivatives of total energy*

To identify the Hamiltonian formulation let us first find those functional derivatives :

$$\mathcal{H}\left[v_i, \mu, S\right] = \int \left(\frac{1}{2}g^{ij}(v_i - R_i)(v_j - R_j) + \Phi + e\left(\frac{J}{\mu}, \frac{S}{\mu},\right)\right)\mu \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$$

$$v_i = g_{ij}u^j + R_i, k = \frac{1}{2}g_{ij}u^i u^j$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = \int \Big(\underset{\overset{\uparrow}{\frac{\delta\mathcal{H}}{\delta v_i}}}{\mu u^i}\partial_t v_i + \underset{\overset{\uparrow}{\frac{\delta\mathcal{H}}{\delta \mu}}}{(k + \Phi + G)}\partial_t \mu + \underset{\overset{\uparrow}{\frac{\delta\mathcal{H}}{\delta S}}}{T}\partial_t S\Big)\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$$

…then transform the equations of motion until they are linear in the functional derivatives

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \mathbf{f}\times\mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

$$\Downarrow \quad (\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nabla\times\mathbf{u}\times\mathbf{u} + \nabla\frac{\mathbf{u}\cdot\mathbf{u}}{2}$$

$$\partial_t \mathbf{u} + (\mathbf{f} + \nabla\times\mathbf{u})\times\mathbf{u} + \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi\right) + \frac{1}{\rho}\nabla p = 0$$

$$\Downarrow \quad \mathrm{d}G = v\mathrm{d}p - s\mathrm{d}T$$

$$\partial_t \mathbf{v} + (\nabla\times\mathbf{v})\times\mathbf{u} + \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + G\right) + s\nabla T = 0$$

$$\Downarrow$$

$$\partial_t v_i + \frac{\partial_j v_i - \partial_i v_j}{\mu}\mu u^j + \partial_i\left(k + \Phi + G\right) + s\partial_i T = 0$$

$$\Downarrow$$

$$\boxed{\begin{aligned}
\partial_t v_i + \frac{\partial_j v_i - \partial_i v_j}{\mu}\frac{\delta\mathcal{H}}{\delta v_i} + \partial_i\frac{\delta\mathcal{H}}{\delta\mu} + s\partial_i\frac{\delta\mathcal{H}}{\delta S} &= 0\\
\partial_t \mu + \partial_i\frac{\delta\mathcal{H}}{\delta v_i} &= 0\\
\partial_t S + \partial_i\left(s\frac{\delta\mathcal{H}}{\delta v_i}\right) &= 0
\end{aligned}}$$

# *Hamiltonian formulation in non-Eulerian vertical coordinates*

*(Dubos & Tort, MWR 2014)*

$$
\begin{aligned}
\partial_t \mu + \partial_\eta (\mu \dot\eta) + \partial_i \frac{\delta \mathcal{H}}{\delta v_i} &= 0, \\
\partial_t \Theta + \partial_\eta (\Theta \dot\eta) + \partial_i \left( \theta \frac{\delta \mathcal{H}}{\delta v_i} \right) &= 0, \\
\partial_t v_i + \dot\eta \partial_\eta v_i + v_3 \partial_i \dot\eta + \frac{\partial_j v_i - \partial_i v_j}{\mu} \frac{\delta \mathcal{H}}{\delta v_j} + \partial_i \frac{\delta \mathcal{H}}{\delta \mu} + \theta \partial_i \left( \frac{\delta \mathcal{H}}{\delta \Theta} \right) &= 0, \\
\partial_t V_3 + \partial_\eta (V_3 \dot\eta) + \frac{\delta \mathcal{H}}{\delta \Phi} &= 0, \\
\partial_t \Phi + \dot\eta \partial_\eta \Phi - \frac{\delta \mathcal{H}}{\delta V_3} &= 0.
\end{aligned}
$$

***Integration by parts***
***+ invariance of Hamiltonian (total energy) w.r.t. vertical coordinate***
***=> conservation of energy***

## *Hamiltonian formulation in non-Eulerian vertical coordinates*

*(Dubos & Tort, MWR 2014)*

$$
\begin{aligned}
\partial_t \mu + \partial_\eta (\mu \dot\eta) + \partial_i \frac{\delta \mathcal{H}}{\delta v_i} &= 0, \\
\partial_t \Theta + \partial_\eta (\Theta \dot\eta) + \partial_i \left( \theta \frac{\delta \mathcal{H}}{\delta v_i} \right) &= 0, \\
\partial_t v_i + \dot\eta \partial_\eta v_i + v_3 \cancel{\partial_i \eta} + \frac{\partial_j v_i - \partial_i v_j}{\mu} \frac{\delta \mathcal{H}}{\delta v_j} + \partial_i \frac{\delta \mathcal{H}}{\delta \mu} + \theta \partial_i \left( \frac{\delta \mathcal{H}}{\delta \Theta} \right) &= 0, \\
\cancel{\partial_t V_3} + \partial_\eta \cancel{(V_3 \dot\eta)} + \frac{\delta \mathcal{H}}{\delta \Phi} &= 0, \\
\cancel{\partial_t \Phi} + \cancel{\dot\eta \partial_\eta \Phi} - \cancel{\frac{\delta \mathcal{H}}{\delta V_3}} &= 0.
\end{aligned}
$$

**Hydrostatic** $v_3 \equiv \dfrac{\partial L}{\partial u^3} = 0$

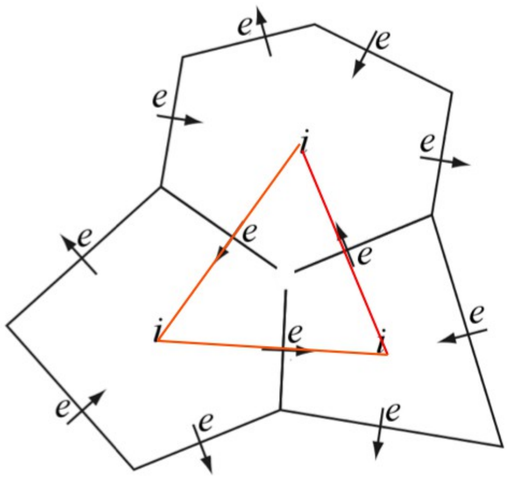
***Integration by parts***

***+ invariance of Hamiltonian (total energy) w.r.t. vertical coordinate***

***=> conservation of energy***

**Computational space $S^2 \times [0,1]$**

$\mathbf{n}$ $\eta$

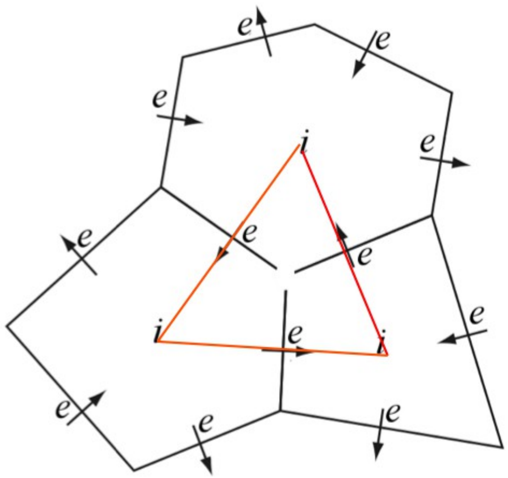



**Computational space $S^2 \times [0,1]$**

$\mathbf{n}$ $\eta$

*Horizontal mesh*

Icosahedral C-grid

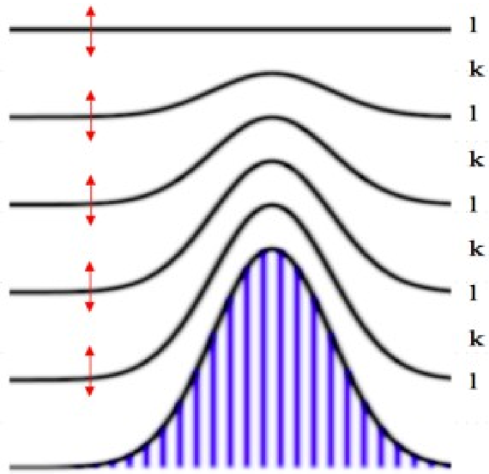


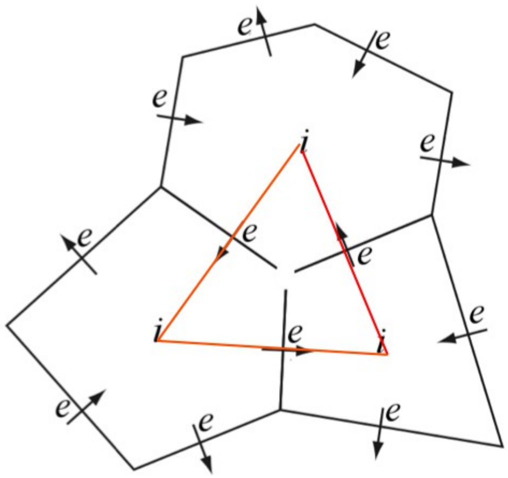

**Computational space $S^2$ x [0,1]**

$\mathbf{n}$ $\eta$

*Horizontal mesh*
Icosahedral C-grid

*Vertical mesh*
Lorenz



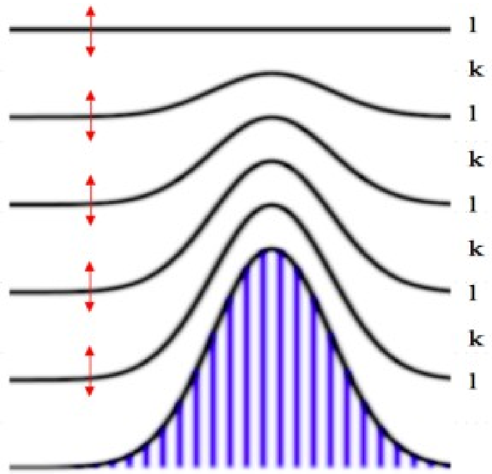



**Computational space $S^2 \times [0,1]$**

$\mathbf{n}$ $\eta$

***Horizontal mesh***
Icosahedral C-grid

***Vertical mesh***
Lorenz



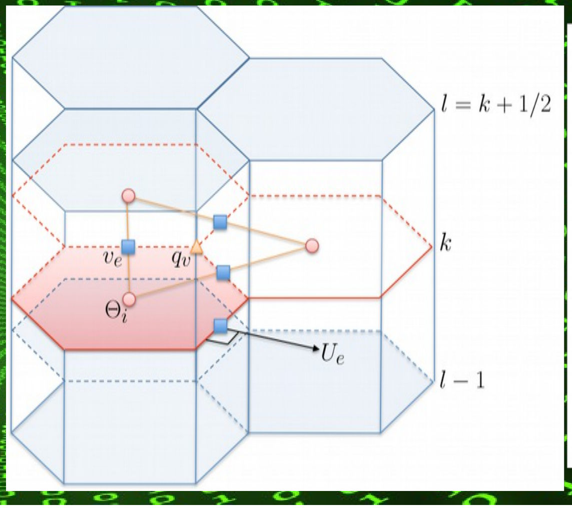

**Discrete representation**

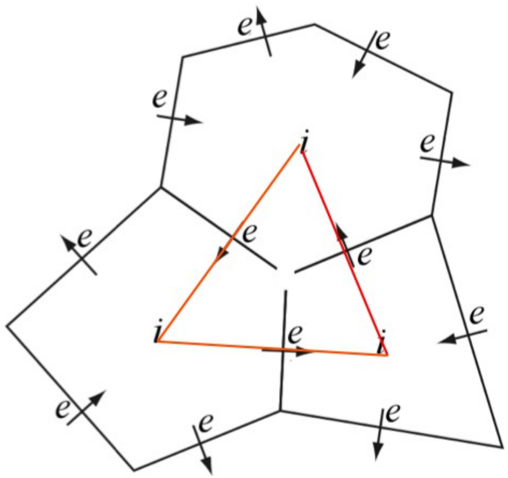
$$m_{ik} = \int\int\int \mu \mathrm{d}\mathbf{n}\mathrm{d}\eta$$

$$W_{il} = \int\int \mu\dot{\eta}\mathrm{d}\mathbf{n}$$

$$v_{ek} = \int \mathbf{v} \cdot \mathrm{d}\mathbf{n}$$

$$\alpha_{ik} = \alpha(p_{ik}, s_{ik}),$$





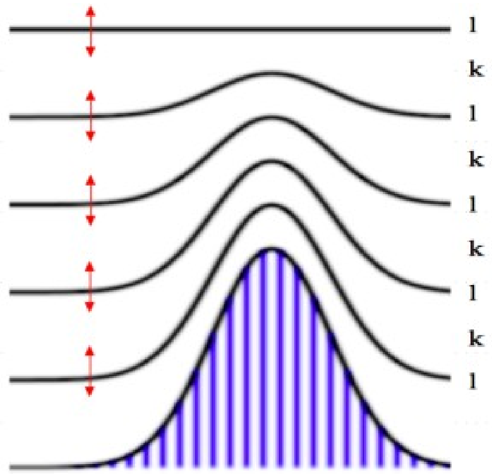


**Computational space $S^2 \times [0,1]$**

$\mathbf{n}$ $\eta$

*Horizontal mesh*
Icosahedral C-grid

*Vertical mesh*
Lorenz





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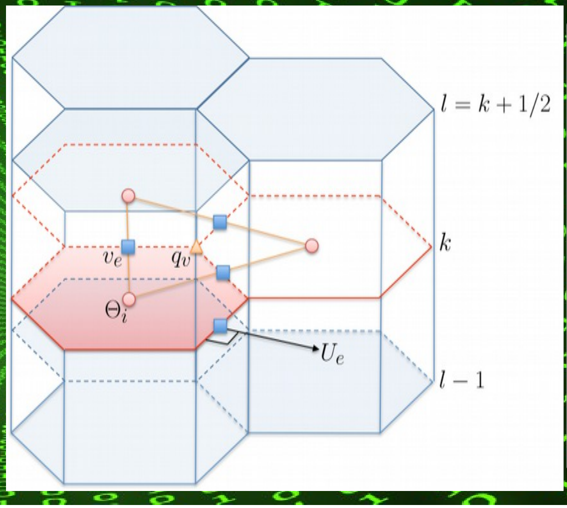
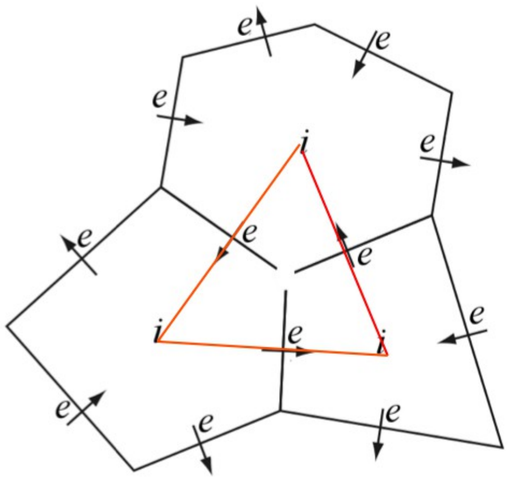




- Discrete Exterior Calculus : discrete exterior derivatives (grad, curl, div) are *exact*
- curl grad = 0

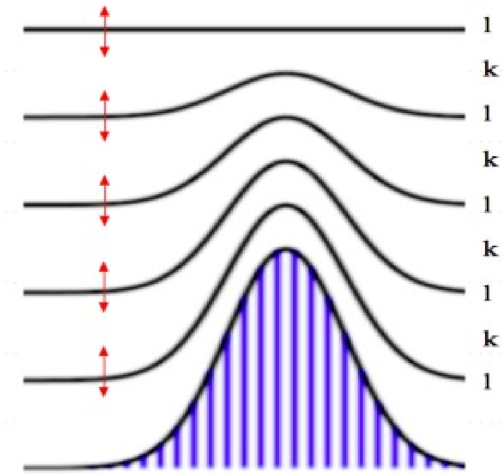



**Computational space $S^2 \times [0,1]$**

$\mathbf{n}$ $\qquad$ $\eta$

*Horizontal mesh*
Icosahedral C-grid

*Vertical mesh*
Lorenz




**Discrete representation**

$$m_{ik} = \int\int\int \mu \mathrm{d}\mathbf{n}\mathrm{d}\eta$$

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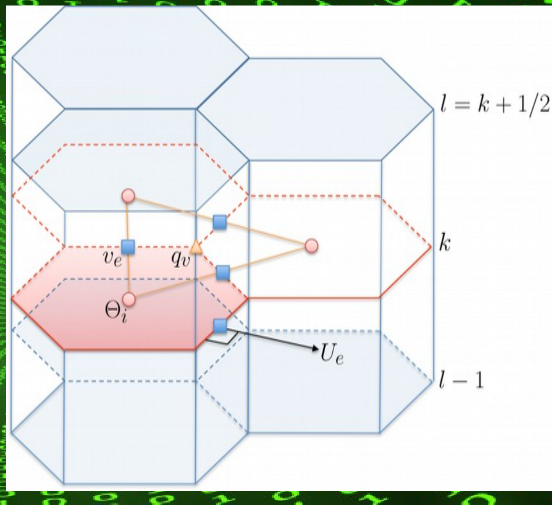
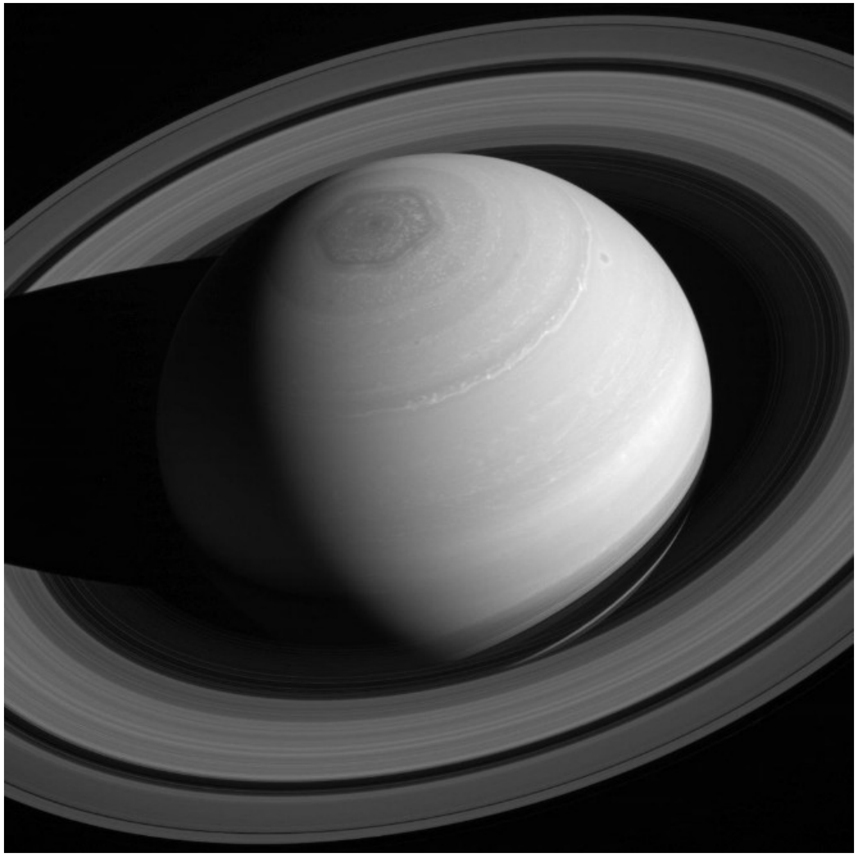
$$\alpha_{ik} = \alpha(p_{ik}, s_{ik}),$$




- Discrete Exterior Calculus : discrete exterior derivatives (grad, curl, div) are *exact*
- curl grad = 0
- Discrete integration by parts (Bonaventura & Ringler, 2005)
- Energy- and vorticity- conserving Coriolis discretization (TRiSK : Thuburn et al., 2009 ; Ringler et al., 2010)

**Energy-conserving 3D solver**



# Saturn GCM simulations

## Grid

- Horizontal resolution: $1/2°$ (+ tests $1/4°$ & $1/8°$)
- Vertical levels: 32 levels from 3 bars to 1 mbar

## Boundary conditions

- Initial: steady-state temperature from 1D run, no winds
- Dissipation (SGS): very strong ($\mathcal{D}^+$), strong ($\mathcal{D}^=$), regular ($\mathcal{D}^-$)
- Type 1: 64 levels + sponge layer on uppermost 4 levels
- Type 2: Bottom drag $|\lambda| > 33°$: $\tau = 100$d ($\mathcal{F}^+$), 1000d ($\mathcal{F}^-$) [Liu and Schneider JAS 2010]

## Machinery

- MPI+openMP code run on Occigen cluster in CINES
- cores: 1200 ($1/2°$), 9000 ($1/4°$), 11520-30000 ($1/8°$)
- Results are shown after 20000 Saturn days integrations

*1 Sat day = 10 h ; 1 Sat year = 30 Earth yr*

## Physical parameterizations ⇒ 1D computations of forcings on each grid point

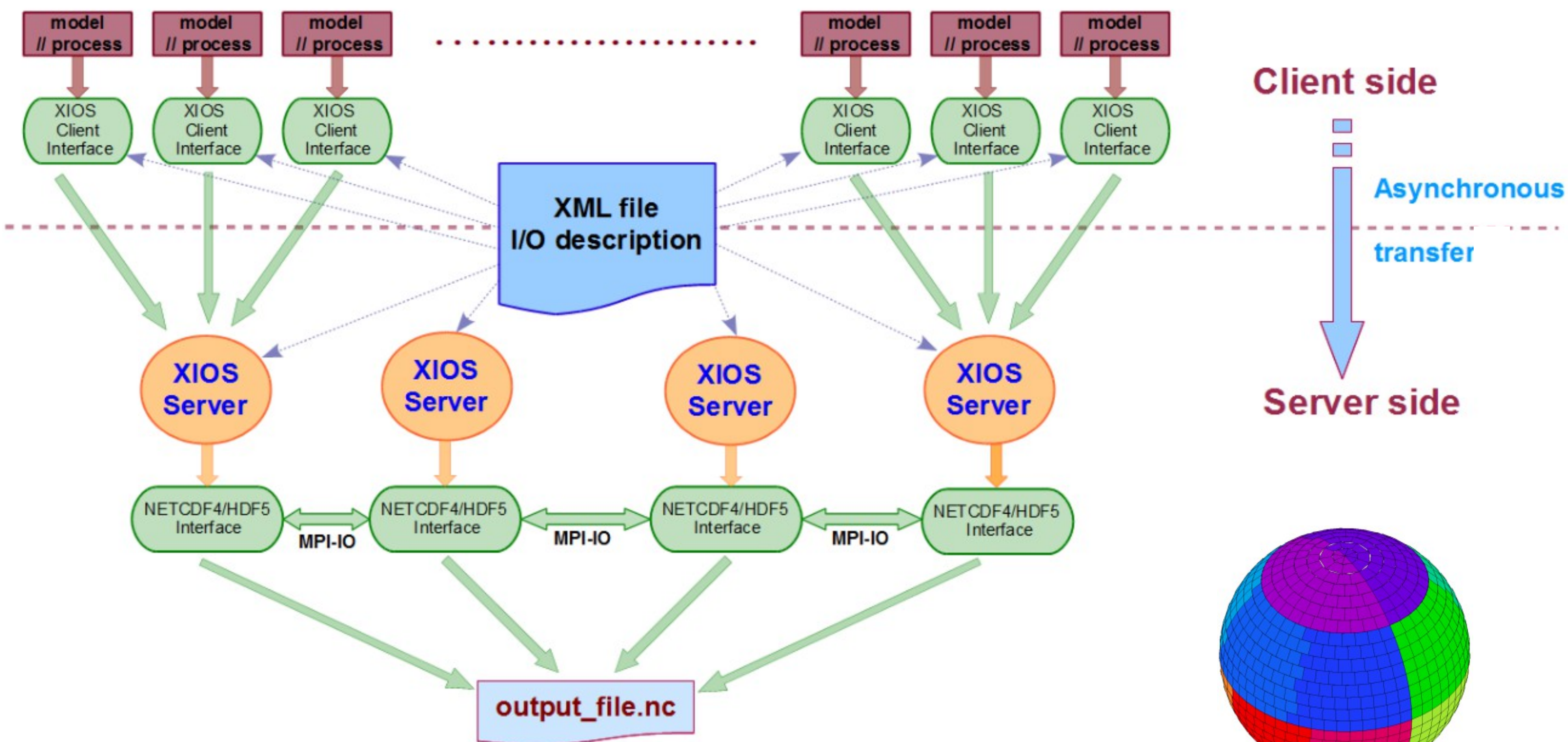
☞ Radiative transfer ⇒ Guerlet et al. Icarus 2014

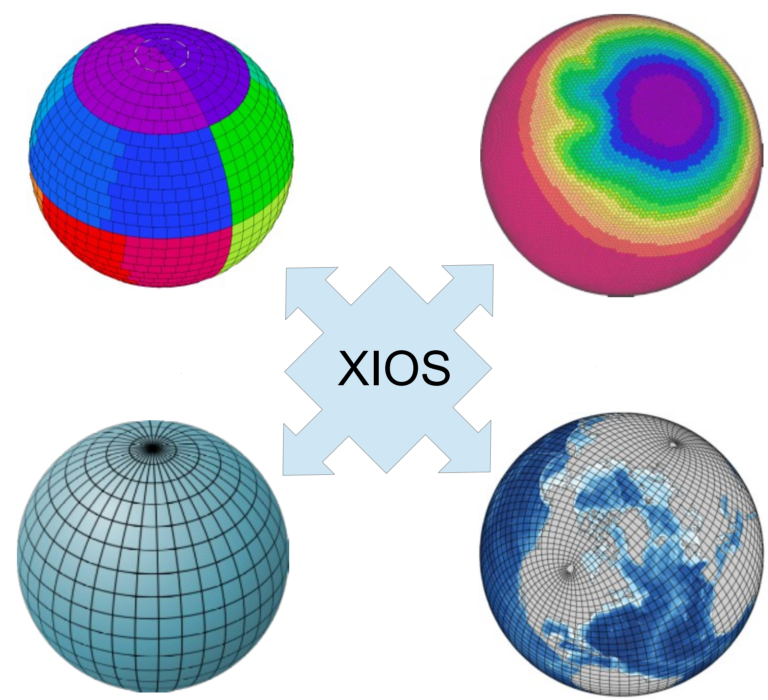
- correlated-$k$ scheme for IR and VIS heating rates [Wordsworth et al. 2010]
- gases $CH_4$, $C_2H_6$, $C_2H_2$ with optimized spectral discretization
- HITRAN 2012 database + Karkoschka and Tomasko 2010 for $CH_4$ around $1\mu$m
- collision-induced absorption $H_2$-$H_2$ and $H_2$-He [Wordsworth et al. 2012]
- Rayleigh scattering $H_2$, He
- simple two-layer aerosol model [constrained by Roman et al. 2013]
  - tropospheric haze layer $180 - 660$ mbar / $\tau \sim 8$ / $r = 2\mu$m
  - stratospheric haze layer $1 - 30$ mbar / $\tau \sim 0.1$ / $r = 0.1\mu$m
- free bottom surface with internal heat flux
- incoming flux: ring shadowing, oblateness

☞ Turbulent diffusion + dry convective adjustment [Hourdin et al. 1993]

# XIOS (Y. MEURDESOIF) : XML I/O SERVER

## PARALLEL ASYNCHRONOUS I/O - ONLINE POST-PROCESSING LIBRARY AND SERVER

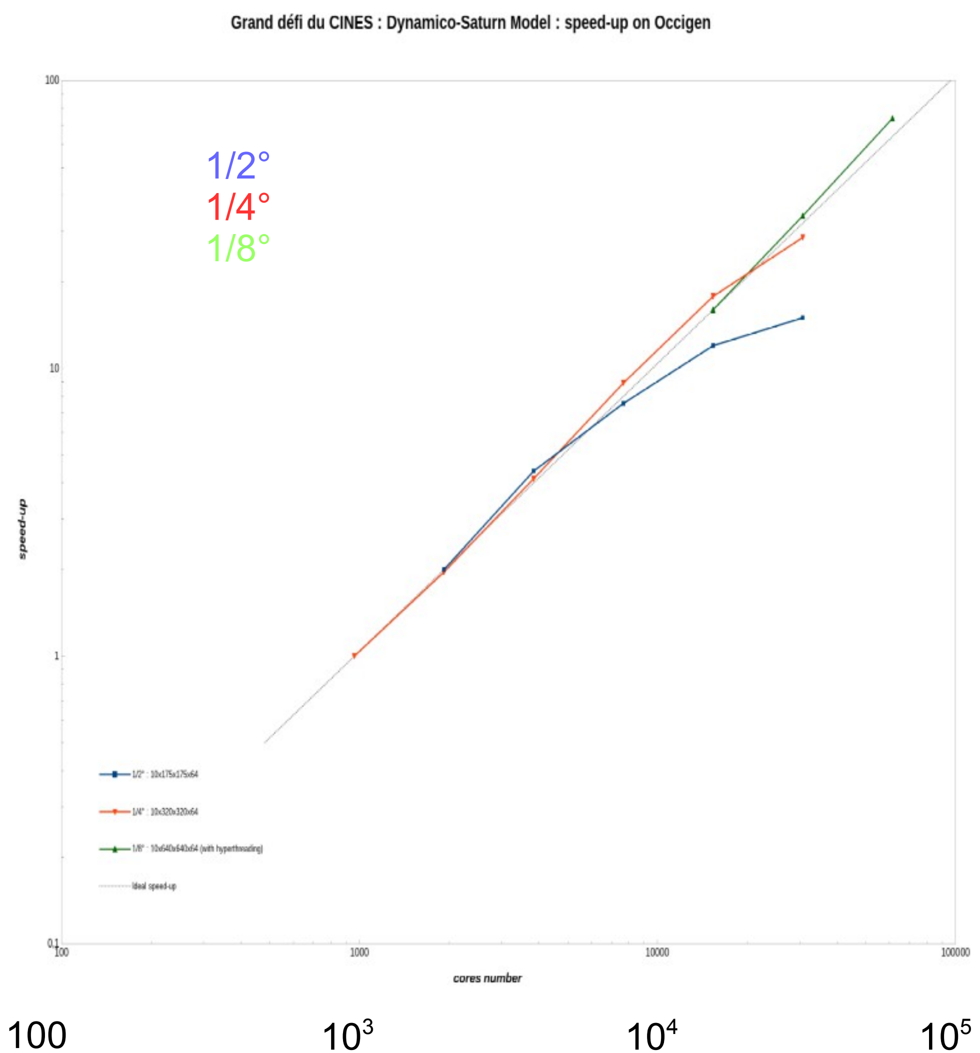


- Throughput on OCCIGEN (dycore only) for 60 vertical levels :

  1° : 500 cores ~ 40yr/day

  1/4° : 8000 cores ~10yr/day, 2Mh/century
- LMDZ CMIP6 physics now coupled, aquaplanet evaluation under way
- expect at least a few yr/day at 1/4 ° for full GCM



## Background on climate modelling

- Weather vs climate
- Characteristic scales
- Equations for atmospheric flow motion

## DYNAMICO, an energy-conserving finite difference/finite volume atmospheric solver

- Conservation of energy : why ?
- Hamiltonian formulation
- Preliminary results

## Finite elements, the path to higher-order accuracy ?

- Finite elements and conservation of energy
- A-HA

- Finite differences on unstructured meshes are low-order (1~2)
- FEM methods can be higher-order
- But will energy be conserved ?

- Illustrative sketch with a minimal model : 1D nonlinear wave equation

$$\partial_t h + \partial_x (Hu) = 0 \qquad \partial_t u + \partial_x F(h) = 0$$

$$\mathcal{H}[h,u] = \int \left( G(h) + H\frac{u^2}{2}\mathrm{d}x \right) \qquad F(h) = \frac{\mathrm{d}G}{\mathrm{d}h}$$

$$\boxed{\partial_t h + \partial_x \frac{\delta \mathcal{H}}{\delta u} = 0 \qquad \partial_t u + \partial_x \frac{\delta \mathcal{H}}{\delta h} = 0}$$

Conservation of energy / antisymmetry of the bracket result from integration by parts :

$$\{\mathcal{F},\mathcal{H}\} \equiv \partial_t \mathcal{F} = -\int \left( \frac{\delta \mathcal{F}}{\delta h}\partial_x \frac{\delta \mathcal{H}}{\delta u} + \frac{\delta \mathcal{F}}{\delta u}\partial_x \frac{\delta \mathcal{H}}{\delta h} \right) \mathrm{d}x$$

# Energy-conserving FEM

$$\boxed{\partial_t h + \partial_x \frac{\delta \mathcal{H}}{\delta u} = 0 \qquad \partial_t u + \partial_x \frac{\delta \mathcal{H}}{\delta h} = 0}$$

- We need to pick finite element spaces $H^\varepsilon, U^\varepsilon, B^\varepsilon, F^\varepsilon$

  for approximations $h^\varepsilon, u^\varepsilon, b^\varepsilon, f^\varepsilon$ of $h, u, b = \dfrac{\delta \mathcal{H}}{\delta h}, f = \dfrac{\delta \mathcal{H}}{\delta u}$
- By definition $$\delta \mathcal{H} = \int \left( \frac{\delta \mathcal{H}}{\delta h} \delta h + \frac{\delta \mathcal{H}}{\delta u} \delta u \right) \mathrm{d}x$$

  hence $b^\varepsilon, f^\varepsilon$ are obtained by projection :

$$\boxed{\begin{aligned} &\forall \hat{b} \in B^\varepsilon, \int \hat{h} b^\varepsilon \mathrm{d}x = \int \hat{h} \frac{u^2}{2} \mathrm{d}x \\ &\forall \hat{u} \in U^\varepsilon, \int \hat{u} f^\varepsilon \mathrm{d}x = \int \hat{u} H u \mathrm{d}x \end{aligned}}$$

- Therefore spaces $H^\varepsilon, U^\varepsilon, B^\varepsilon, F^\varepsilon$ must be in duality ; simplest choice is $B^\varepsilon = H^\varepsilon, F^\varepsilon = U^\varepsilon$
- At least one among $H^\varepsilon, U^\varepsilon$ must be in $H^1$ ; say it is $H^\varepsilon$
- Then define $\partial_x b$ as a weak derivative :

$$\boxed{\begin{aligned} &\forall \hat{h} \in H^\varepsilon \int \left( \hat{H} \partial_t h^\epsilon + \partial_x f^\varepsilon \right) = 0 \\ &\forall \hat{u} \in U^\varepsilon \int \left( \hat{u} \partial_t u - \partial_x \hat{u} u^\varepsilon \right) \mathrm{d}x = 0 \end{aligned}}$$

=> discrete integration by parts => conservation of energy

# Computational efficiency of FEM : the A-HA project

- Many possible choices of spaces, some giving high-order accuracy (e.g. spectral elements)

- Compared to a finite-difference method, a FEM method involves many more operations :
  - Assemble mass / stiffness matrices
  - Assemble right-hand-sides
  - Solve mass matrix (typically N-diagonal with N>1)

- High-order methods require quadrature rules with more quadrature points

- CEMRACS project A-HA explores different approaches to perform matrix assembly / matrix-vector multiply, including an approach proposed by Kirby (2014) which rearranges the computation in order to formulate compute-critical parts as dense matrix-matrix products to be offloaded to an optimized BLAS library

# Summary

- The atmospheric component of a climate model solves approximate equations of compressible fluid motion in a thin nearly-spherical shell.
- For climate applications where the model evolves freely over long time scales, conservation properties are desirable in order to physically constrain its evolution.
- Conservation of energy can be achieved in a systematic way at the discrete level by exploiting the Hamiltonian formulation of the equations
- The Hamiltonian formulation combines very well with mimetic finite differences and with finite element discretization methods
- However computational performance of FEM is potentially an issue, especially with higher-order FEM

Thanks for your attention !

# Extra slides

# Hydrostatic primitive equations : discrete

*(Lagrangian vertical coordinate – extra terms for vertical transport when mass-based)*

mass-weighted potential temperature

$$H = K(m_{ik},\, v_{ek}) + P(m_{ik},\, \Theta_{ik},\, \Phi_{il})$$

TRiSK

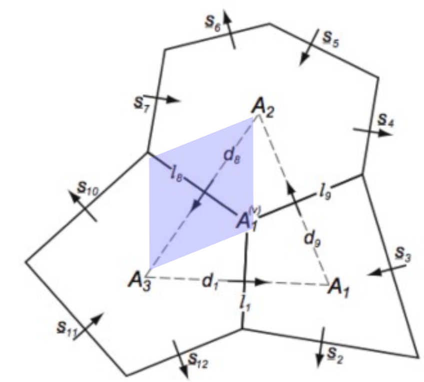
shallow-atmosphere metric

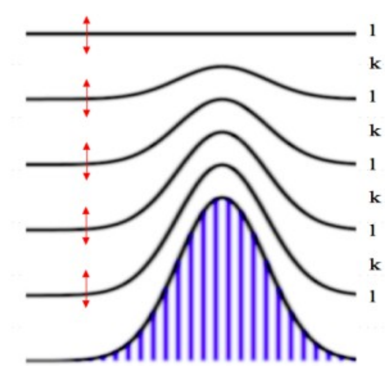
planetary velocity

$$K = \sum_{ike} m_{ik} \frac{A_{ie}}{A_i} \left( \frac{v_{ek} - R_e}{a d_e} \right)^2$$

normal velocity

$$P_{HPE} = \sum_{ik} m_{ik} \left[ \overline{\Phi_i}^k + e \left( \alpha_{ik} = \frac{A_i \delta_k (a^2 \Phi_i / g)}{m_{ik}},\, \theta_{ik} = \frac{\Theta_{ik}}{m_{ik}} \right) \right]$$





mass flux

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0$$

centered/upwind flux

$$\partial_t \Theta_{ik} + \delta_i \left( \theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left( \frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^{\perp} + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort.

Bernoulli function

Exner function

# Discrete energy budget : Lagrangian vertical coordinate

mass flux

centered/upwind

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0 \qquad \partial_t \Theta_{ik} + \delta_i \left( \theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left( \frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^\perp + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort.

Bernoulli function

Exner function

discrete div

discrete integration by parts

discrete grad

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

# Discrete energy budget : Lagrangian vertical coordinate

mass flux

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SW pot. Vort. Bernoulli function Exner function

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$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

Hydrostatic balance

$$\Rightarrow p_{ik} \Rightarrow \alpha_{ik} \Rightarrow \Phi_{il}$$

# Discrete energy budget : Lagrangian vertical coordinate

mass flux

centered/upwind

$$\partial_t m_{ik} + \delta_i \frac{\partial H}{\partial v_{ek}} = 0 \qquad \partial_t \Theta_{ik} + \delta_i \left( \theta^*_{ek} \frac{\partial H}{\partial v_{ek}} \right) = 0$$

$$\partial_t v_{ek} + \left( \frac{\delta_v v_k}{\overline{m_k}^v} \frac{\partial H}{\partial v_{ek}} \right)^\perp + \delta_e \frac{\partial H}{\partial m_{ik}} + \theta^*_{ek} \delta_e \frac{\partial H}{\partial \Theta_{ik}} = 0$$

SW pot. Vort.

Bernoulli function

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discrete div

discrete integration by parts

discrete grad

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum \frac{\partial H}{\partial m_{ik}} \partial_t m_{ik} + \frac{\partial H}{\partial \Theta_{ik}} \partial_t \Theta_{ik} + \frac{\partial H}{\partial v_{ek}} \partial_t v_{ek} + \frac{\partial H}{\partial \Phi_{il}} \partial_t \Phi_{il} = 0$$

Hydrostatic balance

$$\Rightarrow p_{ik} \Rightarrow \alpha_{ik} \Rightarrow \Phi_{il}$$

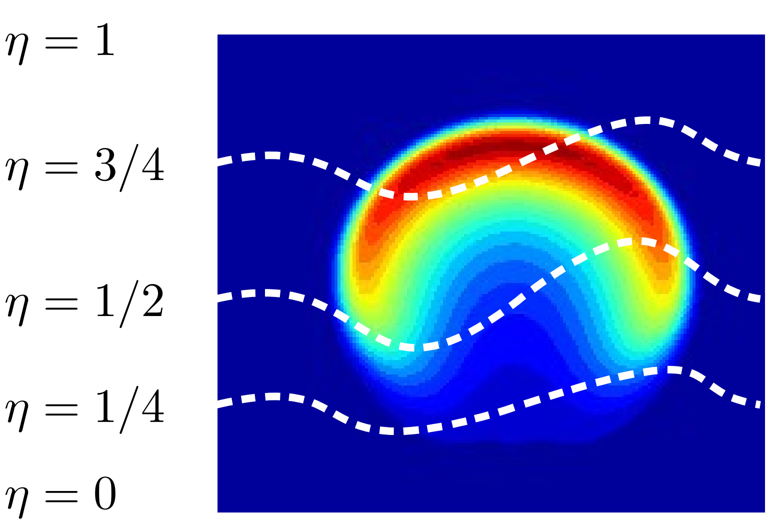
Non-Lagrangian vertical coordinate : also possible to cancel additional contributions from vertical transport (Tort et al., QJRMS 2015)

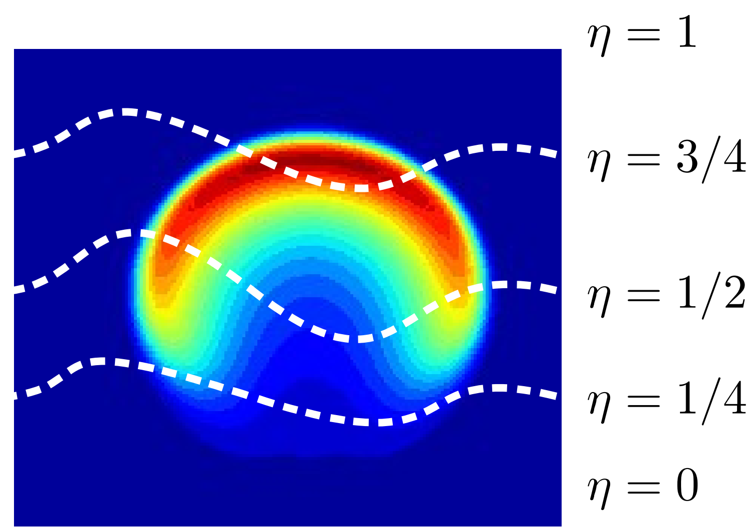
*Functional derivatives and redundancy in the flow description*

*(Dubos & Tort, MWR 2014)*

$$\mathcal{H}\left[\mu,\,\Theta,\,\Phi,\,v_1,\,v_2,\,V_3\right]$$

$$\delta\mathcal{H} = \int \left( \frac{\delta\mathcal{H}}{\delta\mu}\delta\mu + \frac{\delta\mathcal{H}}{\delta\Theta}\delta\Theta + \frac{\delta\mathcal{H}}{\delta\Phi}\delta\Phi + \frac{\delta\mathcal{H}}{\delta v_i}\delta v_i + \frac{\delta\mathcal{H}}{\delta V_3}\delta V_3 \right) \mathrm{d}\xi^i \mathrm{d}\eta$$



# *Functional derivatives and redundancy in the flow description*

*(Dubos & Tort, MWR 2014)*

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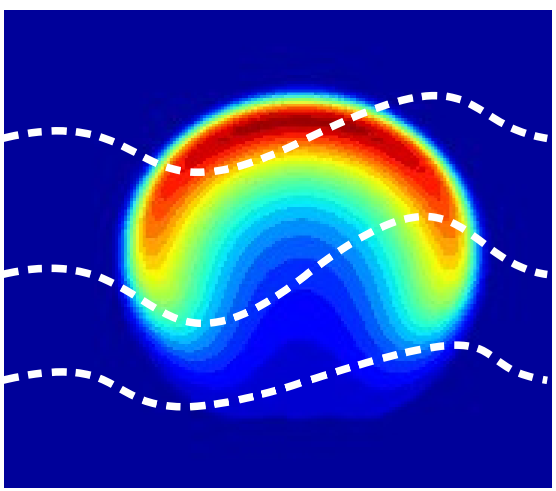
$\eta = 1$

$\eta = 3/4$

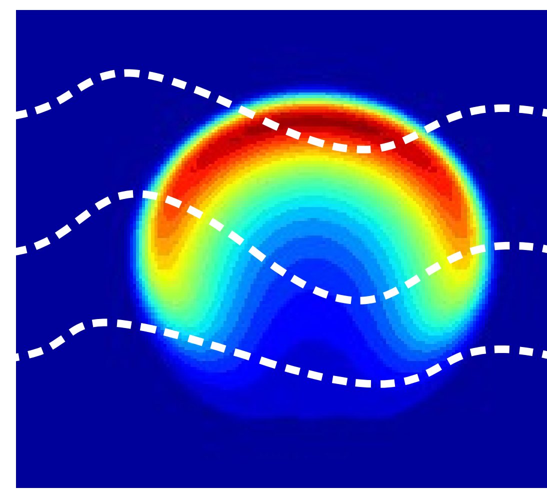
$\eta = 1/2$

$\eta = 1/4$

$\eta = 0$



Invariance w.r.t. vertical remapping



$\eta = 1$

$\eta = 3/4$

$\eta = 1/2$

$\eta = 1/4$

$\eta = 0$

$$\mu\partial_\eta \frac{\delta\mathcal{H}}{\delta\mu} = \left(\partial_\eta v_i\right)\frac{\delta\mathcal{H}}{\delta v_i} - \partial_i\left(v_3\frac{\delta\mathcal{H}}{\delta v_i}\right) + \left(\partial_\eta\Phi\right)\frac{\delta\mathcal{H}}{\delta\Phi} - V_3\partial_\eta\frac{\delta\mathcal{H}}{\delta V_3} - \Theta\partial_\eta\frac{\delta\mathcal{H}}{\delta\Theta}$$