CEMRACS 2016 project: *'ApostDD'* A posteriori estimates and stopping criteria for domain decomposition solvers in porous media flow

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Abstract

Global-in-time, non-overlapping domain decomposition methods can provide parallel efficiency in the numerical solution of porous media flow and transport problems. These iterative algorithms are based on time-dependent Robin-to-Robin-type (or possibly Ventcell-to-Ventcell type) operator that introduces additional coefficients that can be optimized to improve convergence rates. Recently, a posteriori estimates and stopping criteria on these iterative domain decomposition algorithms for one phase flow have been introduced and analyzed. The objective of this project is to extend this approach to a two-phase flow model to obtain significant gains in the total number of iterations of the domain decomposition method and thus in the computational time in parallel numerical simulations of two-phase flows. This work is motivated by ANDRA's CIGEO project and is part of the ANR project DEDALES.

1 Context and state of the art

Two-phase (water/gas) flow in a porous medium [3, 8], for instance for modeling hydrogen propagation in a radioactive waste disposal site, is described by the system

$$\begin{aligned}
\partial_t S_\alpha + \nabla \cdot \boldsymbol{u}_\alpha &= q_\alpha && \text{(mass conservation)}\\
\boldsymbol{u}_\alpha &= -k_{r\alpha}(S_\alpha) K \left(\nabla p_\alpha - \nabla G\right) && \text{(Darcy's law)}\\
S_g + S_e &= 1 && \text{(pore saturation)}\\
p_g - p_e &= \pi(S_e) && \text{(capillary pressure)}
\end{aligned} \tag{1}$$

where $K = K(\boldsymbol{x})$ is the permeability, $G = G(\boldsymbol{x})$ is the gravity field, q_α is a source term, S_α, $\boldsymbol{u}_\alpha$, and p_α are the saturation, the velocity, and the pressure of the phase α (unknowns), respectively, and $k_{r\alpha}$ and π are respectively the relative permeability and the capillary pressure (both nonlinear functions of the saturation, possibly discontinuous in space between different rock types). System (1) is degenerate: the relative permeability may vanish. It is crucial to develop accurate and efficient simulations for system (1) and its generalizations.

1.1 Space-time domain decomposition with local time steps

The computational domain for system (1) is usually made up of subdomains with very different physical properties. Consequently, the time scales vary over several orders of magnitude between these different subdomains. In [9], schemes that enable local time stepping are proposed and analyzed for one-phase flows (linear advection–diffusion equations). In particular, we now have an approach based on Schwarz iterations with optimized transmission conditions [10, 7, 9] to perform efficient *parallel simulations* with *local time steps*.

Part of the work in [9] on space-time domain decomposition with optimized transmission conditions is currently being extended for solving system (1), in the framework of the post-doc of Elyes Ahmed (2015–2017) supported by ANR DEDALES.

1.2 A posteriori error estimates

The theory of *a posteriori error estimates* allows to mesure the size of the error between the exact and approximate solutions in a numerical simulation and to identify areas where the error is located. Usually, an optimal numerical algorithm must be *adaptive* and such that:

1. A given accuracy is obtained at the end of the simulation.
2. The necessary amount of work is as small as possible.

Recently, a *unified framework* for a posteriori estimates has been developed in [5, 6]. It allows both to control the total error and to distinguish its various components and consequently to design *stopping criteria* for different iterative solvers, see [6] and the references therein.

Applications to porous media flows are treated in [12, 2].

Extensions of the theory of a posteriori estimates and stopping criteria on iterative domain decomposition algorithms for one-phase flow (linear avection–diffusion equations) are the objectives of the PhD of Sarah Ali Hassan (2014–2017), supported by ANDRA. The first results are very promising and show the potential of this approach.

2 Objectives of the project and working program

The project is motivated by ANDRA's research program (CIGEO project) and is part of ANR DEDALES. It builds on the results described above. In this project, a simplified scalar model (see [4, 1]), leading to a nonlinear degenerate parabolic equation, will be considered. The objectives for this model are:

- extend the theory of a posteriori estimates and stopping criteria to the parallel iterative domain decomposition algorithm, for a two-phase flow,
- propose stopping criteria to stop iterations of domain decomposition solvers and to choose local time steps for the problem under study,
- implemement these methods in the MRST software[1] [11] and validate them on a typical example.

We expect both to control the numerical error and to obtain significant gains (up to several orders of magnitude) in the total number of iterations and thus in the computational time in parallel numerical simulations of two-phase flows.

3 Partners, participants and funding

Project partners:

- **ANDRA** has expertise in industrial waste disposal, in particular the simulation of the transport of radionuclides around a nuclear waste repository. It aims to develop numerical methods (domain decomposition, a posteriori error estimates) for solving two-phase flow (e.g. water and oil) problems through heterogeneous porous media.
- **INRIA** (team-project SERENA) has an expertise in numerical methods for PDEs, scientific computing, high performance computing, especially in a posteriori error estimates and domain decomposition for porous media flow and transport.
- **LAGA** (University Paris 13) has an expertise in numerical methods for PDEs and scientific computing, especially in domain decomposition methods (optimized Schwarz methods) for fluid dynamics.

This project is part of the **ANR DEDALES** (*Algebraic and Geometric Domain Decomposition for Subsurface/Groundwater Flows*, ANR-14-CE23-0005) led by Michel Kern, which

[1] Matlab Reservoir Simulation Toolbox, `http://www.sintef.no/projectweb/mrst/`

brings together INRIA Paris, ANDRA, LAGA, Maison de la Simulation and INRIA Bordeaux-Sud-Ouest teams, and which aims to bring domain decomposition algorithms to larger classes of parallel architectures, increase the theoretical understanding and expand the domains of application including industrial size problems.

Project participants:

- Sarah Ali Hassan (SERENA, doctoral student, 6 weeks)
- Elyes Ahmed (LAGA, University Paris 13, post-doctoral student, 6 weeks)
- Caroline Japhet (LAGA, University Paris 13, 4 weeks)
- Michel Kern (SERENA, 1 week)
- Martin Vohralík (SERENA, 1 week at cirm and 1 week with skype)
- Marc Leconte (ANDRA)

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References

[1] C. Cancès. Nonlinear parabolic equations with spatial discontinuities. *NoDEA Nonlinear Differential Equations Appl.*, 15(4-5):427–456, 2008.

[2] C. Cancès, F. Nabet, and M. Vohralík. Convergence and a posteriori error analysis for nonlinear finite element approximations of degenerate parabolic equations. *In preparation.*

[3] Z. Chen, G. Huan, and Y. Ma. *Computational methods for multiphase flows in porous media.* Computational Science & Engineering. Society for Industrial and Applied Mathematics, Philadelphia, 2006.

[4] G. Enchéry, R. Eymard, and A. Michel. Numerical approximation of a two-phase flow problem in a porous medium with discontinuous capillary forces. *SIAM J. Numer. Anal.*, 43(6):2402–2422, 2006.

[5] A. Ern and M. Vohralík. A posteriori error estimation based on potential and flux reconstruction for the heat equation. *SIAM J. Numer. Anal.*, 48(1):198–223, 2010.

[6] A. Ern and M. Vohralík. Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs. *SIAM J. Sci. Comput.*, 35(4):A1761–A1791, 2013.

[7] L. Halpern, C. Japhet, and J. Szeftel. Optimized Schwarz waveform relaxation and discontinuous Galerkin time stepping for heterogeneous problems. *SIAM J. Numer. Anal.*, 50(5):2588–2611, 2012.

[8] R. Helmig. *Multiphase flow and transport processes in the subsurface : a contribution to the modeling of hydrosystems.* Environmental engineering. Springer, Berlin, New York, 1997.

[9] T. Hoang, J. Jaffré, C. Japhet, M. Kern, and J. Roberts. Space-time domain decomposition methods for diffusion problems in mixed formulations. *SIAM J. Numer. Anal.*, 51(6):3532–3559, 2013.

[10] C. Japhet and F. Nataf. The best interface conditions for domain decomposition methods: absorbing boundary conditions. *Absorbing boundaries and layers, domain decomposition methods*, pages 348–373, 2001.

[11] K.-A. Lie, S. Krogstad, I. Ligaarden, J. Natvig, H. Nilsen, and B. Skaflestad. Open-source MATLAB implementation of consistent discretisations on complex grids. *Computational Geosciences*, 16(2):297–322, 2012.

[12] M. Vohralík and M. Wheeler. A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows. *Comput. Geosci.*, 17(5):789–812, 2013.