The Geometrical Gyro-Kinetic Approximation

LMBA Laboratoire de Mathématiques de Bretagne Atlantique UMR6205 - CNRS/UBO/UBS

CENTRE HENRI LEBESGUE CENTRE DE MATHÉMATIQUES

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Université Bretagne Sud

see-∂ Sciences et Entreprises - ∂écisions

EP Inria Tonus

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EUROfusion

CfP-WP14-ER-01/IPP-03 & CfP-WP15-ER/IPP-01

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Joint work with Mathieu Lutz

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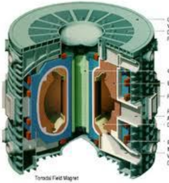
Charge particles submitted to Strong Magnetic Field

In Usual Coordinates : $(\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$

$$\mathbf{X}(t; \mathbf{x}, \mathbf{v}, s), \mathbf{V}(t; \mathbf{x}, \mathbf{v}, s)$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{q}{m}(\mathbf{E}(\mathbf{X}) + \mathbf{V} \times \mathbf{B}(\mathbf{X}))$$

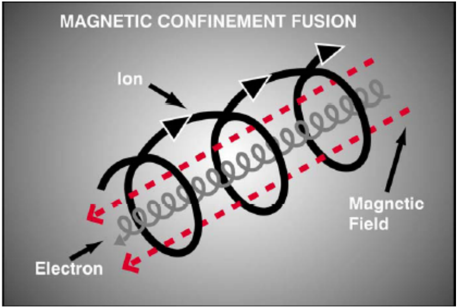


$$\mathbf{B} : \underbrace{\text{Self Induced Perturbations}}_{\text{Forgotten}} + \underbrace{\text{Strong Applied piece}}_{\to \frac{1}{\varepsilon}\mathbf{B}}$$

$$\mathbf{E} : \underbrace{\text{Self Induced piece}}_{\text{Forgotten}}$$

$$\varepsilon \sim \frac{\text{Larmor Radius}}{\text{Tokamak size}}$$

Helicoidal trajectories - Larmor Radius



Source: S. Jardin's Lectures at Cemracs'10

In Tokamak:

Electron Larmor Radius $\sim 5 \cdot 10^{-4} m$

Ion Larmor Radius $\sim 10^{-2} m$

Dimensionless Dynamical System

$$\varepsilon \sim \frac{\text{Ion Larmor Radius}}{\text{Tokamak size}} \sim \frac{10^{-2}m}{10m} \sim 10^{-3}$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{V} \times \frac{\mathbf{B}(\mathbf{X})}{\varepsilon}$$

Simplifications

$$\mathbf{B}(\mathbf{x}) = (0, 0, B(x_1, x_2))$$

$$B > 1, \quad B(x_1, x_2) = \nabla \times \mathbf{A}(x_1, x_2) = \frac{\partial A_2}{\partial x_1}(x_1, x_2) - \frac{\partial A_1}{\partial x_2}(x_1, x_2)$$

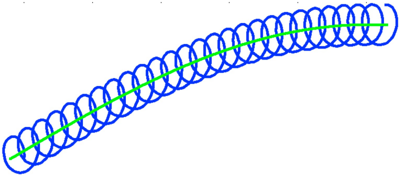
Turn to dimension 2: $\mathbf{x} = (x_1, x_2)$, $\mathbf{v} = (v_1, v_2)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}, \qquad \mathbf{X}(0) = \mathbf{x}_0,$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon} B(\mathbf{X})\,{}^{\perp}\mathbf{V} = \frac{1}{\varepsilon} B(\mathbf{X}) \begin{pmatrix} V_2 \\ -V_1 \end{pmatrix}, \qquad \mathbf{V}(0) = \mathbf{v}_0$$

$$\frac{\partial}{\partial t} \begin{pmatrix} X_1 \\ X_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ \frac{1}{\varepsilon} B(\mathbf{X})\, V_2 \\ -\frac{1}{\varepsilon} B(\mathbf{X})\, V_1 \end{pmatrix}, \quad \begin{pmatrix} X_1 \\ X_2 \\ V_1 \\ V_2 \end{pmatrix}(0) = \begin{pmatrix} x_{01} \\ x_{02} \\ v_{01} \\ v_{02} \end{pmatrix}$$

Gyrokinetic model



$$\frac{\partial \mathbf{Z}}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})}\,{}^{\perp}\nabla B(\mathbf{Z}), \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

$$\frac{\partial}{\partial t}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})}\begin{pmatrix} \frac{\partial B}{\partial x_2}(\mathbf{Z}) \\ -\frac{\partial B}{\partial x_1}(\mathbf{Z}) \end{pmatrix}, \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

for magnetic moment $\mathcal{J}$

What is hidden

$$\frac{\partial \mathbf{Z}}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B(\mathbf{Z})}\, {}^{\perp}\nabla B(\mathbf{Z}), \qquad \mathbf{Z}(0) = \mathbf{z}_0$$

$$\frac{\partial \Gamma}{\partial t} = \frac{B(\mathbf{Z})}{\varepsilon} + \varepsilon \frac{\mathcal{J}}{2B(\mathbf{Z})^2}\left(B(\mathbf{Z})\nabla^2 B(\mathbf{Z}) - 3\left(\nabla B((\mathbf{Z}))\right)^2\right), \quad \Gamma(0) = \gamma_0$$

$$\frac{\partial \mathcal{J}}{\partial t} = 0, \qquad \mathcal{J}(0) = j_0$$

Key result

IF: In coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4)$, a Hamiltonian Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R})\nabla_{\mathbf{r}} H(\mathbf{R}) \qquad \mathcal{P}(\mathbf{r}) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathtt{M}(\mathbf{r})$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{array}\right)$$

with

$$\frac{\partial H}{\partial r_3} = 0$$

$$\textbf{THEN:} \quad \frac{\partial \mathtt{M}}{\partial r_3} = \frac{\partial \mathtt{M}}{\partial r_4} = 0 \qquad \textbf{AND:} \quad \frac{\partial R_4}{\partial t} = 0$$

(Trajectory $\mathbf{R} = (R_1, R_2, R_3, R_4)$)

Key result

IF: In coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4)$, a Hamiltonian Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R}) \nabla_{\mathbf{r}} H(\mathbf{R}) \quad \mathcal{P}(\mathbf{r}) = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbb{M}(\mathbf{r})$}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{array} \right) \begin{pmatrix} \frac{\partial H}{\partial r_1} \\ \frac{\partial H}{\partial r_2} \\ 0 \\ \frac{\partial H}{\partial r_4} \end{pmatrix}$$

with

$$\frac{\partial H}{\partial r_3} = 0$$

$$\textbf{THEN:} \quad \frac{\partial \mathbb{M}}{\partial r_3} = \frac{\partial \mathbb{M}}{\partial r_4} = 0 \quad \textbf{AND:} \quad \frac{\partial R_4}{\partial t} = 0$$

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Key result

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with

$$\frac{\partial H}{\partial r_3} = 0$$

$$\textbf{THEN:} \quad \frac{\partial \mathtt{M}}{\partial r_3} = \frac{\partial \mathtt{M}}{\partial r_4} = 0 \qquad \textbf{AND:} \quad \frac{\partial R_4}{\partial t} = 0$$

(Trajectory $\mathbf{R} = (R_1, R_2, R_3, R_4)$)

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon} B(\mathbf{X})^{\perp}\mathbf{V}$$

1: Hamiltonian? ↓   2 ↑

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon = \breve{H}_\varepsilon(\mathbf{q}, \mathbf{p})$:

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}} \breve{H}_\varepsilon$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}} \breve{H}_\varepsilon$$

3 →

Polar Coordinates

$(\mathbf{x}, \theta, v)$

4: Darboux Method ↓

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, v)$

5: Lie Method ↓

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

2

Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

$$\left(\begin{array}{c|cc} \mathtt{M} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ -\frac{1}{\varepsilon} \end{matrix} & \begin{matrix} \frac{1}{\varepsilon} \\ 0 \end{matrix} \end{array}\right)$$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Canonical Coordinates

Usual Coordinates : $(\mathbf{x},\mathbf{v}) = (x_1, x_2, v_1, v_2)$
Trajectory : $(\mathbf{X}(t;\mathbf{x},\mathbf{v},s), \mathbf{V}(t;\mathbf{x},\mathbf{v},s))$ $\big((\mathbf{X},\mathbf{V}) = (X_1, X_2, V_1, V_2)\big)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V} \qquad B(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{1}{\varepsilon} B(\mathbf{X})\,{}^{\perp}\mathbf{V}$$

Canonical Coordinates : $(\mathbf{q},\mathbf{p}) = (q_1, q_2, p_1, p_2)$
Trajectory : $(\mathbf{Q}(t;\mathbf{q},\mathbf{p},s), \mathbf{P}(t;\mathbf{q},\mathbf{p},s))$ $\big((\mathbf{Q},\mathbf{P}) = (Q_1, Q_2, P_1, P_2)\big)$

$$\mathbf{q} = \mathbf{x}, \qquad \mathbf{p} = \mathbf{v} + \frac{\mathbf{A}(\mathbf{x})}{\varepsilon}$$

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon \qquad \breve{H}_\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2$$

$$\mathcal{S} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

Check of Canonical nature of Canonical Coordinates

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S}\nabla_{\mathbf{q},\mathbf{p}}\breve{H}_\varepsilon, \qquad \breve{H}_\varepsilon(\mathbf{q},\mathbf{p}) = \frac{1}{2}\left|\mathbf{p} - \frac{\mathbf{A}(\mathbf{q})}{\varepsilon}\right|^2$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \nabla_{\mathbf{p}}\breve{H}_\varepsilon(\mathbf{Q},\mathbf{P}) = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla_{\mathbf{q}}\breve{H}_\varepsilon(\mathbf{Q},\mathbf{P}) = \frac{(\nabla\mathbf{A}(\mathbf{Q}))^T}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$

$$(\nabla\mathbf{A})^T(\mathbf{p}-\mathbf{A}) = (\nabla\mathbf{A})(\mathbf{p}-\mathbf{A}) + (\nabla\times\mathbf{A})\ {}^\perp(\mathbf{p}-\mathbf{A})$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla\mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \frac{\nabla\times\mathbf{A}(\mathbf{Q})}{\varepsilon}\ {}^\perp\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$

Check of Canonical nature of Canonical Coord. - 2

$$\mathbf{X} = \mathbf{Q}$$
$$\mathbf{V} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$

$$\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}$$
$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big) = \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon}\ {}^{\perp}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$
$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{P}}{\partial t} - \frac{(\nabla \mathbf{A}(\mathbf{Q}))}{\varepsilon}\Big(\frac{\partial \mathbf{Q}}{\partial t}\Big) = \frac{\partial \Big[\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big]}{\partial t} = \frac{\nabla \times \mathbf{A}(\mathbf{Q})}{\varepsilon}\ {}^{\perp}\Big(\mathbf{P} - \frac{\mathbf{A}(\mathbf{Q})}{\varepsilon}\Big)$$
$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{V}$$
$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\nabla \times \mathbf{A}(\mathbf{X})}{\varepsilon}\ {}^{\perp}\mathbf{V}$$

Change of Coordinates Formula

In any coordinate system $\mathbf{r} = (r_1, r_2, r_3, r_4)$, the Dynamical System writes:

$$\frac{\partial \mathbf{R}}{\partial t} = \mathcal{P}(\mathbf{R})\nabla_{\mathbf{r}} H(\mathbf{R})$$

Another coordinate system $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)$ with $\tilde{\mathbf{r}} = \boldsymbol{\rho}(\mathbf{r})$, $\mathbf{r} = \tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}) = \boldsymbol{\rho}^{-1}(\tilde{\mathbf{r}})$

$$\frac{\partial \tilde{\mathbf{R}}}{\partial t} = \tilde{\mathcal{P}}(\tilde{\mathbf{R}})\nabla_{\tilde{\mathbf{r}}} \tilde{H}(\tilde{\mathbf{R}})$$

$$\boxed{\tilde{H}(\tilde{\mathbf{r}}) = H(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}})) \qquad (\tilde{\mathcal{P}}(\tilde{\mathbf{r}}))_{ij} = \{\boldsymbol{\rho}_i, \boldsymbol{\rho}_j\}(\tilde{\boldsymbol{\rho}}(\tilde{\mathbf{r}}))}$$

where: $\{f, g\}(\mathbf{r}) = (\nabla_{\mathbf{r}} f(\mathbf{r})) \cdot (\mathcal{P}(\mathbf{r})(\nabla_{\mathbf{r}} g(\mathbf{r})))$ $\quad$ (f and $g : \mathbb{R}^4 \to \mathbb{R}$)

Hamiltonian Function and Poisson Matrix in Usual Coordinates

$$H_\varepsilon(\mathbf{x},\mathbf{v}) = \frac{1}{2}\left|\mathbf{v}\right|^2$$

$$\mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v}) = \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{(\nabla \mathbf{A}(\mathbf{x}))^T - (\nabla \mathbf{A}(\mathbf{x}))}{\varepsilon} \end{pmatrix}$$

Panorama

Usual Coordinates

$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

2

3

Polar Coordinates

$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

$$\begin{pmatrix} \mathtt{M} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & \frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & 0 \end{matrix} \end{pmatrix}$$

Darboux Almost Canonical Coordinates

$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$

$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

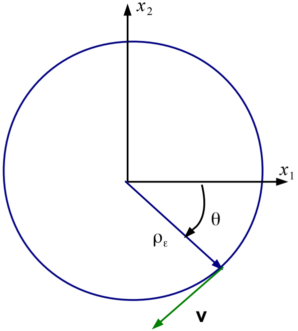
Canonical Coordinates

$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

Polar Coordinates (in velocity)



$$(\mathbf{x}, \theta, v): \qquad v = |\mathbf{v}|, \qquad \theta \text{ s.t. } \mathbf{v} = v \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix}$$

Hamiltonian Function and Poisson Matrix in Polar Coordinates

$$\widetilde{H}_\varepsilon(\mathbf{x}, \theta, v) = \frac{v^2}{2}$$

$$\tilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v) = \begin{pmatrix} 0 & 0 & -\frac{\cos(\theta)}{v} & -\sin(\theta) \\ 0 & 0 & \frac{\sin(\theta)}{v} & -\cos(\theta) \\ \frac{\cos(\theta)}{v} & -\frac{\sin(\theta)}{v} & 0 & \frac{B(\mathbf{x})}{\varepsilon v} \\ \sin(\theta) & \cos(\theta) & -\frac{B(\mathbf{x})}{\varepsilon v} & 0 \end{pmatrix}$$

Panorama

Usual Coordinates
$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{X}}{\partial t} \\ \dfrac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

3

Polar Coordinates
$(\mathbf{x}, \theta, v)$
$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

1: Hamiltonian?

2

Canonical Coordinates
$(\mathbf{q}, \mathbf{p})$

$\check{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \check{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \dfrac{\partial \mathbf{Q}}{\partial t} \\ \dfrac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \check{H}_\varepsilon$$

4: Darboux Method

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathtt{M}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates
$(\mathbf{y}, \theta, k)$
$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates
$(\mathbf{z}, \gamma, j)$
$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$
$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Darboux Method Target

Find a Coordinate System $(\mathbf{y},\theta,k)$ s.t. Poisson Matrix $(\overline{\mathcal{P}}_\varepsilon)$ shape:

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathbb{M}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)$$

$$(\mathbf{y},\theta,k) = \boldsymbol{\Upsilon}(\mathbf{x},\theta,v), \qquad (\mathbf{x},\theta,v) = \boldsymbol{\xi}(\mathbf{y},\theta,k), \qquad (\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1})$$

$$(\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k))_{ij} = \{\boldsymbol{\Upsilon}_i,\boldsymbol{\Upsilon}_j\}(\boldsymbol{\xi}(\mathbf{y},\theta,k)), \{\boldsymbol{\Upsilon}_i,\boldsymbol{\Upsilon}_j\} = (\nabla\boldsymbol{\Upsilon}_i)\cdot(\widetilde{\mathcal{P}}_\varepsilon(\nabla\boldsymbol{\Upsilon}_j))$$

Needed: $\{\boldsymbol{\Upsilon}_4,\boldsymbol{\Upsilon}_3\} = -\frac{1}{\varepsilon}$

$\{\boldsymbol{\Upsilon}_1,\boldsymbol{\Upsilon}_3\} = 0$

$\{\boldsymbol{\Upsilon}_1,\boldsymbol{\Upsilon}_4\} = 0$

$\{\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}_3\} = 0$

$\{\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}_4\} = 0$

First equation processing - 1: Exact solution

$$\{\boldsymbol{\Upsilon}_4, \boldsymbol{\Upsilon}_3\} = -\tfrac{1}{\varepsilon} \text{ or } \{\boldsymbol{\Upsilon}_3, \boldsymbol{\Upsilon}_4\} = \tfrac{1}{\varepsilon} \qquad (\bullet)$$

$$\boxed{\nabla \boldsymbol{\Upsilon}_3 = (0, 0, 1, 0)^T}$$

$\{\boldsymbol{\Upsilon}_3, \boldsymbol{\Upsilon}_4\} = (\nabla \boldsymbol{\Upsilon}_3) \cdot (\widetilde{\mathcal{P}}^\varepsilon(\nabla \boldsymbol{\Upsilon}_4))$: penultimate comp. of $(\widetilde{\mathcal{P}}^\varepsilon(\nabla \boldsymbol{\Upsilon}_4))$
$(\bullet) \rightarrow$

$$\cos(\theta)\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_1} - \sin(\theta)\frac{\partial \boldsymbol{\Upsilon}_4}{\partial x_2} + \frac{B(\mathbf{x})}{\varepsilon v}\frac{\partial \boldsymbol{\Upsilon}_4}{\partial v} = \frac{1}{\varepsilon}$$

Method of Characteristics

First equation processing - 1: Exact solution - 2

$$\cos(\theta)\frac{\partial \Upsilon_4}{\partial x_1} - \sin(\theta)\frac{\partial \Upsilon_4}{\partial x_2} + \frac{B(\mathbf{x})}{\varepsilon v}\frac{\partial \Upsilon_4}{\partial v} = \frac{1}{\varepsilon}$$

$$\frac{\partial \Upsilon_4}{\partial v} + \varepsilon \frac{v\cos(\theta)}{B(\mathbf{x})}\frac{\partial \Upsilon_4}{\partial x_1} - \varepsilon \frac{v\sin(\theta)}{B(\mathbf{x})}\frac{\partial \Upsilon_4}{\partial x_2} = \frac{v}{B(\mathbf{x})}$$

$$\Upsilon_{4|v=0} = 0$$

$$\mathcal{X}_1(\theta; v; \mathbf{x}, u) \text{ s.t. } \frac{\partial \mathcal{X}_1}{\partial v} = \varepsilon \frac{v\cos(\theta)}{B(\mathcal{X}_1, \mathcal{X}_2)}, \qquad \mathcal{X}_1(\theta; u; \mathbf{x}, u) = x_1$$

$$\mathcal{X}_2(\theta; v; \mathbf{x}, u) \text{ s.t. } \frac{\partial \mathcal{X}_2}{\partial v} = -\varepsilon \frac{v\sin(\theta)}{B(\mathcal{X}_1, \mathcal{X}_2)}, \qquad \mathcal{X}_2(\theta; u; \mathbf{x}, u) = x_2$$

$$\Upsilon_4(\mathbf{x}, \theta, v) = \Upsilon_4(\boldsymbol{\mathcal{X}}(\theta; 0; \mathbf{x}, v), \theta, 0) + \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, v))} ds$$
$$= \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, v))} ds$$

Gives explicit expression of k in terms of $(\mathbf{x}, \theta, v)$

First equation processing - 2: Asymptotic expansion

$$\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, u) = \mathbf{x} + \varepsilon s \boldsymbol{\mathcal{X}}^1 + \varepsilon^2 s^2 \boldsymbol{\mathcal{X}}^2 + \ldots\ldots$$

$$\boldsymbol{\Upsilon}_4(\mathbf{x}, \theta, v) = \int_0^v \frac{s}{B(\boldsymbol{\mathcal{X}}(\theta; s; \mathbf{x}, u))} ds = \int_0^v \frac{s}{B(\mathbf{x})} ds + \varepsilon \int_0^v s^2\, \mathcal{T}^1\Big(\frac{1}{B(\mathbf{x})}\Big) \cdot \boldsymbol{\mathcal{X}}^1 ds + + \varepsilon^2 \int_0^v s^3 \left(\mathcal{T}^2\Big(\frac{1}{B(\mathbf{x})}\Big) \cdot \boldsymbol{\mathcal{X}}^1 + \mathcal{T}^1\Big(\frac{1}{B(\mathbf{x})}\Big) \cdot \boldsymbol{\mathcal{X}}^2\right) ds + \ldots = \frac{v^2}{2B(\mathbf{x})} + \ldots$$

($\mathcal{T}^i$ linked with the Taylor expansion coefficients)

Gives new variable k as an expansion in ε

On other equations - Poisson Matrix in Darboux Coordinates

$\{\Upsilon_4, \Upsilon_3\} = -\frac{1}{\varepsilon}$

Processed. Gave k

$\{\Upsilon_1, \Upsilon_3\} = 0$, $\{\Upsilon_1, \Upsilon_4\} = 0$
$\{\Upsilon_2, \Upsilon_3\} = 0$, $\{\Upsilon_2, \Upsilon_4\} = 0$

To be Processed.

Check $\boldsymbol{\Upsilon}$ and $\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1}$: one to one, regular and invertible.

Will give $\mathbf{y}$ and k in terms of $(\mathbf{x}, \theta, v)$ and expansions in ε:

$$\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^0 + \varepsilon \boldsymbol{\Upsilon}^1 + \varepsilon^2 \boldsymbol{\Upsilon}^2 + \dots$$

Hence: $(\mathbf{y}, \theta, k)$ gotten
Last term of new Poisson matrix $\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k)$:
$(\overline{\mathcal{P}}_\varepsilon)_{12} = -(\overline{\mathcal{P}}_\varepsilon)_{21} = \{\Upsilon_1, \Upsilon_2\}$,

$$\overline{\mathcal{P}}_\varepsilon(\mathbf{y}, \theta, k) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 \\ \frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix}$$

Hamiltonian Function in Darboux Coordinates

We know:

- $\widetilde{H}_\varepsilon(\mathbf{x}, \theta, v) = \dfrac{v^2}{2}$
- $\overline{H}_\varepsilon(\mathbf{y}, \theta, k) = \widetilde{H}_\varepsilon(\boldsymbol{\xi}(\mathbf{y}, \theta, k))$ with $\boldsymbol{\xi} = \boldsymbol{\Upsilon}^{-1}$
- $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}^0 + \varepsilon \boldsymbol{\Upsilon}^1 + \varepsilon^2 \boldsymbol{\Upsilon}^2 + \dots$

We do :

- $\boldsymbol{\xi} = \boldsymbol{\xi}^0 + \varepsilon \boldsymbol{\xi}^1 + \varepsilon^2 \boldsymbol{\xi}^2 + \dots$
- $\widetilde{H}_\varepsilon(\boldsymbol{\xi}^0 + \varepsilon \boldsymbol{\xi}^1 + \varepsilon^2 \boldsymbol{\xi}^2 + \dots) = \widetilde{H}_\varepsilon(\boldsymbol{\xi}^0) + \varepsilon \mathcal{T}^1(\widetilde{H}_\varepsilon)(\boldsymbol{\xi}^0) \cdot \boldsymbol{\xi}^1 + \dots$

$$\overline{H}_\varepsilon(\mathbf{y}, \theta, k) = B(\mathbf{y})k + \varepsilon \overline{H}^1(\mathbf{y}, \theta, k) + \varepsilon^2 \overline{H}^2(\mathbf{y}, \theta, k) + \dots$$

First term : Independent of θ

Let us take stock

Usual Coordinates
$(\mathbf{x}, \mathbf{v})$

$H_\varepsilon(\mathbf{x}, \mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x}, \mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

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Polar Coordinates
$(\mathbf{x}, \theta, v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x}, \theta, v)$

4: Darboux Method

Darboux Almost Canonical Coordinates
$(\mathbf{y}, \theta, k)$

$\overline{H}_\varepsilon(\mathbf{y}, \theta, k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathtt{M}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)$$

Canonical Coordinates
$(\mathbf{q}, \mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q}, \mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q}, \mathbf{p}) = \mathcal{S}$
s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

5: Lie Method

Lie Coordinates
$(\mathbf{z}, \gamma, j)$

$\widehat{H}_\varepsilon(\mathbf{z}, j), \widehat{\mathcal{P}}_\varepsilon(\mathbf{z})$
$= \overline{\mathcal{P}}_\varepsilon(\mathbf{z})$

Lie Transform based Method Target - 1

We have $\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k)$: sought shape. But:

$$\overline{H}_\varepsilon(\mathbf{y},\theta,k) = \overline{H}^0(\mathbf{y},\not\theta,k) + \varepsilon\overline{H}^1(\mathbf{y},\theta,k) + \varepsilon^2\overline{H}^2(\mathbf{y},\theta,k) + \dots$$

depends on θ.

Key result $\longleftarrow$ $\theta-$independent Hamiltonian Function.
Target: Change of coordinates

$$(\mathbf{y},\theta,k) \mapsto (\mathbf{z},\gamma,j) = \boldsymbol{\zeta}(\mathbf{y},\theta,k)$$

leaving $\overline{\mathcal{P}}_\varepsilon$ unchanged,

$(\widehat{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j) = \overline{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j))$

$\varepsilon-$parametrized, close to identity, i.e.:

$\boldsymbol{\zeta}(\mathbf{y},\theta,k) = (\mathbf{y},\theta,k) + \varepsilon$ Something

$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) = \overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \widehat{H}^0(\mathbf{z},j) + \varepsilon\widehat{H}^1(\mathbf{z},j) + \varepsilon^2\widehat{H}^2(\mathbf{z},j) + \dots$

$(\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$

Lie Transform based Method Target - 2

We have $\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k)$: sought shape. But:

$$\overline{H}_\varepsilon(\mathbf{y},\theta,k) = \overline{H}^0(\mathbf{y},\not\theta,k) + \varepsilon\overline{H}^1(\mathbf{y},\theta,k) + \varepsilon^2\overline{H}^2(\mathbf{y},\theta,k) + \dots$$

depends on θ.

Key result $\longleftarrow$ $\theta-$independent Hamiltonian Function.
Target: Change of coordinates

$$(\mathbf{y},\theta,k) \mapsto (\mathbf{z},\gamma,j) = \boldsymbol{\zeta}(\mathbf{y},\theta,k)$$

leaving $\overline{\mathcal{P}}_\varepsilon$ almost unchanged (up to order $N-1$ in ε)

$$(\widehat{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j) = \overline{\mathcal{P}}_\varepsilon(\mathbf{z},\gamma,j) + \varepsilon^{N-1}\text{Something})$$

$\varepsilon-$parametrized, close to identity, i.e.:

$$\boldsymbol{\zeta}(\mathbf{y},\theta,k) = (\mathbf{y},\theta,k) + \varepsilon \text{ Something}$$

$$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) = \overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \widehat{H}^0(\mathbf{z},j) + \varepsilon\widehat{H}^1(\mathbf{z},j) + \varepsilon^2\widehat{H}^2(\mathbf{z},j) + \varepsilon^N\widehat{H}^N(\mathbf{z},j) + \varepsilon^{N+1}\widehat{H}^{N+1}(\mathbf{z},\gamma,j)$$

$(\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$

A remark

$$\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}} = \varepsilon\bar{\mathcal{P}}_{\varepsilon}\nabla\bar{f};$$

$$ij \geq N \Rightarrow \quad \left(\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^{n}\right)\cdot\{g,h\}\right) =$$
$$\left\{\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^{n}\right)\cdot g, \left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^{n}\right)\cdot h\right\} + \varepsilon^{N}\text{Something}$$

$$\boldsymbol{\vartheta}^{i,j}_{\varepsilon,\bar{f}}(\mathbf{y},\theta,k) = \left(\left(\sum_{n=0}^{i}\frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^{n}\right)\cdot\begin{pmatrix}\bar{\mathbf{r}}_1\\ \bar{\mathbf{r}}_2\\ \bar{\mathbf{r}}_3\\ \bar{\mathbf{r}}_4\end{pmatrix}\right)(\mathbf{y},\theta,k)$$

$$\bar{\mathbf{r}}_1 : (\mathbf{y},\theta,k)\mapsto y_1, \bar{\mathbf{r}}_2 : (\mathbf{y},\theta,k)\mapsto y_2, \bar{\mathbf{r}}_3 : (\mathbf{y},\theta,k)\mapsto \theta, \bar{\mathbf{r}}_4 : (\mathbf{y},\theta,k)\mapsto k$$

Consequence of the remark

For $\bar{g}_1, \dots, \bar{g}_N$, $\alpha_i = \min\left\{k \in \mathbb{N} \text{ s.t. } ki \geq N\right\} \left(= \mathbb{E}(\frac{N}{i}) + 1\right)$

$$\boldsymbol{\zeta} = \boldsymbol{\vartheta}^{\alpha_1,1}_{\varepsilon,-\bar{g}_1} \circ \boldsymbol{\vartheta}^{\alpha_2,2}_{\varepsilon,-\bar{g}_2} \circ \dots \circ \boldsymbol{\vartheta}^{\alpha_N,N}_{\varepsilon,-\bar{g}_N}, \qquad (\boldsymbol{\lambda} = \boldsymbol{\zeta}^{-1})$$

Since for i, j s.t. $ij \geq N$ $\left(\left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right) \cdot \{g, h\}\right) =$ $\left\{\left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right) \cdot g, \left(\sum_{n=0}^{i} \frac{\varepsilon^{jn}}{n!}\left(\bar{\mathbf{X}}^{\varepsilon}_{\varepsilon\bar{f}}\right)^n\right) \cdot h\right\} + \varepsilon^N$Something

$(\hat{\mathcal{P}}_\varepsilon(\mathbf{z},\theta,j))_{k,l} = \{\zeta_k, \zeta_l\}\left(\boldsymbol{\lambda}(\mathbf{z},\theta,j)\right) = (\bar{\mathcal{P}}_\varepsilon(\mathbf{z},\theta,j))_{k,l} + \varepsilon^{N-1}$Something

The game to play

Build $\bar{g}_1, \dots, \bar{g}_N$ s.t.

$$\widehat{H}^0(\mathbf{z},j)+\varepsilon\widehat{H}^1(\mathbf{z},j)+\varepsilon^2\widehat{H}^2(\mathbf{z},j)+\varepsilon^N\widehat{H}^N(\mathbf{z},j)+\varepsilon^{N+1}\widehat{H}^{N+1}(\mathbf{z},\gamma,j) =$$

$$\widehat{H}_\varepsilon(\mathbf{z},\gamma,j) =$$

$$\overline{H}_\varepsilon(\boldsymbol{\lambda}(\mathbf{z},\gamma,j)) = \left(\sum_{n=0}^{\alpha_1} \frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^n\right)\cdot\left(\sum_{n=0}^{\alpha_2} \frac{\varepsilon^{2n}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\right)^n\right)\cdot\dots\cdot$$

$$\left(\sum_{n=0}^{\alpha_N} \frac{\varepsilon^{Nn}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_N}\right)^n\right)\cdot\ \overline{H}_\varepsilon(\mathbf{z},\gamma,j) + \varepsilon^{N+1}\text{Something}$$

$$= \left(\sum_{n=0}^{\alpha_1} \frac{\varepsilon^n}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_1}\right)^n\right)\cdot\left(\sum_{n=0}^{\alpha_2} \frac{\varepsilon^{2n}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_2}\right)^n\right)\cdot\dots\cdot$$

$$\left(\sum_{n=0}^{\alpha_N} \frac{\varepsilon^{Nn}}{n!}\left(\bar{\mathbf{X}}^\varepsilon_{\varepsilon\bar{g}_N}\right)^n\right)\cdot\left(\overline{H}^0(\mathbf{z},\not{\gamma},j)+\varepsilon\overline{H}^1(\mathbf{z},\gamma,j)+\varepsilon^2\overline{H}^2(\mathbf{z},\gamma,j)+\dots\right)$$

$$+\,\varepsilon^{N+1}\text{Something},$$

If you play the game ...

... with:

$$\overline{\mathcal{P}}_\varepsilon(\mathbf{y},\theta,k) = \begin{pmatrix} 0 & -\frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 \\ \frac{\varepsilon}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{pmatrix} = \frac{1}{\varepsilon}\bar{\mathcal{T}}_0 + \varepsilon\bar{\mathcal{T}}_2(\mathbf{y})$$

$$\bar{\mathcal{T}}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}; \quad \bar{\mathcal{T}}_2(\mathbf{y}) = \begin{bmatrix} 0 & \frac{-1}{B(\mathbf{y})} & 0 & 0 \\ \frac{1}{B(\mathbf{y})} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

... you have

$$\widehat{H}^0(\mathbf{z},j) = \overline{H}^0(\mathbf{z},\not\gamma,j),$$

$$\widehat{H}^1(\mathbf{z},j) = -\frac{1}{2\pi}\int_0^{2\pi}\overline{H}^1 d\gamma,$$

$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_1\right)\cdot\nabla\overline{H}^0 = \overline{H}^1 - \frac{1}{2\pi}\int_0^{2\pi}\overline{H}^1 d\gamma$$

$$\widehat{H}^2(\mathbf{z},j) = -\frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_2(\overline{H}^1,\overline{H}^2,\bar{g}_1)d\gamma$$

$$\left(\bar{\mathcal{T}}_0\nabla\bar{g}_2\right)\cdot\nabla\overline{H}^0 = \mathcal{V}_2(\overline{H}^1,\overline{H}^2,\bar{g}_1) - \frac{1}{2\pi}\int_0^{2\pi}\mathcal{V}_2(\overline{H}^1,\overline{H}^2,\bar{g}_1)d\gamma$$

etc.

At the end of the day

Usual Coordinates

$(\mathbf{x},\mathbf{v})$

$H_\varepsilon(\mathbf{x},\mathbf{v}), \mathcal{P}_\varepsilon(\mathbf{x},\mathbf{v})$ s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{X}}{\partial t} \\ \frac{\partial \mathbf{V}}{\partial t} \end{pmatrix} = \mathcal{P}_\varepsilon \nabla_{\mathbf{x},\mathbf{v}} H_\varepsilon$$

1: Hamiltonian?

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Polar Coordinates

$(\mathbf{x},\theta,v)$

$\widetilde{H}_\varepsilon(v), \widetilde{\mathcal{P}}_\varepsilon(\mathbf{x},\theta,v)$

4: Darboux Method

Canonical Coordinates

$(\mathbf{q},\mathbf{p})$

$\breve{H}_\varepsilon(\mathbf{q},\mathbf{p}), \breve{\mathcal{P}}_\varepsilon(\mathbf{q},\mathbf{p}) = \mathcal{S}$

s.t:

$$\begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial t} \\ \frac{\partial \mathbf{P}}{\partial t} \end{pmatrix} = \mathcal{S} \nabla_{\mathbf{q},\mathbf{p}} \breve{H}_\varepsilon$$

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathtt{M}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & -\frac{1}{\varepsilon} & 0 \end{array}\right)$$

Darboux Almost Canonical Coordinates

$(\mathbf{y},\theta,k)$

$\overline{H}_\varepsilon(\mathbf{y},\theta,k), \overline{\mathcal{P}}_\varepsilon(\mathbf{y})$

5: Lie Method

Lie Coordinates

$(\mathbf{z},\gamma,j)$

$$\widehat{H}^{\mathbf{0}}(\mathbf{z},j) + \cdots + \varepsilon^N \widehat{H}^N(\mathbf{z},j) + \varepsilon^{N+\mathbf{1}} \widehat{H}^{N+\mathbf{1}}(\mathbf{z},\gamma,j)$$

$\widehat{\mathcal{P}}_\varepsilon(\mathbf{z}) = \overline{\mathcal{P}}_\varepsilon(\mathbf{z}) + \varepsilon^{N-\mathbf{1}}$Something

Hence

In Lie Coordinates: Trajectories $(\mathbf{Z}, \Gamma, \mathcal{J})$

$$\frac{\partial \mathbf{Z}}{\partial t} = \text{Something independent of } \Gamma + \varepsilon^{N+1}\text{Remainder}(\mathbf{Z}, \Gamma, \mathcal{J})$$

$$\frac{\partial \Gamma}{\partial t} = \text{Something complicated}$$

$$\frac{\partial \mathcal{J}}{\partial t} = \varepsilon^{N-1}\text{Something}(\mathbf{Z}, \Gamma, \mathcal{J})$$

$(\mathbf{Z}^T, \Gamma^T, \mathcal{J}^T)$:

$$\frac{\partial \mathbf{Z}^T}{\partial t} = \text{Something independent of } \Gamma^T$$

$$\frac{\partial \Gamma^T}{\partial t} = \text{Something complicated}$$

$$\frac{\partial \mathcal{J}^T}{\partial t} = 0$$

$$|(\mathbf{Z}^T, \Gamma^T, \mathcal{J}^T)(t) - (\mathbf{Z}, \Gamma, \mathcal{J})(t)| \leq C\varepsilon^{N-1}$$

Implementing with N=3

$$\frac{\partial \mathbf{Z}^T}{\partial t} = -\frac{\varepsilon \mathcal{J}}{B\left(\mathbf{Z}^T\right)} {}^{\perp}\nabla B\left(\mathbf{Z}^T\right),$$

$$\frac{\partial \Gamma^T}{\partial t} = \frac{B\left(\mathbf{Z}^T\right)}{\varepsilon} + \varepsilon \frac{\mathcal{J}^T}{2\left(B\left(\mathbf{Z}^T\right)\right)^2}\left(B\left(\mathbf{Z}^T\right)\nabla^2 B\left(\mathbf{Z}^T\right) - 3\left(\nabla B\left(\mathbf{Z}^T\right)\right)^2\right)$$

$$\frac{\partial \mathcal{J}^T}{\partial t} = 0,$$

Thank for your attention