Fictitious domain methods for finite element methods, application to structural mechanics

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- Motivation
- Introduction
- Existence, Uniqueness and Optimality
- *A posteriori* error estimators
- Conclusion and Perspectives

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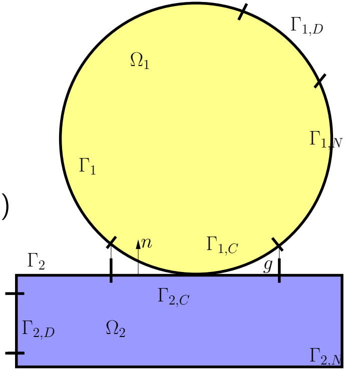
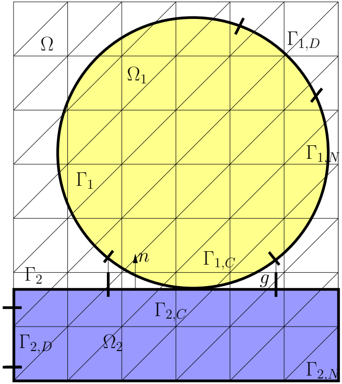
We consider the unilateral contact problem between two elastics bodies without friction, as follows:

$$\left\{\begin{array}{ccc} \text{Find } u = (u_1, u_2) \text{ such as:} & & \\ -\text{div}\sigma(u_i) = f_i & \text{in} & \Omega_i, \\ \sigma(u_i) = A\varepsilon(u_i) & \text{in} & \Omega_i, \\ u_i = u_{i,D} & \text{on} & \Gamma_{i,D}, \\ \sigma(u_i)n_i = l_i & \text{on} & \Gamma_{i,N}, \\ C.C. & \text{on} & \Gamma_{i,C} \end{array}\right. \quad (1)$$

where $\varepsilon(v) = \dfrac{1}{2}(\nabla v + \nabla v^T)$.



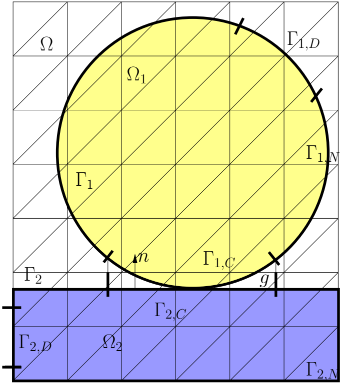
Example of contact problem.



Example of fictitious domain method for an unilateral contact problem between two bodies.

- E. Bécache, P.Joly and G. Scarella, 2001.
- S. Tahir, 2006.

Nitsche's method: consistency and avoid using of lagrangien multiplier



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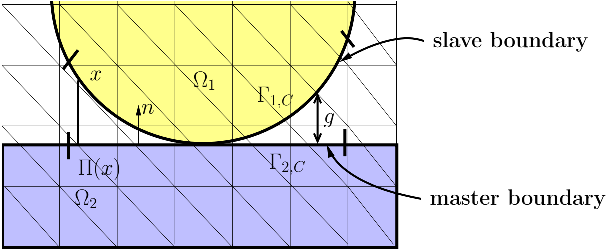


Example of projection, gap and unit normal vector.

Orthogonale projection: Initial gap: unit normal vector:

$$\Pi : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \Gamma_{2,C} \\ x & \mapsto & \Pi(x) \end{array}, \quad g : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R} \\ x & \mapsto & \|x - \Pi(x)\| \end{array}, \quad n : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R}^d \\ x & \mapsto & n_2(\Pi(x)) \end{array}.$$

The contact conditions in small deformation can be expressed, as follows:

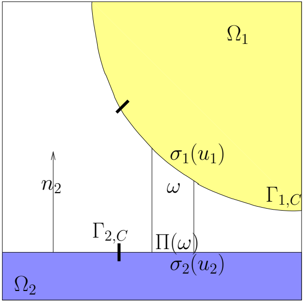
$$\left\{\begin{array}{cll} [\![u\cdot n]\!] \leqslant g & \text{on } \Gamma_{1,C} & (i), \\ \sigma_n(u_1) \leqslant 0 & \text{on } \Gamma_{1,C} & (ii), \\ \sigma_n(u_1)([\![u\cdot n]\!] - g) = 0 & \text{on } \Gamma_{1,C} & (iii), \\ [\![\sigma(u)n]\!] = 0 & \text{on } \Gamma_{1,C} & (iv), \\ \sigma_t(u_1) = 0 & \text{on } \Gamma_{1,C} & (v), \end{array}\right. \tag{2}$$

with

$$[\![u\cdot n]\!] = (u_2 \circ \Pi - u_1)\cdot n,$$

and

$$[\![\sigma(u)n]\!] = \sigma(u_1)n_1 + \sigma(u_2\circ\Pi)n_2\circ\Pi\ \left|\det(J_\Pi)\right|.$$



Example of action-reaction principle.

$$\forall \omega \subset \Gamma_{1,C}, \qquad \int_\omega \sigma(u_1)\cdot n_1 \, \mathrm{d}\Gamma = -\int_\omega \sigma(u_2 \circ \Pi) \; |\det(J_\Pi)| \cdot n_2 \circ \Pi \, \mathrm{d}\Gamma.$$

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We define the convex cone K_c of admissible displacement

$$K_c = \{v = (v_1, v_2) \in H^1(\Omega_1)^d \times H^1(\Omega_2)^d \mid v_1 = u_{1,D} \text{ on } \Gamma_{1,D}$$

$$\text{and } v_2 = u_{2,D} \text{ on } \Gamma_{2,D} \mid [\![v \cdot n]\!] - g \leqslant 0 \text{ on } \Gamma_{1,C}\}.$$

The weak formulation of the contact problem as variational inequality, reads as:

$$\left\{ \begin{array}{l} \text{Find } u \in K_c \text{ such as} \\ a(u, v-u) \geqslant L(v-u) \qquad \forall v \in K_c. \end{array} \right. \quad (3)$$

Stampacchia's theorem

It exists a unique solution of the weak formulation (3)

Using Green's formula and equilibrium equation, it holds:

$$a(u,v) - \sum_{i=1,2} \int_{\Gamma_{i,D}} \sigma_n(u_i) n_i \cdot v_i \, \mathrm{d}\Gamma - \int_{\Gamma_{1,C}} \sigma_n(u_1) [\![v \cdot n]\!] \, \mathrm{d}\Gamma = L(v).$$

Reminds: $a(u,v) = \sum_{i=1,2} \int_{\Omega_i} \sigma(u_i) : \varepsilon(v_i) \, \mathrm{d}\Omega,$

$$L(v) = \sum_{i=1,2} \int_{\Omega_i} f_i v_i \, \mathrm{d}\Omega + \sum_{i=1,2} \int_{\Gamma_{i,N}} l_i v_i \, \mathrm{d}\Gamma.$$

proposition

The classical reformulation of contact conditions of (i) -(iii) for $\gamma > 0$ is:

$$\sigma_n(u) = -\frac{1}{\gamma}[[\![u \cdot n]\!] - g - \gamma \sigma_n(u)]_+ \qquad a.e.$$

- Remark:

$$[\![v \cdot n]\!] = [\![v \cdot n]\!] - \theta\gamma\sigma_n(v) + \theta\gamma\sigma_n(v) \qquad , \forall \theta \in \mathbb{R}$$

- For the contact conditions, we replace $-\int_{\Gamma_{1,C}} \sigma_n(u_1)[\![v \cdot n]\!] \, \mathrm{d}\Gamma$ by

$$\int_{\Gamma_{1,C}} \frac{1}{\gamma}[[\![u \cdot n]\!] - g - \gamma\sigma_n(u)]_+([\![v \cdot n]\!] - \theta\gamma\sigma_n(v)) \, \mathrm{d}\Gamma$$
$$-\int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u)\sigma_n(v) \, \mathrm{d}\Gamma$$

- For the contact conditions, we replace $-\int_{\Gamma_{i,D}} \sigma(u_i)n_i \cdot v_i \, \mathrm{d}\Gamma$ by

$$\int_{\Gamma_{i,D}} \frac{1}{\gamma}(u_i - u_{i,D} - \gamma\sigma(u_i)n_i) \cdot (v_i - \gamma\theta\sigma(v_i)n_i) \, \mathrm{d}\Gamma - \int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i)\sigma(v_i) \, \mathrm{d}\Gamma$$

We obtain the following formulation based on Nitsche's method

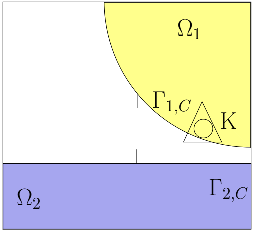
$$\begin{cases} a(u,v) \\ + \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma}(u_i - u_{i,D} - \gamma\sigma(u_i)n_i) \cdot (v_i - \gamma\theta\sigma(v_i)n_i)\, \mathrm{d}\Gamma \\ - \sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i) \cdot \sigma(v_i)\, \mathrm{d}\Gamma \\ + \int_{\Gamma_{\mathbf{1},C}} \frac{1}{\gamma}[[\![u \cdot n]\!] - g - \gamma\sigma_n(u)]_+ ([\![v \cdot n]\!] - \theta\gamma\sigma_n(v))\, \mathrm{d}\Gamma \\ - \int_{\Gamma_{\mathbf{1},C}} \theta\gamma\sigma_n(u)\sigma_n(v)\, \mathrm{d}\Gamma = L(v). \end{cases} \quad (4)$$

The parameter $\theta \in \mathbb{R}$ determines the choice of the method:

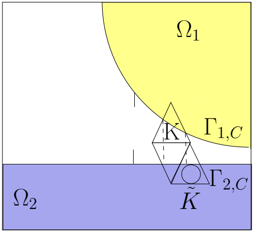
- $\theta = 1$ symmetric method, derived of a potentiel.
- $\theta = 0$ non symmetric method.
- $\theta = -1$ skew symmetric method.

Using standard Galerkin method, we obtain the following discrete formulation based on Nitsche's method

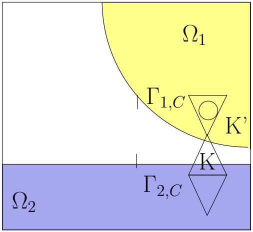
$$
\left\{
\begin{aligned}
& a(u^h, v^h) \\
& + \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma}(u_i^h - u_{i,D}^h - \gamma\sigma(u_i^h)n_i) \cdot (v_i^h - \gamma\theta\sigma(v_i^h)n_i)\ \mathrm{d}\Gamma \\
& - \sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i^h) \cdot \sigma(v_i^h)\ \mathrm{d}\Gamma \\
& + \int_{\Gamma_{1,C}} \frac{1}{\gamma}[[\![u^h \cdot n]\!] - g - \gamma\sigma_n(u^h)]_+ ([\![v^h \cdot n]\!] - \theta\gamma\sigma_n(v^h))\ \mathrm{d}\Gamma \\
& - \int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u^h)\sigma_n(v^h)\ \mathrm{d}\Gamma = L(v^h).
\end{aligned}
\right.
\tag{5}
$$



a) If $\Omega_1 \cap K$ is sufficiently large



b) If $\exists \tilde{K} \in S_K$ such that $\Omega_2 \cap \tilde{K}$ is sufficiently large



c) Otherwise

J. Haslinger and Y. Renard, 2009.

E. Burman and P. Hansbo, 2010.

E. Burman and P. Hansbo, 2012.

Contact

$$P_\gamma^h : \begin{array}{ccc} V_1^h \times V_2^h & \rightarrow & L^2(\Gamma_{1,C}) \\ v & \mapsto & [\![v \cdot n]\!] - \gamma R_{\hat{\rho}}(v) \end{array},$$

Dirichlet

$$\overline{P}_{i,\gamma}^h : \begin{array}{ccc} V_i^h & \rightarrow & L^2(\Gamma_{i,D})^d \\ v_i & \mapsto & v_i - \gamma \overline{R}_{\hat{\rho}}(v_i) \end{array}.$$

Using finite element methods, it holds:

$$\left\{ \begin{array}{l} a(u^h, v^h) \\ + \displaystyle\sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma} (\overline{P}_{i,\gamma}^h(u_i^h) - u_{i,D}^h) \cdot \overline{P}_{i,\gamma\theta}^h(v_i^h) \, \mathrm{d}\Gamma \\ - \displaystyle\sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma \overline{R}_{\hat{\rho}}(u_i^h) \cdot \overline{R}_{\hat{\rho}}(v_i^h) \, \mathrm{d}\Gamma \\ + \displaystyle\int_{\Gamma_{1,C}} \frac{1}{\gamma} [P_\gamma^h(u^h) - g]_+ P_{\gamma\theta}^h(v^h) \, \mathrm{d}\Gamma \\ - \displaystyle\int_{\Gamma_{1,C}} \theta\gamma R_{\hat{\rho}}(u^h) R_{\hat{\rho}}(v^h) \, \mathrm{d}\Gamma = L(v^h) \qquad \forall v^h \in V^h. \end{array} \right. \tag{6}$$

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Theorem : Existence, Uniqueness

We take $\gamma = \gamma_0 h_K$ and $\gamma_0 > 0$. If $\theta \neq -1$, then we suppose γ_0 sufficiently small. it exits a unique solution $v^h \in V^h$ of discrete problem (6).

Theorem : Consistence

We define u a sufficiently regular solution of the continuous problem (1) and the contact conditions (2), then u is solution of our discrete problem (6).

M. Fabre, J. Pousin, Y. Renard, 2014.

Theorem : Optimal *a priori* error estimate

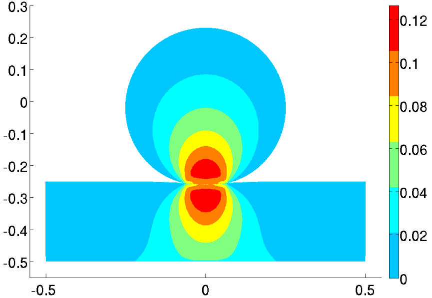
We define u a solution of the contact problem in $H^{\frac{3}{2}+\nu}(\Omega_1) \times H^{\frac{3}{2}+\nu}(\Omega_2)$ with $1/2 \geqslant \nu > 0$ if $k = 1$ and $1 > \nu > 0$ if $k = 2$. If $\theta \neq -1$, we suppose $\gamma_0 > 0$ sufficiently small. The solution u^h of the problem (6) satisfy the following estimation of *a priori* error:

$$\sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i}^2 + \left\| \gamma^{\frac{1}{2}} (\sigma_n(u) + \frac{1}{\gamma}[P_\gamma^{h,\hat{\rho}}(u^h) - g]_+) \right\|_{0,\Gamma_{1,C}}^2 + \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} (\overline{R}_{\hat{\rho}}(u_i^h) - \sigma(u_i)) \right\|_{0,\Gamma_{i,D}}^2 \leqslant Ch^{1+2\nu} \sum_{i=1,2} \|u\|_{\frac{3}{2}+\nu,\Omega_i}^2 \quad (7)$$

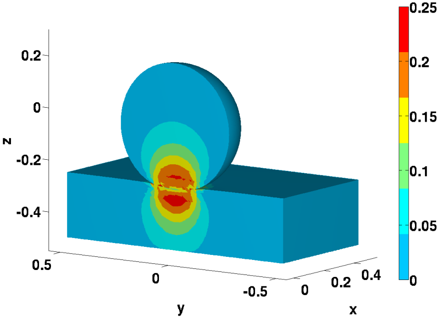
with $C > 0$ a constant independent of h and u.

M. Fabre, J. Pousin, Y. Renard, 2014.

Referential solution $\theta = -1$, $\gamma_0 = \frac{1}{200}$ and Lagrange's elements P_2



$2D$ and $h = \frac{1}{400}$.



$3D$ and $h = \frac{1}{30}$.

Numerical experiments: getfem++

Table: rate of convergence with $\gamma_0 = \frac{1}{200}$, Lagrange's elements P_1 in $2D$

méthode	$\theta = 1$	$\theta = 0$	$\theta = -1$	optimal rate
$\left\|u - u^h\right\|_{0,\Omega_1}$	2.25	2.10	2.00	2 ?
$\left\|u - u^h\right\|_{0,\Omega_2}$	1.72	1.73	1.73	2 ?
$\left\|u - u^h\right\|_{1,\Omega_1}$	1.11	1.12	1.12	1
$\left\|u - u^h\right\|_{1,\Omega_2}$	0.97	0.97	0.97	1

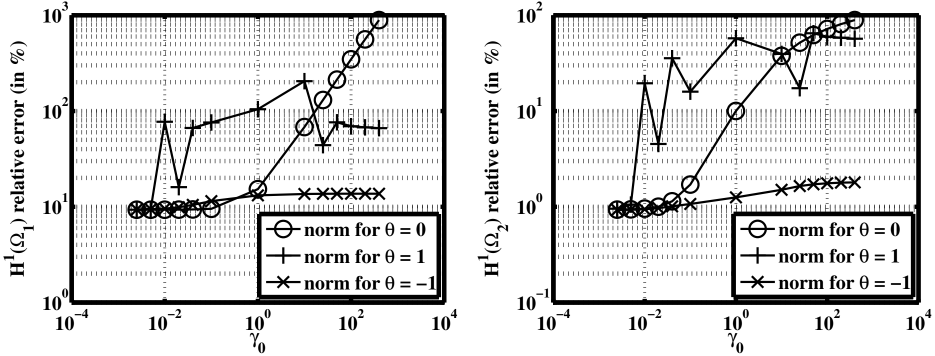
Table: rate of convergence with $\gamma_0 = \frac{1}{200}$, Lagrange's elements P_2 in $2D$

méthode	$\theta = 1$	$\theta = 0$	$\theta = -1$	optimal rate
$\left\|u - u^h\right\|_{0,\Omega_1}$	1.87	2.23	2.90	2.5 ?
$\left\|u - u^h\right\|_{0,\Omega_2}$	2.03	2.14	2.21	2.5 ?
$\left\|u - u^h\right\|_{1,\Omega_1}$	1.58	1.80	1.70	1.5
$\left\|u - u^h\right\|_{1,\Omega_2}$	1.30	1.31	1.31	1.5

Numerical experiments: getfem++

Influence of γ_0 on the relative H^1-norm

$h = \frac{1}{90}$, elements P_2 in $2D$



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1. $\left\|u-u^h\right\| \lesssim C_1\eta+\zeta \qquad \text{and} \qquad \eta \lesssim C_2\left\|u-u^h\right\|+\zeta.$

2. A local error estimator allow to locale the error and adopt a method of adaptive refinement.

M. Ainsworth and J.T. Oden, 1997.

M. Ainsworth and J. T. Oden, 2000.

A. Ern and J.-L. Guermond, 2004. ...

P. Hild and S. Nicaise, 2007.

We define the unilateral contact problem Without fictitious domain approach between an elastic body and a rigid body:

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h \text{ such as :} \\ a(u^h, v^h) - \displaystyle\int_{\Gamma_C} \theta\gamma \, \sigma_n(u^h)\sigma_n(v^h) \ \mathrm{d}\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma} [P^h_\gamma(u^h)]_+ P^h_{\theta\gamma}(v^h) \ \mathrm{d}\Gamma = L(v^h), \quad \forall\, v^h \in V^h, \end{array} \right. \tag{8}$$

with $a(u,v) = \int_\Omega \sigma(u) : \varepsilon(v) \, \mathrm{d}\Omega$, $L(v) = \int_\Omega fv \, \mathrm{d}\Omega + \int_{\Gamma_N} \ell v \, \mathrm{d}\Gamma$.
We take

$$f_K = \int_K f(x) \, \mathrm{d}x / |K| \, .$$

We introduce the local error estimator η_K and the global η defined by

definition of *a posteriori* error estimators

$$
\begin{aligned}
\eta_K &= \left(\sum_{i=1}^{4} \eta_{iK}^2\right)^{1/2} \quad \text{and} \quad \eta = \left(\sum_{K\in T_h} \eta_K^2\right)^{1/2}, \\
\eta_{1K} &= h_K \|\mathrm{div}\,\sigma(u^h) + f_K\|_{0,K}, \\
\eta_{2K} &= h_K^{1/2} \left(\sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(u^h)\|_{0,E}^2\right)^{1/2}, \\
\eta_{3K} &= h_K^{1/2} \left(\sum_{E\in E_K^C} \|\sigma_t(u^h)\|_{0,E}^2\right)^{1/2}, \\
\eta_{4K} &= h_K^{1/2} \left(\sum_{E\in E_K^C} \left\|\frac{1}{\gamma}[P_\gamma^h(u^h) - g]_+ + \sigma_n(u^h)\right\|_{0,E}^2\right)^{1/2}.
\end{aligned} \tag{9}
$$

$J_{E,n}(u^h)$ means the constraint jump of u^h in the normal direction, i.e.

$$J_{E,n}(u^h) = \begin{cases} [\![\sigma(u^h)\nu_E]\!], & \forall E \in E_h^{int}, \\ \sigma(u_i^h)\nu_E - \ell_{i,E}, & \forall E \in E_h^N. \end{cases} \quad (10)$$

The local and global approximation terms are given by

$$\zeta_K = \left(h_K^2 \sum_{K' \subset \omega_K} \|f_i - f_{i,K'}\|_{0,K'}^2 + h_E \sum_{E \in E_K^N} \|\ell_i - \ell_{i,E}\|_{0,E}^2 \right)^{1/2},$$

$$\zeta = \left(\sum_{K \in T_h} \zeta_K^2 \right)^{1/2}.$$

saturation assumption:

The solution u of the weak problem and the solution u^h of discrete problem (8) are such as:

$$\left\|\sigma_n(u - u^h)\right\|_{0,\Gamma_C} \lesssim h^{-1/2}\|u - u^h\|_{1,\Omega}. \tag{11}$$

R. Becker, P. Hansbo and R. Stenberg, 2003.

B.I. Wohlmuth, 1999.

Theorem : Upper error bound (Reliability) :

We take u the solution of variational problem with $u \in (H^{\frac{3}{2}+\nu}(\Omega))^d$ ($\nu > 0$ and $d = 2, 3$) and u^h the solution of the discrete problem (8). We suppose that if $\theta \neq -1$ then $\gamma_0 > 0$ is sufficiently small. Assume that the saturation assumption holds as well. Then we obtain:

$$\|u - u^h\|_{1,\Omega} + h^{1/2} \left\| \sigma_n(u) + \frac{1}{\gamma}[P_\gamma(u^h)]_+ \right\|_{0,\Gamma_C} + h^{1/2}\|\sigma_n(u) - \sigma_n(u^h)\|_{0,\Gamma_C} \lesssim (1+\gamma_0)\eta + \zeta. \tag{12}$$

Theorem : Lower error bound (Efficacy) :

For all elements $K \in T_h$, the following local lower error bounds hold:

$$\eta_{1K} \lesssim \|u - u^h\|_{1,K} + \zeta_K, \quad (13)$$

$$\eta_{2K} \lesssim \|u - u^h\|_{1,\omega_K} + \zeta_K. \quad (14)$$

For all elements K such as $K \cap E_K^C \neq \emptyset$, the following local lower error bounds hold:

$$\eta_{3K} \lesssim \|u - u^h\|_{1,K} + \zeta_K, \quad (15)$$

$$\eta_{4K} \lesssim \sum_{E \in E_K^C} h_K^{1/2} \left(\left\| \sigma_n(u) + \frac{1}{\gamma}[P_\gamma(u^h)]_+ \right\|_{0,E} + \left\| \sigma_n(u - u^h) \right\|_{0,E} \right) \quad (16)$$

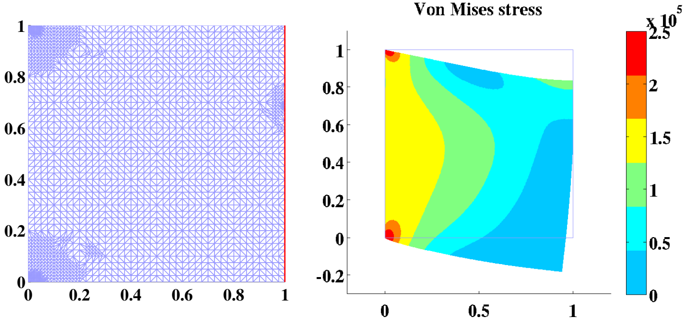

Figure: Left panel: mesh with adaptive refinement and contact boundary. Right panel: plot of Von Mises stress. Parameters $\gamma_0 = 1/E$, $\theta = -1$ and elements P_2.

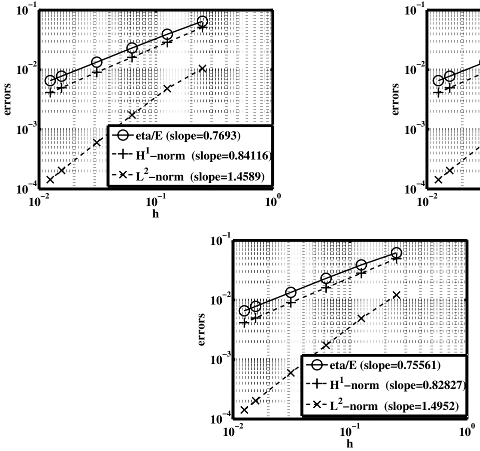

Figure: Convergence curves of the error estimator η, the L^2 and H^1-norms of the error $u - u^h$, for $\gamma_0 = 1/E$ ($\theta = 1, 0$ or -1).

Table: $\theta = -1$ and $\gamma_0 = 1/E$.

size of mesh h	1/8	1/16	1/32	1/64	1/80	slope
degrees of freedom	128	512	2048	8192	12800	
$\left\|u - u^h\right\|_{0,\Omega}$ $(\times 10^{-4})$	48.9718	17.3613	5.9619	2.0360	1.4255	1.4952
$\left\|u - u^h\right\|_{1,\Omega}$ $(\times 10^{-3})$	28.1269	16.0087	9.0385	4.9714	4.1467	0.8283
η_1	8359.9	4179.95	2089.97	1044.99	835.99	1
η_2	37649.9	22607.7	13213.2	7723.58	6506.99	0.7428
η_3	1464.81	558.637	192.194	70.7559	53.7733	1.3544
η_4	2854.93	832.228	229.683	62.842	44.0949	1.8004
η	38700.1	23012.7	13380.8	7794.52	6560.84	0,7779
Eff_E	1.3759	1.4375	1.4804	1.5677	1.5820	

We introduce the local error estimator η_K and the global η defined by

definition of *a posteriori* error estimators

$$\eta_K = \left(\sum_{i=1}^{4} \eta_{iK}^2\right)^{1/2}, \eta = \left(\sum_{K\in T_h} \eta_K^2\right)^{1/2},$$

$$\eta_{1K} = h_K \|\mathrm{div}\,\sigma(u_i^h) + f_{i,K}\|_{0,K},\ \eta_{3K} = h_K^{1/2}\left(\sum_{E\in E_K^C} \|\sigma_t(u^h)\|_{0,E}^2\right)^{1/2},$$

$$\eta_{2K} = h_K^{1/2}\left(\sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(u^h)\|_{0,E}^2\right)^{1/2},$$

$$\eta_{4K} = h_K^{1/2}\left(\sum_{E\in E_K^C} \left\|\frac{1}{\gamma}[P_\gamma^{h,\hat{\rho}}(u^h) - g]_+ + \sigma_n(u^h)\right\|_{0,E}^2\right)^{1/2},$$

$$\eta_{5K} = h_K^{1/2}\left(\sum_{E\in E_K^D} \left\|\frac{1}{\gamma}\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D} + \sigma(u_i^h)\right\|_{0,E}^2\right)^{1/2},$$

saturations assumptions:

The solution u of the weak problem and the solution u^h of discrete problem (8) are such as:

$$\begin{aligned} \left\|\sigma_n(u-u^h)\right\|_{0,\Gamma_{1,C}}^2 &\lesssim h^{-1}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2 \\ \left\|\sigma_n(u)-R_{\hat{\rho}}(u^h)\right\|_{0,\Gamma_{1,C}}^2 &\lesssim h^{-1}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2, \end{aligned} \tag{17}$$

$$\begin{aligned} \sum_{i=1,2}\left\|\sigma(u_i-u_i^h)\right\|_{0,\Gamma_{i,D}}^2 &\lesssim h^{-1}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2 \\ \sum_{i=1,2}\left\|\sigma(u_i)-\overline{R}_{\hat{\rho}}(u_i^h)\right\|_{0,\Gamma_{i,D}}^2 &\lesssim h^{-1}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2, \end{aligned} \tag{18}$$

$$\sum_{i=1,2}\left\|\sigma(u_i)n_i-\sigma(u_i^h)n_i\right\|_{0,\Gamma_{i,N}}^2 \lesssim h^{-1}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2, \tag{19}$$

$$\sum_{i=1,2}\left\|\sigma(u_i)-\sigma(u_i^h)\right\|_{0,\Omega_i}^2 \lesssim h^{-2}\sum_{i=1,2}\|u_i-u_i^h\|_{1,\Omega_i}^2. \tag{20}$$

Upper error bound :

We suppose u the solution of variational problem with $u \in (H^{\frac{3}{2}+\nu}(\Omega_1))^d \times (H^{\frac{3}{2}+\nu}(\Omega_2))^d$ ($\nu > 0$ and $d = 2, 3$) and u^h the solution of the discrete problem. We suppose that if $\theta \neq -1$ then $\gamma_0 > 0$ is sufficiently small. Assume that the saturation assumptions (17)-(20) hold as well. Then we obtain:

$$
\begin{aligned}
&\sum_{i=1,2} \|u_i - u_i^h\|_{1,\Omega_i} + h^{1/2} \left\| \sigma_n(u) + \frac{1}{\gamma}[P_\gamma^{h,\hat{\rho}}(u^h) - g]_+ \right\|_{0,\Gamma_{1,C}} \\
&+h^{1/2}\|\sigma_n(u) - R_{\hat{\rho}}(u^h)\|_{0,\Gamma_{1,C}} + h^{1/2} \sum_{i=1,2} \left\| \sigma(u_i) + \frac{1}{\gamma}\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D} \right\|_{0,\Gamma_{i,D}} \\
&+h^{1/2} \sum_{i=1,2} \|\sigma(u_i) - \overline{R}_{\hat{\rho}}(u_i^h)\|_{0,\Gamma_{i,D}} \lesssim (1+\gamma_0)\eta + \zeta.
\end{aligned}
$$

Lower error bound :

For all elements $K \in T_h$, the following local lower error bounds hold:

$$\eta_{1K} \lesssim \|u_i - u_i^h\|_{1,K} + \zeta_K, \eta_{2K} \lesssim \|u_i - u_i^h\|_{1,\omega_K} + \zeta_K. \tag{22}$$

For all elements K such as $K \cap E_K^C \neq \emptyset$ then $K \cap E_K^D \neq \emptyset$, the following local lower error bounds hold:

$$\eta_{3K} \lesssim \|u_i - u_i^h\|_{1,K} + \zeta_K, \tag{23}$$

$$\eta_{4K} \lesssim \sum_{E \in E_K^C} h_K^{1/2}\left(\left\|\sigma_n(u) + \frac{1}{\gamma}[P_\gamma^h(u^h) - g]_+\right\|_{0,E} + \left\|\sigma_n(u - u^h)\right\|_{0,E}\right) \tag{24}$$

$$\eta_{5K} \lesssim \sum_{E \in E_K^D} h_K^{1/2}\left(\left\|\frac{1}{\gamma}\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D} + \sigma(u_i)\right\|_{0,E} + \left\|\sigma(u_i - u_i^h)\right\|_{0,E}\right) \tag{25}$$

We define a new stabilized operator R on all domains.

a new stabilized problem

We can define a new stabilized problem :

$$\left\{\begin{array}{l} \text{Find } u^h \in V^h \text{ such as} \\ \displaystyle\sum_{i=1,2}\int_{\Omega_i} R(u_i^h) : \varepsilon(v_i^h)\, \mathrm{d}\Omega - \int_{\Gamma_{1,C}} \theta\gamma R_{\hat{\rho}}(u^h) R_{\hat{\rho}}(v^h)\, \mathrm{d}\Gamma \\ \displaystyle -\sum_{i=1,2}\int_{\Gamma_{i,D}} \theta\gamma \overline{R}_{\hat{\rho}}(u_i^h)\cdot \overline{R}_{\hat{\rho}}(v_i^h)\, \mathrm{d}\Gamma + \int_{\Gamma_{1,C}} \frac{1}{\gamma}[P_{\gamma}^{h,\hat{\rho}}(u^h) - g]_+ P_{\theta\gamma}^{h,\hat{\rho}}(v^h)\, \mathrm{d}\Gamma \\ \displaystyle +\sum_{i=1,2}\int_{\Gamma_{i,D}} \frac{1}{\gamma}(\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D})\cdot \overline{P}_{i,\gamma\theta}^{h,\hat{\rho}}(v_i^h)\, \mathrm{d}\Gamma = L(v^h) \qquad \forall v^h \in V^h. \end{array}\right.$$

We introduce the local error estimator η_K and the global η defined by

definition of *a posteriori* error estimators

$$\eta_K = \left(\sum_{i=1}^{4} \eta_{iK}^2\right)^{1/2}, \eta = \left(\sum_{K\in T_h} \eta_K^2\right)^{1/2}, \eta_{1K} = h_K \|\mathrm{div}\, R(u_i^h) \;+\; f_{i,K}\|_{0,K},$$

$$\eta_{2K} = h_K^{1/2}\left(\sum_{E\in E_K^{int}\cup E_K^N} \|J_{E,n}(u^h)\|_{0,E}^2\right)^{1/2}, \eta_{3K} = h_K^{1/2}\left(\sum_{E\in E_K^C} \|R_t(u^h)\|_{0,E}^2\right.$$

$$\eta_{4K} = h_K^{1/2}\left(\sum_{E\in E_K^C} \left\|\frac{1}{\gamma}[P_\gamma^{h,\hat\rho}(u^h) - g]_+ + R_{\hat\rho}(u^h)\right\|_{0,E}^2\right)^{1/2},$$

$$\eta_{5K} = h_K^{1/2}\left(\sum_{E\in E_K^D} \left\|\frac{1}{\gamma}\overline{P}_{i,\gamma}^{h,\hat\rho}(u_i^h) - u_{i,D} + \overline{R}_{\hat\rho}(u_i^h)\right\|_{0,E}^2\right)^{1/2}.$$

saturations assumptions

The solution u of the weak problem and the solution u^h of discrete problem (26) are such as:

$$\left\|\sigma_n(u) - R_{\hat{\rho}}(u^h)\right\|^2_{0,\Gamma_{1,C}} \lesssim h^{-1} \sum_{i=1,2} \|u_i - u_i^h\|^2_{1,\Omega_i}, \quad (28)$$

$$\sum_{i=1,2} \left\|\sigma(u_i) - \overline{R}_{\hat{\rho}}(u_i^h)\right\|^2_{0,\Gamma_{i,D}} \lesssim h^{-1} \sum_{i=1,2} \|u_i - u_i^h\|^2_{1,\Omega_i}, \quad (29)$$

$$\sum_{i=1,2} \left\|\sigma(u_i)n_i - R(u_i^h)n_i\right\|^2_{0,\Gamma_{i,N}} \lesssim h^{-1} \sum_{i=1,2} \|u_i - u_i^h\|^2_{1,\Omega_i}, \quad (30)$$

$$\sum_{i=1,2} \left\|\sigma(u_i) - R(u_i^h)\right\|^2_{0,\Omega_i} \lesssim h^{-2} \sum_{i=1,2} \|u_i - u_i^h\|^2_{1,\Omega_i}. \quad (31)$$

Upper error bound:

We suppose u the solution of variational problem with $u \in (H^{\frac{3}{2}+\nu}(\Omega_1))^d \times (H^{\frac{3}{2}+\nu}(\Omega_2))^d$ ($\nu > 0$ and $d = 2, 3$) and u^h the solution of the discrete problem. We suppose that if $\theta \neq -1$ then $\gamma_0 > 0$ is sufficiently small. Assume that the saturation assumptions (28)-(29) hold as well. Then we obtain:

$$\sum_{i=1,2} \|u_i - u_i^h\|_{1,\Omega_i} + \left\| \sigma_n(u) + h^{1/2}\frac{1}{\gamma}[P_\gamma^{h,\hat{\rho}}(u^h) - g]_+ \right\|_{0,\Gamma_{1,C}}$$
$$+h^{1/2}\|\sigma_n(u) - R_{\hat{\rho}}(u^h)\|_{0,\Gamma_{1,C}} + h^{1/2}\sum_{i=1,2}\left\| \sigma(u_i) + \frac{1}{\gamma}\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D} \right\|_{0,\Gamma_{i,D}}$$
$$+h^{1/2}\sum_{i=1,2}\|\sigma(u_i) - \overline{R}_{\hat{\rho}}(u_i^h)\|_{0,\Gamma_{i,D}} \lesssim (1+\gamma_0)\eta + \zeta.$$

Lower error bound:

For all elements $K \in T_h$, the following local lower error bounds hold:

$$\eta_{1K} \lesssim \|u_i - u_i^h\|_{1,K} + \zeta_K, \eta_{2K} \lesssim \|u_i - u_i^h\|_{1,\omega_K} + \zeta_K. \quad (32)$$

For all elements K such as $K \cap E_K^C \neq \emptyset$ then $K \cap E_K^D \neq \emptyset$, the following local lower error bounds hold:

$$\eta_{3K} \lesssim \|u_i - u_i^h\|_{1,K} + \zeta_K, \quad (33)$$

$$\eta_{4K} \lesssim \sum_{E \in E_K^C} h_K^{1/2} \left(\left\| \sigma_n(u) + \frac{1}{\gamma}[P_\gamma^h(u^h) - g]_+ \right\|_{0,E} + \left\| \sigma_n(u) - R_{\hat{\rho}}(u^h) \right\|_{0,E} \right. \quad (34)$$

$$\eta_{5K} \lesssim \sum_{E \in E_K^D} h_K^{1/2} \left(\left\| \frac{1}{\gamma} \overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D} + \sigma(u_i) \right\|_{0,E} + \left\| \sigma(u_i) - \overline{R}_{\hat{\rho}}(u_i^h) \right\|_{0,E} \right. \quad (35)$$

Plan

- Motivation
- Introduction
- Existence, Uniqueness and Optimality
- *A posteriori* error estimators
- Conclusion and Perspectives

Conclusion and Perspectives

Conclusion:

- Existence, Uniqueness and Consistence.
- Optimal *a priori* error estimate and theoretical and numerical results.
- *A posteriori* error estimators.

Perspectives:

- Numerical validation of the *a posteriori* error estimators in the Fictitious domain approach.
- Extension of the model (dynamic, non linear elasticity, great deformation).
- To delate the saturations assumptions.

A. Hansbo, P. Hansbo and M.G. Larson, 2003.

Thank you for your attention.