Numerics for Hydromorphodynamics Processes Relaxation Solvers and Stochastic Aspects

E. Audusse

LAGA, UMR 7569, Univ. Paris 13
ANGE group (CETMEF – INRIA – UPMC - CNRS)

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Saint-Venant – Exner Model : A CEMRACS Story

- CEMRACS 2011 : Numerical Simulation
  Sediment transport modelling : Relaxation schemes for Saint-Venant Exner and three layer models.
  Emmanuel Audusse, Christophe Berthon, Christophe Chalons, Olivier Delestre, Nicole Goutal, Magali Jodeau, Jacques Sainte-Marie, Jan Giesselmann and Georges Sadaka.
  ESAIM Proc., Vol. 38, pp 78-98, 2012.
- CEMRACS 2013 : Stochastic Aspects
  Numerical simulation of the dynamics of sedimentary river beds with a stochastic Exner equation.
  Emmanuel Audusse, Sébastien Boyaval, Nicole Goutal, Magali Jodeau and Philippe Ung.
  ESAIM Proc., Vol. 48, pp 312-340, 2015.

Morphodynamic processes

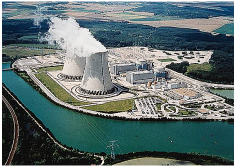
Coastal erosion

River Morphodynamics

Dunes formation

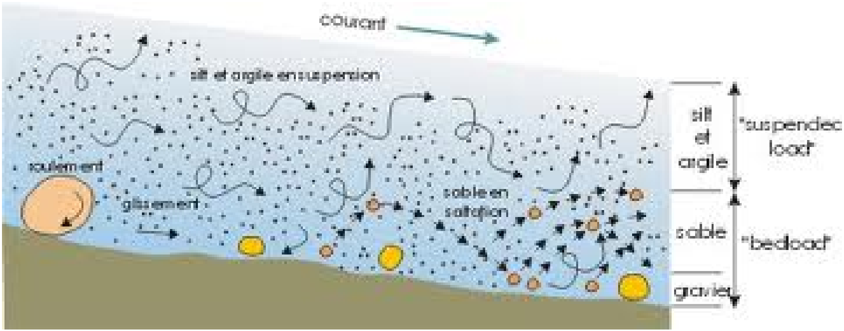
Soil erosion

Dam drain



Industrial sites

Morphodynamics processes



Morphodynamics process
Suspended- and bedload

Modelling for bedload transport

- Saint-Venant – Exner model
  - Implemented at the industrial level
  - Empirical derivation, no energy
  - Reasonable results for a large class of experiences
  - A lot of parameters to tune...
- Two-layer (Saint-Venant ?) models
  - Well-known in the hyperbolic community for bi-fluid modelling
  - Validity of the extension to bedload transport ?
  - Definition of the layers ?
  - Interfaces conditions ?
  - Rheology law in the solid part ?

Saint-Venant – Exner model

- Equations

$$
\begin{aligned}
\partial_t h + \partial_x q_w &= 0, \\
\partial_t q_w + \partial_x \left(\frac{q_w^2}{h} + \frac{g}{2} h^2 \right) &= -gh\partial_x b - \frac{\tau_b}{\rho_w}, \\
\rho_s (1-p) \partial_t b + \partial_x q_s &= 0,
\end{aligned}
$$

(Exner[25], Paola-Voller[05], VanRijn[93,06], Parker[06], Cordier[11], Garegnani[13]...)

- Comments
  - Fluid quantities : Water depth (h) and discharge (q_w)
  - Sediment quantity : Position of the interface (b)
  - 2 conservation equations + 1 dynamic equation

Saint-Venant – Exner model

- Equations

$$
\begin{aligned}
\partial_t h + \partial_x q_w &= 0, \\
\partial_t q_w + \partial_x \left(\frac{q_w^2}{h} + \frac{g}{2} h^2 \right) &= -gh\partial_x b - \frac{\tau_b}{\rho_w}, \\
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$$

(Exner[25], Paola-Voller[05], VanRijn[93,06], Parker[06], Cordier[11], Garegnani[13]...)

- Comments
  - No dynamics in the solid phase
  - No sediment transport in the fluid phase
  - Closure relations for friction term τ_b and sediment flux q_s

Friction coefficient formulae

- Laminar flow

$$\tau_b = \kappa(h)u$$

Gerbeau[01]

- Engineers formulae

$$\tau_b = \kappa(h)u^2$$

  - Chezy formula

$$\kappa(h) = \kappa$$

Formule pour trouver la vitesse uniforme que l'eau aura dans un fossé ou dans un canal dont la pente est connue. Applications pour la Seine et l'Yvette [Chezy, 1776]

  - Manning formula

$$\kappa(h) = \frac{\kappa}{h^{1/3}}$$

Manning[1891]

Sediment flux formulae

- Power laws (Exner[25], Grass[81])

$$q_s = A_g |\bar{u}|^m \bar{u}$$

- Shields parameter

$$\theta = \frac{|\tau_b|/\rho_w}{g(s-1)d_m}, \qquad s = \rho_s/\rho_w, \qquad \tau_b = \frac{u^2}{Kh^\alpha}$$

- Threshold sediment flux formulae

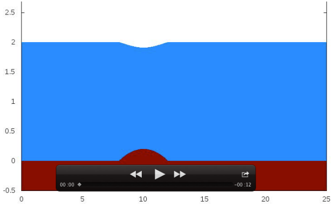
$$q_s = \Phi\sqrt{g(s-1)d_m^3}$$

  - Meyer-Peter & Müller [48] : $\Phi = 8\,(\theta - \theta_c)_+^{3/2}$
  - Engelund & Fredsoe [76]: $\Phi = 18.74(\theta - \theta_c)_+\,(\theta^{1/2} - 0.7\theta_c^{1/2})$
  - Nielsen [92] : $\Phi = 12\,\theta^{1/2}\,(\theta - \theta_c)_+ \ldots$

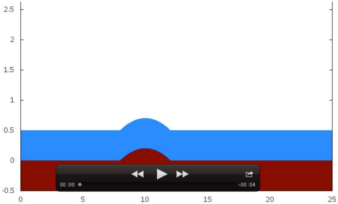
"Steady" flow over a movable bump

Dunes

Antidunes

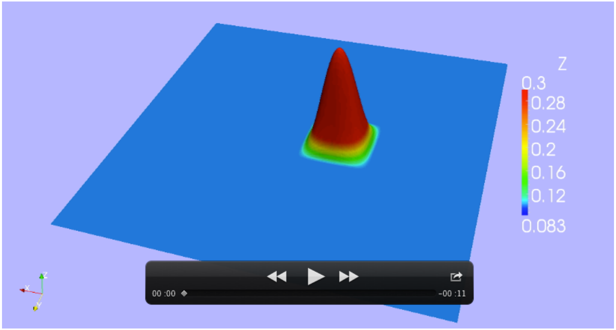


Fluvial flow



Torrential flow

"Steady" flow over a movable bump in 2d



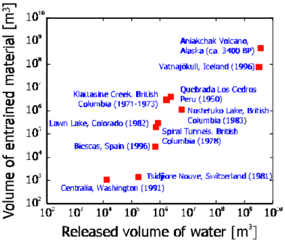
(O. Delestre, CEMRACS 2012)

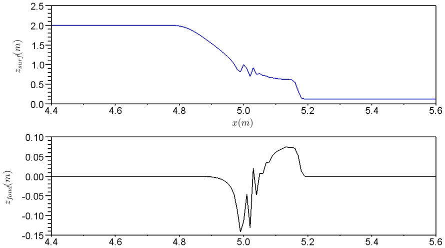
Numerical strategies for bedload transport

- Steady state strategy
  - Hydrodynamic computation on fixed topography
    $\rightsquigarrow$ Steady state
  - Evolution of topography forced by hydrodynamic steady state
  - Efficient for phenomena involving very different time scales
- External coupling (time splitting)
  - Use of two different softwares for hydro- and morphodynamics
  - Allow to use existing solvers and different numerical strategies
  - Actual strategy in industrial softwares
  - Efficient for low coupling (?)
- Internal coupling
  - Solution of the whole system at once
  - Need for a new solver
  - Efficient for general coupling

(Hudson[05], Castro[08], Delis[08], Benkhaldoun[09], Murillo[10]...)

Dam break : Failure for time splitting

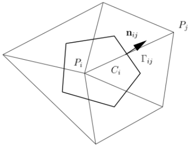




External coupling (M. Jodeau, EDF)

Finite Volume Method

$$\partial_t u + \nabla \cdot f(u) = 0, \qquad u \in \mathbb{R}^p, \quad f : \mathbb{R}^p \to \mathbb{R}^{2\times p}$$

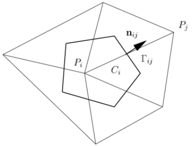


- Integration on the prism $C_i \times [t^n, t^{n+1}]$

$$\int_{C_i} u(t^{n+1}, x)dx = \int_{C_i} u(t^n, x)dx - \sum_{j\in V(i)} \int_{\Gamma_{ij}} \int_{t^n}^{t^{n+1}} f(u(t,s)).n_{ij} dtds$$

Finite Volume Method

$$\partial_t u + \nabla \cdot f(u) = 0, \qquad u \in \mathbb{R}^p, \quad f : \mathbb{R}^p \to \mathbb{R}^{2\times p}$$



- Integration on the prism $C_i \times [t^n, t^{n+1}]$

$$U_i^{n+1} = U_i^n - \sum_{j\in V(i)} \sigma_{ij}^n F(U_i^n, U_j^n, n_{ij}), \qquad \sigma_{ij}^n = \frac{\Delta t^n |\Gamma_{ij}|}{|C_i|}$$

with

$$F(U_i^n, U_j^n, n_{ij}) \approx \frac{1}{\Delta t^n |\Gamma_{ij}|} \int_{\Gamma_{ij}} \int_{t^n}^{t^{n+1}} f(u(t,s)).n_{ij} dtds$$

Godunov scheme

- Riemann problem

$$\partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = \begin{cases} u_l & \text{if} \quad x \le 0 \\ u_r & \text{if} \quad x > 0 \end{cases}$$

$\rightsquigarrow$ Much more easy to solve that the general IBVP

- Godunov scheme
  - Start from piecewise constant initial data
  - Solve the Riemann pb at each interface $x_{i+1/2}$: $u^r_{i+1/2}(t,x)$
  - Fix the time step so that the Riemann problems do not interact
  - Construct a global solution by merging the solutions of all the local Riemann problems

$$u^g(t,x) = u^r_{i+1/2}(t,x) \quad \text{if} \quad x \in [x_i, x_{i+1}]$$

  - Take the meanvalue of this solution at time Δt^n

$$U_i^{n+1} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u^g(\Delta t^n, x) dx$$

Godunov scheme

- Riemann problem

$$\partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = \begin{cases} u_l & \text{if} \quad x \leq 0 \\ u_r & \text{if} \quad x > 0 \end{cases}$$

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$$u^g(t,x) = u^r_{i+1/2}(t,x) \quad \text{if} \quad x \in [x_i, x_{i+1}]$$

  - For conservative equations, equivalent with FV approach

$$F_{i+1/2} = \frac{1}{\Delta t^n} \int_0^{t^n} u^r_{i+1/2}(t, x_{i+1/2}) dt$$

Relaxation solver: General idea

- Godunov scheme
  - Consistency, Stability
  - Complexity : Solution of the Riemann problem (rarefaction waves, shocks, contact discontinuities...)
- Relaxation models : Introduction of a larger system
  - that is hyperbolic (with LD fields)
  - that formally converges to the physical one
  - that ensures some stability properties
  - for which the (homogeneous) Riemann problem is easy to solve
- Numerical algorithm
  - Definition of auxiliary variables from physical ones
  - Solution of the (homogeneous) Riemann problem
  - Computation of the physical variables at the next time step

(Suliciu [92], Chen et al. [94], Jin-Xin [95],
Nonlinear Stability of FVM for Hyperbolic Conservation Laws [Bouchut, 04]...)

Relaxation solver : Scalar conservation law

- SCL to solve

$$\partial_t u + \partial_x f(u) = 0$$

- Relaxation model

$$\begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + c^2 \partial_x u &= \frac{1}{\epsilon}(f(u) - v) \end{aligned}$$

- Homogeneous part : Linear wave equation

$$\partial_{tt} u - c^2 \partial_{xx} u = 0$$

- Rusanov (or HLL) scheme

Relaxation solver : Stability criterion (Wave celerities)

- Wave celerities

$$f'(u) \quad \text{vs.} \quad c, -c$$

- Information goes faster in the relaxation model

$$c \geq |f'(u)|$$

- Same argument as for explicit numerical schemes (cone of dependence)

Relaxation solver : Stability criterion (Diffusive correction)

- "Chapman-Enskog" expansion

$$v = f(u) + \epsilon v_1 + O(\epsilon^2)$$

- Auxiliary equation

$$v_1 = \partial_t f(u) + c^2 \partial_x u + O(\epsilon)$$

- Estimation of the time derivative

$$\begin{aligned} \partial_t f(u) = f'(u)\partial_t u = -f'(u)\partial_x v &= -f'(u)\partial_x f(u) + O(\epsilon) \\ &= -f'(u)^2 \partial_x u + O(\epsilon) \end{aligned}$$

- Modified equation with viscous correction

$$\partial_t u + \partial_x f(u) = \epsilon\, \partial_x \left((-f'(u)^2 + c^2) \partial_x u \right) + O(\epsilon^2)$$

Relaxation solver : Numerical strategy

- Entries

$$(u_i^n)_i$$

- Initialization of auxiliary unknowns

$\rightsquigarrow$ Solution of EDO system with "$\epsilon = 0$"

$$v_i^n = f(u_i^n)$$

- Advance in time

$\rightsquigarrow$ Solution of the homogeneous Riemann problems at each interface

$$w^R_{[W_l, W_r]}(t,x), \quad W_=(u,v)^T$$

$\rightsquigarrow$ Computation of physical unknowns : Averaging of the solution on each cell

$$(u_i^{n+1}) = \frac{1}{\Delta x}\int_{C_i} u^R(\Delta t, x)dx$$

Homogeneous SW system

- Equations

$$\begin{aligned} \partial_t h + \partial_x(h\bar{u}) &= 0 \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + \frac{gh^2}{2}\right) &= 0 \end{aligned}$$

- Energy

$$\partial_t(hE) + \partial_x(\bar{u}(HE + p)) \leq 0, \qquad hE = \frac{h\bar{u}^2}{2} + \frac{gh^2}{2}$$

- Eigenvalues

$$\lambda_\pm = \bar{u} \pm \sqrt{gh}$$

- Riemann problem

  Multiple waves and complex structure

Homogeneous SW system

- Relaxation model (v1) : Direct extension

$$
\begin{aligned}
\partial_t h + \partial_x \tilde{Q} &= 0 \\
\partial_t \tilde{Q} + c_1^2 \, \partial_x h &= \frac{1}{\epsilon}\left(h\bar{u} - \tilde{Q}\right) \\
\partial_t (h\bar{u}) + \partial_x \tilde{H} &= 0 \\
\partial_t \tilde{H} + c_2^2 \, \partial_x (h\bar{u}) &= \frac{1}{\epsilon}\left(h\bar{u}^2 + \frac{gh^2}{2} - \tilde{H}\right)
\end{aligned}
$$

- Eigenvalues (distincts and associated to LD fields)

$$(\lambda_1)_\pm = \pm c_1, \quad (\lambda_2)_\pm = \pm c_2$$

- Stability criterion

$$max(c_1, c_2) > |\bar{u}| + \sqrt{gh}$$

Homogeneous SW system

- Relaxation model (v2) : Suliciu approach

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) &= 0 \\
\partial_t h\pi + \partial_x (h\pi\bar{u}) + c^2 \partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right)
\end{aligned}
$$

- Eigenvalues (distincts and associated to LD fields)

$$
\lambda_\pm = \bar{u} \pm \frac{c}{h}, \quad \lambda_u = \bar{u}
$$

- Stability criterion

$$
c \geq h\sqrt{gh}
$$

Homogeneous SW system

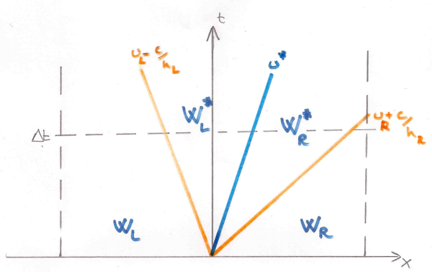
- Initialization

  $\rightsquigarrow$ Computation of auxiliary variables : System with "$\epsilon = 0$"

$$\pi_i^n = g(h_i^n)^2/2$$

- Solution of the (homogeneous) Riemann problem

$$w^R_{[W_l, W_r]}(t,x), \quad W_=(h, \bar{u}, \pi)^T$$



Homogeneous SW system

- Computation of the intermediate states

$$u^* = u_M - \frac{1}{2c}\Delta\pi, \qquad \pi^* = \pi_M - \frac{c}{2}\Delta u$$

$$\tau_l^* = \tau_l + \frac{1}{2c}\Delta u - \frac{1}{2c^2}\Delta\pi, \quad \tau_r^* = \tau_r + \frac{1}{2c}\Delta u + \frac{1}{2c^2}\Delta\pi, \quad "\tau = \frac{1}{h}"$$

- Computation of the physical solution

  $\rightsquigarrow$ Projection of the solution onto piecewise constant space

$$h_i^{n+1}, (h\bar{u})_i^{n+1}$$

- Stability criterion

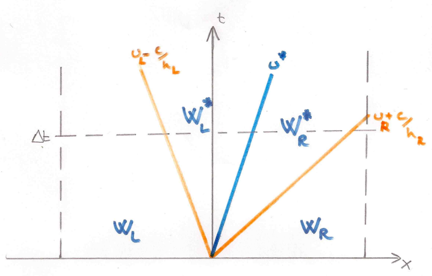
$$c = max\left(h_l\sqrt{gh_l}, h_r\sqrt{gh_r}\right)$$

- Extension to vacuum

(Bouchut [04])

Homogeneous SW system

- Conservative system
  - Averaging process
  - Flux computation
- Properties of the Suliciu solver
  Consistency, Positivity, Entropy inequality
- Riemann problem
  Eigenvalues always ordered $\rightsquigarrow$ One single Riemann pb to solve



SW system with source

- Equations

$$
\begin{aligned}
\partial_t h + \partial_x (h\bar{u}) &= 0 \\
\partial_t (h\bar{u}) + \partial_x \left(h\bar{u}^2 + \frac{gh^2}{2} \right) &= -gh\partial_x B \\
\partial_t B &= 0
\end{aligned}
$$

- Stationary States

$$\bar{u} = 0, \qquad h + B = 0$$

- Eigenvalues

$$\lambda_\pm = \bar{u} \pm \sqrt{gh}, \quad \lambda_0 = 0$$

- Riemann problem
  Multiple waves and (very !) complex structure

SW system with source

- Relaxation Model v1

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + gh\partial_x B &= 0 \\
\partial_t h\pi + \partial_x (h\pi\bar{u}) + c^2 \partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t B &= 0
\end{aligned}
$$

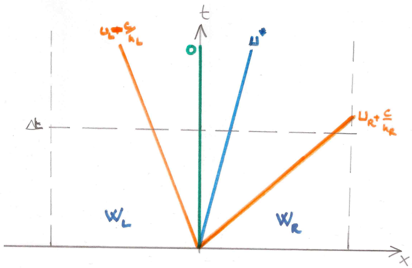
- Eigenvalues (associated to LD fields)

$$
\lambda_{\pm} = u \pm \frac{c}{h}, \quad \lambda_u = u, \quad \lambda_0 = 0
$$

(Bouchut [04])

SW system with source

- Non conservative system
  Integration process
- Properties
  Consistency, WB, Positivity, Entropy inequality
- Riemann problem
  Eigenvalues not ordered $\rightsquigarrow$ Multiple Riemann pbs to solve
  Possibility of resonance $\rightsquigarrow$ Difficulty to define the solution



SW system with source

- Relaxation model v2

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + gh\partial_x \tilde{B} &= 0 \\
\partial_t h\pi + \partial_x(h\pi\bar{u}) + c^2\partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t \tilde{B} + \bar{u}\partial_x \tilde{z} &= \frac{1}{\epsilon}(B - \tilde{B}) \\
\partial_t B &= 0
\end{aligned}
$$

- Eigenvalues (associated to LD fields)

$$
\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u \,(\text{double}), \qquad (\lambda_0 = 0)
$$

Riemann problem is under-determined...

SW system with source

- Relaxation model v3

$$\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t h\bar{u} + \partial_x \left(h\bar{u}^2 + \pi\right) + g\bar{h}\partial_x \tilde{B} &= 0 \\
\partial_t h\pi + \partial_x (h\pi\bar{u}) + c^2\partial_x \bar{u} &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t \tilde{B} + \bar{u}\partial_x \tilde{z} &= \frac{1}{\epsilon}(B - \tilde{B}) \\
\partial_t B &= 0
\end{aligned}$$

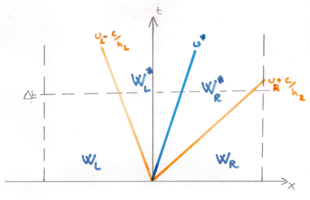
- Eigenvalues (associated to LD fields)

$$\lambda_\pm = u \pm \frac{c}{h}, \quad \lambda_u = u \; (\text{double}), \qquad (\lambda_0 = 0)$$

Riemann problem is well-posed !

SW system with source

- Non conservative system
  Integration process
- Properties
  Consistency, WB ($\bar{h} = h_M$), Positivity
- Riemann problem
  Eigenvalues always ordered $\rightsquigarrow$ One single Riemann pb to solve
  Same structure as homogeneous case, $\neq$ intermediate states



(Introduced as a wb simple Riemann solver : Galice [02])

SW-Exner system

- Equations

$$
\begin{aligned}
\partial_t h + \partial_x q_w &= 0, \\
\partial_t q_w + \partial_x \left(\frac{q_w^2}{h} + \frac{g}{2} h^2 \right) &= -gh\partial_x b - \frac{\tau_b}{\rho_w}, \\
\partial_t b + \partial_x q_s &= 0,
\end{aligned}
$$

- Properties
  - Hyperbolicity depends on the choice of q_s
  - Eigenvalues hard to compute except for special choices of q_s
  - No energy associated to the present version

SW-Exner system

- Relaxation model v1 (CEMRACS 2011)

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t (h\bar{u}) + \partial_x \left(h\bar{u}^2 + \pi\right) + g\bar{h}\partial_x b &= 0 \\
\partial_t \pi + \bar{u}\partial_x \pi + \frac{\alpha^2}{h}\partial_x u &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t b + \partial_x \omega &= 0 \\
\partial_t \omega + \left(\frac{\beta^2}{h^2} - \bar{u}^2\right)\partial_x b + 2\bar{u}\partial_x \omega &= \frac{1}{\epsilon}(q_s - \omega)
\end{aligned}
$$

- Definitions
  - (π, ω) : Auxiliary variables (fluid pressure, sediment flux)
  - $\epsilon > 0$: (Small) relaxation parameter
  - $(\alpha, \beta) > 0$: Have to be fixed to ensure stability

SW-Exner system

- Mathematical properties
  - Formally tends to SW-Exner model when ϵ tends to 0
  - Stability requirement (diffusive correction)

$$\alpha^2 \geq h^2 p'(h) = gh^3, \qquad \beta^2 \geq (hu)^2 + gh^2 \partial_u Q_s$$

  - Relaxation parameters ratio

$$\frac{\beta^2}{\alpha^2} = Fr^2 + \frac{1}{h} \partial_u Q_s$$

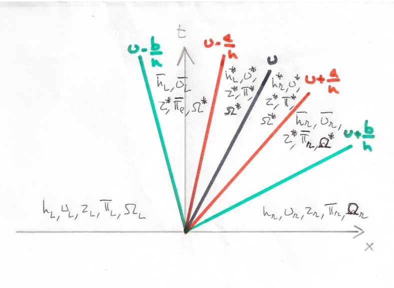
    - Sub- or supercritical flow
    - Low or strong erosion coefficient
- Physical properties
  - No explicit dependency on sediment flux q_S
  - Same as previous model when $u = 0 \rightsquigarrow$ Stationary states
  - Not the same as previous model when $Q = 0$...

SW-Exner system

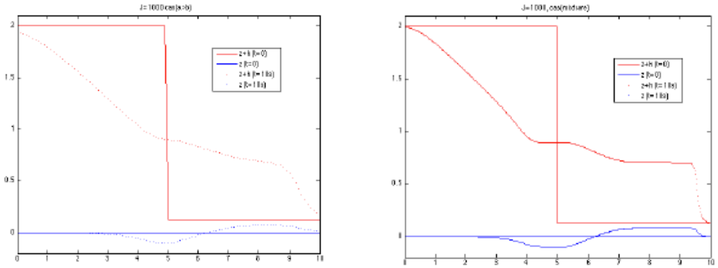
- Numerical properties
  - Always hyperbolic ($h \neq 0$) and LD fields
  - Ordered eigenvalues (two possibilities)

$$u - \frac{\beta}{h} < u - \frac{\alpha}{h} < u < u + \frac{\alpha}{h} < u + \frac{\beta}{h}$$

$\rightsquigarrow$ Two Riemann problems to solve



Dam break on moveable flat bottom



- Stable computation (no oscillations)
- Strong numerical diffusion

SW-Exner system

- Relaxation model v2

$$
\begin{aligned}
\partial_t h + \partial_x h\bar{u} &= 0 \\
\partial_t(h\bar{u}) + \partial_x \left(h\bar{u}^2 + \pi\right) + g\bar{h}\partial_x \tilde{b} &= 0 \\
\partial_t \pi + \bar{u}\partial_x \pi + \frac{\alpha^2}{h}\partial_x u &= \frac{1}{\epsilon}\left(\frac{gh^2}{2} - \pi\right) \\
\partial_t \tilde{b} + \partial_x \omega &= \frac{1}{\epsilon}\left(b - \tilde{b}\right) \\
\partial_t \omega + \left(\frac{\beta^2}{h^2} - \bar{u}^2\right)\partial_x \tilde{b} + 2\bar{u}\partial_x \omega &= \frac{1}{\epsilon}(q_s - \omega) \\
\partial_t b + \partial_x q_s &= 0,
\end{aligned}
$$

- Comments
  - Three auxiliary variables $(\pi, \omega, \tilde{b})$
  - Same Riemann problem as before (b is "not" coupled)

Time Splitting : The come back !

- Usual Time Splitting : Unstable

  Fluid software + Exner equation

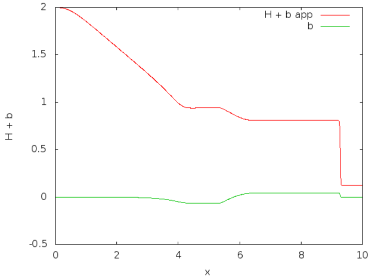
- Relaxation solver : Stable

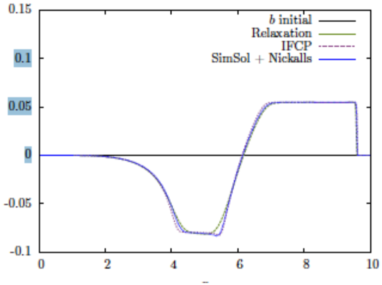
  Modified Fluid software + Exner equation

  $\rightsquigarrow$ The fluid solver has to take into account some information from the sediment

- Modified Time splitting : Stable
  - Consider a fluid solver using a bound of the largest eigenvalue of the SW Jacobian (Rusanov, HLL, Relaxation...)
  - Replace it by a bound of the largest eigenvalue of the SW-Exner Jacobian

Dam break on moveable flat bottom





- Stable computation (no oscillations)
- Accurate results

Uncertainty quantification for SW-Exner system

- Friction coefficient formula

$$\tau_b = \kappa(h)u^2$$

- Sediment flux formula

$$q_s = A\,(\theta - \theta_c)_+^{3/2}\sqrt{g(s-1)d_m^3}$$

- Uncertain parameters

$$\kappa, A, \tau_c$$

- Uncertainties on q_s directly impact bottom topography b

$\rightsquigarrow$ Study of SW system with uncertain topography

Uncertainty quantification for SW system

- Equations

$$\begin{aligned}\partial_t h + \partial_x q &= 0,\\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{g}{2}h^2\right) &= -gh\partial_x b - \frac{\tau_b}{\rho},\end{aligned}$$

- Flow on a constant slope S

$$h = H_0, \quad q = Q_0, \quad Q_0 = K_s S^{1/2} H_0^{5/3}$$

- Study of perturbations of this stationary state

Uncertainty quantification for SW system

- Perturbated bottom topography

$$
\begin{aligned}
B^n_{i+1/2} &= B^0_{i+1/2} + \tilde{B}_{i+1/2}, \\
\tilde{B}_{i+1/2} &= \alpha\sqrt{\Delta x}\sum_{k=1}^{N/2}\frac{1}{k^\beta}\left(a_k \cos\left(2k\pi\frac{i+1/2}{N}\right)\right. \\
&\qquad \left. + \, b_k \sin\left(2k\pi\frac{i+1/2}{N}\right)\right),
\end{aligned}
$$

- Characteristics of the noise
  - Amplitude α
  - Regularity β
  - Random coefficients a_k and b_k (Normal law)

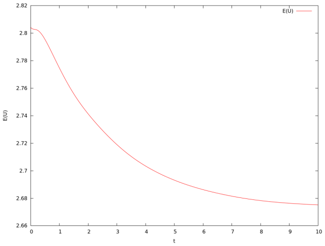
Uncertainty quantification for SW system

- Monte Carlo simulations
- Water depth

  $\rightsquigarrow$ Mass conservation

$$\frac{1}{L}\int_x \mathbb{E}(h) = H_0$$

- Discharge



Uncertainty quantification for SW system

- Objective
  Preservation of the unperturbed stationary state

$$\frac{1}{L}\int_x \mathbb{E}(q) \approx Q_0$$

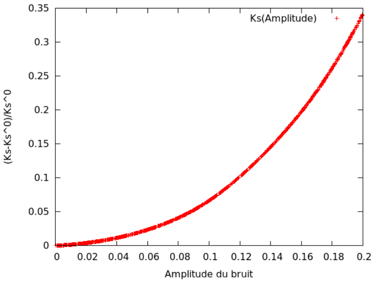
- Modification of the friction coefficient

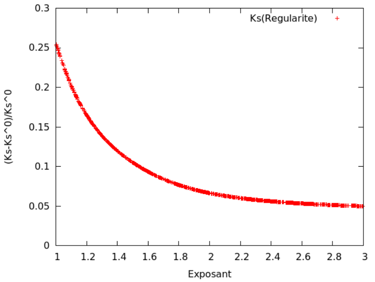
$$K_s \to \tilde{K}_s$$

- Physical motivation
  Friction coefficient takes into account roughness of the bottom

Uncertainty quantification for SW system

▸ Friction coefficient as a function of the noise





$\rightsquigarrow$ May help to characterize suitable bounds for noise

▸ More results in P. Ung PhD thesis (december 2015)