Derivation of multi-fluid models

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Part 3 - last part of the lecture

Works published in :

M. Hillairet, *JMFM* '07
D. B. & X. Huang, *ARMA* '14
D. B. & M. Hillairet, *Proc AMS* '15
D.B. & M. Hillairet: work in progress '15

Thanks to the organizers for the invitation.

Compressible multiphase flows

"Definition" [A. Murrone, PhD '04]
Mixture of several compressible phases at equilibrium in which the topology, composition and transfers are parameters that may constantly vary in time and space.

Comestible examples : soda, champagne, emulsions

Applications :

- Simulating aerated flows (Wave breaking)
- Nuclear industry

Modeling of compressible multiphase flows

Compressible multiphase flows

Modeling [M. Ishii '75, D. Drew & S. Passman '98]

Phase variables :

density ρ_k, velocity u_k, pressure p_k, strain tensor τ_k

Constitutive equations for phase $k\,(=+,-)$

$$\left\{\begin{array}{rcll} \partial_t \rho_k + \mathrm{div}(\rho_k u_k) & = & 0 & \\ \partial_t(\rho_k u_k) + \mathrm{div}(\rho_k u_k \otimes u_k) & = & \mathrm{div}\tau_k - \nabla p_k & \text{on } \mathcal{F}_k(t) \\ p_k & = & \mathcal{P}_k(\rho_k)\,. & \end{array}\right.$$

Extended constitutive equations : $X_k = \mathbf{1}_{\mathcal{F}_k}$, velocity σ

$$\left\{\begin{array}{rcl} \partial_t(\rho_k X_k) + \mathrm{div}(\rho_k u_k X_k) & = & \rho_k(u_k - \sigma)\cdot\nabla X_k \\ \partial_t(\rho_k u_k X_k) + \mathrm{div}(X_k \rho_k u_k \otimes u_k) & = & \mathrm{div}(X_k \tau_k) - \nabla X_k p_k \\ & & +\rho_k(u_k - \sigma)\otimes u_k \nabla X_k + (p_k \mathbb{I} - \tau_k)\nabla X_k \\ p_k & = & \mathcal{P}_k(\rho_k)\,. \end{array}\right.$$

Compressible multiphase flows

Modeling [M. Ishii '75, D. Drew & S. Passman '98]

Mean operator : $\langle \cdot \rangle$

$$\alpha_k = \langle X_k \rangle \quad \bar{\rho}_k = \frac{\langle X_k \rho_k \rangle}{\langle X_k \rangle} \quad \bar{p}_k = \frac{\langle X_k p_k \rangle}{\langle X_k \rangle} \quad \bar{\tau}_k = \frac{\langle X_k \tau_k \rangle}{\langle X_k \rangle} \quad \tilde{u}_k = \frac{\langle X_k \rho_k u_k \rangle}{\langle X_k \rho_k \rangle} .$$

Homogenized system for phase $k \, (= +, -)$

$$\left\{ \begin{array}{rcl} \partial_t(\alpha_k \bar{\rho}_k) + \mathrm{div}(\alpha_k \bar{\rho}_k \tilde{u}_k) & = & \Gamma_k \\ \partial_t(\alpha_k \bar{\rho}_k \tilde{u}_k) + \mathrm{div}(\alpha_k \bar{\rho}_k \tilde{u}_k \otimes \tilde{u}_k) & = & \mathrm{div}(\alpha_k(\bar{\tau}_k + \tau_k^T)) - \nabla(\alpha_k \bar{p}_k) \\ & & + M_k^\Gamma + p_k^{int} \nabla \alpha_k + F_k \\ \bar{p}_k & = & \bar{\mathcal{P}}_k(\bar{\rho}_k) + p_k^T . \end{array} \right.$$

where :

$$\Gamma_k = \langle \rho_k (u_k - \sigma) \cdot \nabla X_k \rangle , \quad F_k = \langle \tau_k \nabla X_k \rangle , \quad \ldots$$

Modeling assumptions :

$$\Gamma_k = 0, \quad M_k^\Gamma = 0, \quad F_k = \frac{1}{\mu}(u_{k'} - u_k) \quad \tau_k^T = 0 \quad p_k^T = 0\,.$$

Closure law : $p_+ = p_-$

System :

$$\left\{ \begin{array}{rcl} \partial_t(\alpha_k \rho_k) + \mathrm{div}(\alpha_k \rho_k u_k) & = & 0 \\ \partial_t(\alpha_k \rho_k u_k) + \mathrm{div}(\alpha_k \rho_k u_k \otimes u_k) + \alpha_k \nabla p & = & \dfrac{1}{\mu}(u_{k'} - u_k) + (p_k^{int} - p)\nabla \alpha_k \\ p & = & \mathcal{P}_k(\rho_+) = \mathcal{P}_k(\rho_-)\,. \end{array} \right.$$

- $0 \leq \alpha_\pm$,
- $\alpha_+ + \alpha_- = 1$

Modeling assumptions :

$$\Gamma_k = 0, \quad M_k^\Gamma = 0, \quad F_k = \frac{1}{\mu}(u_{k'} - u_k) \quad \tau_k^T = 0 \quad p_k^T = 0\,.$$

Closure law : $p_+ = p_-$

System :

$$\left\{ \begin{array}{rcl} \partial_t(\alpha_+\rho_+) + \mathrm{div}(\alpha_+\rho_+ u) & = & 0 \\ \partial_t \rho + \mathrm{div}(\rho u) & = & 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p & = & 0 \\ p & = & \bar{\mathcal{P}}_+(\rho_+) = \bar{\mathcal{P}}_-(\rho_-)\,. \end{array} \right.$$

with

- $0 \leq \alpha_\pm$,
- $\alpha_+ + \alpha_- = 1$
- $\rho = \alpha_+\rho_+ + \alpha_-\rho_-$

Modeling assumptions :

$$\Gamma_k = 0, \quad M_k^{\Gamma} = 0, \quad F_k = \frac{1}{\mu}(u_{k'} - u_k) \quad \tau_k^T = 0 \quad p_k^T = 0\,.$$

Closure law : $\partial_t \alpha_+ + u^{int} \cdot \nabla \alpha_+ = \frac{1}{\lambda}(p_+ - p_-)$

System :

$$\left\{ \begin{array}{rcl} \partial_t(\alpha_k \rho_k) + \mathrm{div}(\alpha_k \rho_k u_k) & = & 0 \\ \partial_t(\alpha_k \rho_k u_k) + \mathrm{div}(\alpha_k \rho_k u_k \otimes u_k) + \nabla(\alpha_k p_k) & = & \frac{1}{\mu}(u_{k'} - u_k) + p_k^{int} \nabla \alpha_k \\ p_k & = & \mathcal{P}_k(\rho_k)\,. \end{array} \right.$$

- $0 \leq \alpha_{\pm}$,
- $\alpha_+ + \alpha_- = 1$

Modeling assumptions :

$$\Gamma_k = 0, \quad M_k^{\Gamma} = 0, \quad F_k = \frac{1}{\mu}(u_{k'} - u_k) \quad \tau_k^T = 0 \quad p_k^T = 0\,.$$

Closure law : $\partial_t \alpha_+ + u \cdot \nabla \alpha_+ = \frac{1}{\lambda}(p_+ - p_-)$

System :

$$\left\{ \begin{array}{rcl} \partial_t(\alpha_+\rho_+) + \mathrm{div}(\alpha_+\rho_+ u) & = & 0 \\ \partial_t \rho + \mathrm{div}(\rho u) & = & 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p & = & 0 \\ p & = & \alpha_+ \mathcal{P}_+(\rho_+) + \alpha_- \mathcal{P}_-(\rho_-)\,. \end{array} \right.$$

with

- $0 \leq \alpha_\pm$,
- $\alpha_+ + \alpha_- = 1$
- $\rho = \alpha_+\rho_+ + \alpha_-\rho_-$

Toward a rigorous derivation

Composite problem

Composite unknowns :

$$\rho = \rho_+ X_+ + \rho_-(1 - X_+) \quad u = u_+ X_+ + u_-(1 - X_+) \quad p = p_+ X_+ + p_-(1 - X_+).$$

Composite systems :

(NS) $$\left\{ \begin{array}{rcl} \partial_t \rho + \mathrm{div}(\rho u) & = & 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p & = & \mathrm{div}\tau \end{array} \right. \quad \text{on } (0, T) \times \Omega$$

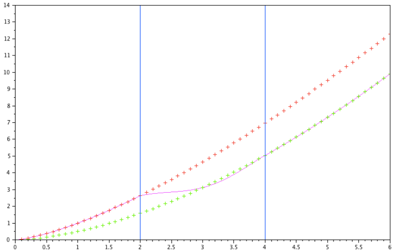
Constitutive equations :

(Cl1) $\tau = 2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}$

(Cl2) $p = \mathcal{P}(\rho)$

Boundary conditions :

$$u = 0 \quad \text{on } \partial\Omega.$$



Composite pressure diagram

New statement of our problem

Cauchy problem

System (CNS) = (NS) +(CI1)+(CI2) + (BC) is completed with initial condition :

$$\text{(IC)} \qquad \begin{cases} \rho(0,x) &= \rho^0(x) \\ u(0,x) &= u^0(x) \end{cases} \quad \text{on } \Omega$$

where u^0 is given and

$$\rho^0(x) = X_+\left(x, \frac{x}{\varepsilon}\right)\rho^0_+(x) + \left(1 - X_+\left(x, \frac{x}{\varepsilon}\right)\right)\rho^0_-(x) \quad \varepsilon << 1.$$

Open question :

Given initial data $(\rho^0_\varepsilon, u^0_\varepsilon)_{\varepsilon\to 0}$, of the above form, and $(\rho_\varepsilon, u_\varepsilon)_{\varepsilon \to 0}$ the associated solutions to (CNS)+(IC), can we :

- recover (ρ_+, ρ_-), and (u_+, u_-) ?
- compute equations satisfied by these unknowns ?

On compactness of solutions to (CNS)

References : D. Serre '91, P.-L. Lions, '98,
E. Feireisl & H. Petzeltova, '00, E. Feireisl '01'02

Question : Let $(\rho_\varepsilon, u_\varepsilon)_{\varepsilon\to 0}$ be a sequence of solution to (CNS) on $(0, T)$ such that for arbitrary ε :

$$(\mathrm{Diss}_\varepsilon) \qquad \sup_{t\in(0,T)} \left\{ \int_\Omega \left[\frac{1}{2}\rho_\varepsilon |u_\varepsilon|^2 + \mathcal{Q}(\rho_\varepsilon) \right] \right\} + \int_0^T \int_\Omega \mu |\nabla u_\varepsilon|^2 + \lambda |\mathrm{div} u_\varepsilon|^2 \leq M$$

Can we extract a limit (ρ, u) solution to (CNS) ?

Remarks

- $(\mathrm{Diss}_\varepsilon)$ means that we have a sequence of bounded-energy solutions.
- $\mathcal{Q}$ is defined by $(\mathcal{Q}(z)/z)' = \mathcal{P}(z)/z^2$. In particular,

$$\mathcal{P}(z) = az^\gamma \Longrightarrow \mathcal{Q}(z) = \frac{a}{\gamma - 1} z^\gamma.$$

Supplementary assumptions :

- Ω is smooth and bounded
- $\mu > 0$ and $\lambda + 2\mu/3 \geq 0$

Uniform integrability I : from the energy estimate

- u_ε uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$
- ρ_ε uniformly bounded in $L^\infty(0,T;L^\gamma(\Omega))$

Multiply momentum equation with $\mathcal{B}_\theta$ solution to

$$\left\{ \begin{array}{rcll} \mathrm{div}\, \mathcal{B}_\theta &=& \rho^\theta - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \rho^\theta & \text{in } \Omega \\ \mathcal{B}_\theta \cdot n &=& 0\,, & \text{on } \partial\Omega\,. \end{array} \right.$$

Uniform integrability II :

- ρ_ε uniformly bounded in $L^{\alpha_\gamma}((0,T)\times\Omega)$ with $\alpha_\gamma = \gamma + \dfrac{2}{3}\gamma - 1.$

Second issue

Can we write an equation for the limit pressure ?

Existence/properties of a weak limit

- $u_\varepsilon \rightharpoonup u$ in $L^2((0,T);H^1_0(\Omega)) - w$ and $L^\infty((0,T);L^2(\Omega)) - w*$
- $\rho_\varepsilon \rightharpoonup \rho$ in $L^{\alpha\gamma}((0,T)\times\Omega) - w$ and $L^\infty((0,T);L^\gamma) - w*$
- $p_\varepsilon \rightharpoonup p$ in $L^{\alpha\gamma/\gamma}((0,T)\times\Omega) - w$

solution to

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}) \end{aligned}$$

Difficulty : Recover $p = \mathcal{P}(\rho)$

Alternative method : Obtaining an equation for p

$$\partial_t \rho_\varepsilon + \mathrm{div} \rho_\varepsilon u_\varepsilon = 0$$

Second issue

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- $p_\varepsilon \rightharpoonup p$ in $L^{\alpha\gamma/\gamma}((0,T)\times\Omega) - w$

solution to

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}) \end{aligned}$$

Difficulty : Recover $p = \mathcal{P}(\rho)$

Alternative method : Obtaining an equation for p

$$\partial_t \beta(\rho_\varepsilon) + \mathrm{div}(\beta(\rho_\varepsilon)u_\varepsilon) + (\beta'(\rho_\varepsilon)\rho_\varepsilon - \beta(\rho_\varepsilon))\mathrm{div} u_\varepsilon = 0$$

for all $\beta : [0,\infty) \to \mathbb{R}$

Second issue

Can we write an equation for the limit pressure ?

Existence/properties of a weak limit

- $u_\varepsilon \rightharpoonup u$ in $L^2((0,T);H_0^1(\Omega)) - w$ and $L^\infty((0,T);L^2(\Omega)) - w*$
- $\rho_\varepsilon \rightharpoonup \rho$ in $L^{\alpha\gamma}((0,T)\times\Omega) - w$ and $L^\infty((0,T);L^\gamma) - w*$
- $p_\varepsilon \rightharpoonup p$ in $L^{\alpha\gamma/\gamma}((0,T)\times\Omega) - w$

solution to

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u \mathbb{I}) \end{aligned}$$

Difficulty : Recover $p = \mathcal{P}(\rho)$

Alternative method : Obtaining an equation for p

$$\partial_t \overline{\beta} + \mathrm{div}(\overline{\beta} u_\varepsilon) + \overline{(\beta'(\rho_\varepsilon)\rho_\varepsilon - \beta(\rho_\varepsilon))\mathrm{div} u_\varepsilon} = 0$$

for all $\beta : [0,\infty) \to \mathbb{R}$ when $\varepsilon \to 0$.

Convention : $\quad \bar{\cdot} = \lim_{\varepsilon\to 0} \cdot_\varepsilon$

Further compactness properties

- The divergence of the momentum equation reads :

$$\Delta((\lambda + 2\mu)\mathrm{div} u_\varepsilon - \mathcal{P}(\rho_\varepsilon)) = \mathrm{div}(\rho_\varepsilon(\partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon))$$

- Lemma [E. Feireisl, P.L. Lions] Given $\beta : [0, \infty) \mapsto \mathbb{R}^+$ then, for arbitrary $\varphi \in C_c^\infty((0, T) \times \Omega)$ there holds :

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega ((\lambda + 2\mu)\mathrm{div} u_\varepsilon - \mathcal{P}(\rho_\varepsilon))\beta(\rho_\varepsilon)\varphi = \int_0^T \int_\Omega ((\lambda + 2\mu)\mathrm{div} u - \mathcal{P}(\rho))\beta(\rho)\varphi$$

Conclusion : There holds :

$$\partial_t \overline{\beta} + \mathrm{div}(\overline{\beta} u) + (\overline{\beta'\rho - \beta})\mathrm{div} u = \frac{\overline{(\beta'\rho - \beta)p} - \overline{(\beta'\rho - \beta)}\,\overline{p}}{\lambda + 2\mu}$$

for all $\beta : [0, \infty) \to \mathbb{R}$

Conclusion

Construction of composite unknowns :

We define $\nu^\varepsilon(t,x,\xi) \in \mathcal{M}_+((0,T)\times\Omega\times[0,\infty))$ s.t. :

$$\langle \nu^\varepsilon, \beta(\xi)\otimes\varphi(t,x)\rangle = \int_0^T\int_\Omega \beta(\rho_\varepsilon(t,x))\varphi(t,x)\mathrm{d}t\mathrm{d}x\,, a.e..$$

Then $\nu_\varepsilon \to \nu$ s.t.

$$\int_0^\infty \beta(z)\mathrm{d}\nu(z) = \overline{\beta}.$$

Full system :

$$\begin{aligned}
\partial_t\rho + \mathrm{div}(\rho u) &= 0\\
\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p &= \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u \mathbb{I})\\
\partial_t\nu + \mathrm{div}_x(\nu u) &= \frac{1}{\lambda+2\mu}\left(\partial_\xi[\xi\nu]G - \partial_\xi[\xi\mathcal{P}(\xi)\nu]\right).
\end{aligned}$$

with :

$$G = (\lambda+2\mu)\mathrm{div}_x u - p \quad p = \int_0^\infty \mathcal{P}(z)\mathrm{d}\nu(z) \quad \rho = \int_0^\infty z\mathrm{d}\nu(z).$$

Back to the homogenization problem

Composite system (HCNS)

Remark : With initial data $\rho^0 = X_+(x, x/\varepsilon)\rho_+(x) + (1 - X_+(x, x/\varepsilon))\rho_-(x)$ we have :

$$\nu_{0,x} = \alpha_+(x)\delta_{\rho_+(x)} + (1 - \alpha_+(x))\delta_{\rho_-(x)}\,, \quad \alpha_+(x) = \frac{1}{|cell|}\int_{cell} X_+(x,y)\mathrm{d}y$$

Assumption : $\nu = \alpha_+\delta_{\rho_+} + \alpha_-\delta_{\rho_-}$ with $\rho_+(t,x) \neq \rho_-(t,x)$.

$$\begin{aligned}
\partial_t\alpha_+ + u\cdot\nabla\alpha_+ &= \frac{\alpha_+(\mathcal{P}(\rho_+) - p)}{\lambda + 2\mu} \\
\alpha_+(\partial_t\rho_+ + \mathrm{div}(\rho_+ u)) &= \alpha_+\frac{\rho_+(p - \mathcal{P}(\rho_+))}{\lambda + 2\mu} \\
\partial_t\rho + \mathrm{div}\rho u &= 0 \\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u \mathbb{I})
\end{aligned}$$

where :

- $0 \leq \alpha_\pm$ and $\alpha_+ + \alpha_- = 1$
- $\rho = \alpha_+\rho_+ + \alpha_-\rho_-$, and $p = \alpha_+\mathcal{P}(\rho_+) + \alpha_-\mathcal{P}(\rho_-)$.

Remark : With initial data $\rho^0 = X_+(x, x/\varepsilon)\rho_+(x) + (1 - X_+(x, x/\varepsilon))\rho_-(x)$ we have :

$$\nu_{0,x} = \alpha_+(x)\delta_{\rho_+(x)} + (1 - \alpha_+(x))\delta_{\rho_-(x)}\,, \quad \alpha_+(x) = \frac{1}{|cell|}\int_{cell} X_+(x, y)\mathrm{d}y$$

Assumption : $\nu = \alpha_+\delta_{\rho_+} + \alpha_-\delta_{\rho_-}$ with $\rho_+(t, x) \neq \rho_-(t, x)$.

$$\begin{aligned} \partial_t(\alpha_+\rho_+) + \mathrm{div}(\alpha_+\rho_+ u) &= 0 \\ \partial_t\rho + \mathrm{div}\rho u &= 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= 0 \\ p &= \mathcal{P}_+(\rho_+) = \mathcal{P}_-(\rho_-) \end{aligned}$$

where :

- $0 \leq \alpha_\pm$ and $\alpha_+ + \alpha_- = 1$
- $\rho = \alpha_+\rho_+ + \alpha_-\rho_-$,

Remark : When $\mathcal{P}(z) = az^\gamma$, we can justify that $\nu^0 = \sum_{i=0}^m \alpha_i^0\delta_{\rho_i^0} \Rightarrow \nu = \sum_{i=0}^m \alpha_i\delta_{\rho_i}$

$$\text{Ingredient :} \quad M_{m,e}[\nu] := \int_\Omega \det[(\overline{\rho^{(i+j)e}})_{1\leq(i,j)\leq m}],$$

Rigorous statements

Cauchy theory for (CNS) + (IC)

Weak solutions : P.-L. Lions '98, E. Feireisl, A. Novotný & H. Petzeltovà '01

Theorem. Assume Ω is smooth and bounded and

- $\mathcal{P}(z) = az^\gamma$ with $\gamma > 3/2$ and $a > 0$
- $\mu > 0$ and $\lambda + 2\mu/3 > 0$

Then, given a positive $\rho^0 \in L^\gamma(\Omega)$ and $q^0 (= \rho^0 u^0)$ compatible with ρ^0 there exists a finite-energy weal solution (ρ, u) to (CNS)+(IC) on arbitrary large times.

Semi-strong solutions : D. Hoff '95, B. Desjardins '97,

Theorem. Assume $\Omega = \mathbb{T}^3$. Given a positive $\rho^0 \in L^\infty(\Omega)$ and $u^0 \in H^1(\mathbb{T}^3)$ there exists $T_0 > 0$ and a finite-energy weak solution (ρ, u) to (CNS)+(IC) such that :

- $\rho \in L^\infty((0, T_0) \times \mathbb{T}^3) \quad \nabla u \in L^\infty((0, T); L^2(\mathbb{T}^3))$
- $\sqrt{\rho}\partial_t u \in L^2((0, T_0) \times \mathbb{T}^3)) \quad Pu \in L^2((0, T_0); H^2(\mathbb{T}^3))$
- $G := (\lambda + 2\mu)\mathrm{div} u - p \in L^2(0, T_0; H^1(\mathbb{T}^3))$

Remark : T_0 depends on $\|\rho_0; L^\infty(\mathbb{T}^3)\|$ and $\|u : H^1(\mathbb{T}^3)\|$ only

Main result

Theorem [D. Bresch & M.H. and D. Bresch & X. Huang '13]

Let initial data $(\rho_n^0, u_n^0) \in L^\infty(\mathbb{T}^3) \times H^1(\mathbb{T}^3)$ satisfy

- $\|\rho_n^0; L^\infty(\mathbb{T}^3)\| + \|u_n^0; H^1(\mathbb{T}^3)\| \leq C$,
- $0 < 1/C \leq \rho_n^0(x)$
- the Young measures ν_n^0 associated with ρ_n^0 converge weakly to

$$\nu^0 = \alpha_+^0(x)\delta_{\rho_+^0(x)} + (1 - \alpha_+^0(x))\delta_{\rho_-^0(x)} \quad \text{on } \Omega.$$

Then, given $\mathcal{P}(z) = az^\gamma$ with $\gamma > 1$

- there exists $T > 0$ and a semi-strong solution (ρ_n, u_n) to (CNS)+(IC) on $(0, T)$
- Up to the extraction of a subsequence

$$\nu_n \rightharpoonup \nu = \alpha_+\delta_{\rho_+} + (1 - \alpha_+)\delta_{\rho_-}\ , \quad u_n \rightharpoonup u \quad p_n \rightharpoonup p$$

- (α_+, ρ_+, u, p) is a solution to (HCNS).

Main steps of the proof

Step 1 : Show that in the limit process

- $\mathrm{div} u \in L^1(0, T; L^\infty(\mathbb{T}^3))$
- control the support of ν

Step 2 : Given (u, p), construct bounded solutions to :

$$\begin{aligned} \partial_t \alpha_k + u \cdot \nabla \alpha_k &= \frac{\alpha_k(\mathcal{P}(\rho_k) - p)}{\lambda + 2\mu} \\ \partial_t \rho_k + \mathrm{div}(\rho_k u)) &= \frac{\rho_k(p - \mathcal{P}(\rho_k))}{\lambda + 2\mu} \end{aligned}$$

Step 3 : Given u and p prove weak-strong uniqueness for the Young measure system :

$$\partial_t \nu + \mathrm{div}_x(\nu u) = \frac{1}{\lambda + 2\mu} \left(\partial_\xi[\xi \nu] G - \partial_\xi[\xi \mathcal{P}(\xi)\nu]\right) = 0\,,$$

where G and u are given as above.

Note. By construction we have a uniform bound on

$$\rho_n \in L^\infty((0,T) \times \mathbb{T}^3)$$
$$\sup_{(0,T)} \|\nabla u_n; L^2(\mathbb{T}^3)\| + \int_0^T \int_{\mathbb{T}^3} \rho_n |\dot{u}_n|^2 .$$

where $\dot{u}_n = \partial_t u_n + u_n \cdot \nabla u_n$.

1. Apply the Hodge-decomposition : $\rho \dot{u} = \nabla G - \mu \nabla \times \omega$

$$\sup_{t\in(0,T)} \|G(t,\cdot); L^2(\mathbb{T}^3)\| + \|\omega; L^2(\mathbb{T}^3)\| \leq C$$
$$\|\nabla G(t,\cdot); L^6(\mathbb{T}^3)\| + \|\omega(t,\cdot); L^2(\mathbb{T}^3)\| \leq C(\|\sqrt{\rho}\dot{u}; L^2(\mathbb{T}^3)\| + \|\nabla \dot{u}; L^2(\mathbb{T}^3)\|)$$

2. Multiply momentum equation with $\sigma(\partial_t + \mathrm{div}(u\cdot))$ where $\sigma = \min(1,t)$:

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^3} \sigma\rho|\dot{u}|^2 + \int_0^T \int_{\mathbb{T}^3} \sigma|\nabla \dot{u}|^2 \leq C$$

3. Conclusion : write

$$\mathrm{div} u = (\lambda + 2\mu)(G + p)$$

+ interpolation + above estimates ...

Details for Step 2+3

1. Construct α_i and ρ_i solution for u and q given.

2. Define $\overline{\nu} = \sum_{i=1}^m \alpha_i \delta_{\rho_i}$ and set ν another solution to the young-measure equation

3. Prove by induction that

$$\int_0^\infty z^{1-k\gamma} \mathrm{d}\nu(z) = \int_0^\infty z^{1-k\gamma} \mathrm{d}\bar{\nu}(z), \qquad \forall k \in \mathbb{N} \cup \{0\} \quad a.e.$$

4. Conclusion :
The above steps entail that

$$\int_0^\infty b(z)\mathrm{d}\nu(z) = \int_0^\infty b(z)\mathrm{d}\bar{\nu}(z) \text{ for all } b \text{ of the form } b(z) = k + z\beta(z^{-\gamma}) \ .$$

As ν and $\overline{\nu}$ have compact support in $(0, \infty)$ we complete the proof by applying

- a density argument
- $s \mapsto s^\gamma$ realizes homeomorphism of $(0, +\infty)$.

D.B., M. Hillairet '2015 (in progress - redaction) :
Starting with compressible Navier-Stokes equations with density dependent viscosity ?

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) & = & 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p - 2\mathrm{div}(\mu(\rho) D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) & = & 0 \\ p = p(\rho) \end{cases}$$

In the one-dimensionall in space case, write

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) & = & 0 \\ \partial_t(\rho u) + \partial_x(\rho u \otimes u) + \partial_x p - \partial_x(\mu(\rho)\partial_x(u)) & = & 0 \\ p = p(\rho) \end{cases}$$

We then search for two-scale solutions under the following form :

$$\begin{aligned} \rho(t,x) &= \sum_{i=+,-} \alpha_i\left(t,\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)\rho_i^{\varepsilon}(t,x)\,, \\ u(t,x) &= u_0(t,x) + \varepsilon u_1\left(t,\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(t,\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right) + O(\varepsilon^3) \end{aligned}$$

assuming that

$$\rho_i^{\varepsilon}(t,x) = \rho_i^0(t,x) + O(\varepsilon)$$

After some calculations in the general setting, we consider in a first calculation the viscosity μ constant and in a second calculation we focus on the case $\mu = \mu(\rho)$. The general system that we obtained on $(\overline{\alpha_\pm}, u_0, \rho^0_\pm, \overline{p})$ reads in the general setting

$$
\begin{aligned}
&\overline{\alpha_+} + \overline{\alpha_-} = 1,\\
&\partial_t \overline{\alpha_+} + u_0 \partial_x \overline{\alpha_+} = \frac{\overline{\alpha_+}\,\overline{\alpha_-}}{\overline{\alpha_-}\mu^0_+ + \overline{\alpha_+}\mu^0_-}[(p^0_+ - p^0_-) + (\mu^0_- - \mu^0_+)\partial_x u_0]\\
&\partial_t(\overline{\alpha_+}\rho^0_+) + \partial_x(\overline{\alpha_+}\rho^0_+ u_0) = 0\,,\\
&\overline{\rho}(\partial_t u_0 + u_0 \partial_x u_0) - \partial_x(\mu \partial_x u_0) + \partial_x \overline{p} = 0\,,\\
&p^0_+ = p(\rho^0_+), \qquad p^0_- = p(\rho^0_-), \qquad \overline{\rho} = \overline{\alpha_+}\rho^0_+ + \overline{\alpha_-}\rho^0_-, \qquad \overline{p} = \overline{\alpha_+}p^0_+ + \overline{\alpha_-}p^0_-,\\
&\mu = \overline{\alpha_+}\mu^0_+ + \overline{\alpha_-}\mu^0_-, \qquad \mu^0_\pm = \mu(\rho^0_\pm)
\end{aligned}
$$

where $\overline{\alpha_\pm}$ denotes the average with respect to the fast variables of $\alpha_\pm$.

Mathematical justification :

- Global strong solution for density far from vacuum : BD entropy (κ-entropy)
- Young measure theory similarly to the constant viscosity case.