Hierachy of fluid models and environmental problem.

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Part 2

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Main objective

- Extend the frame covered by P.–L. Lions and E. Feireisl' theory with constant viscosities (or temperature dependent).

Motivations:

No monotonicity assumption on $\partial_\varrho P(\varrho)$ or $\partial_\varrho P(\varrho, \vartheta)$

Pressure in mind (virial) in temperature dependent case:

$$P(\varrho, \theta) = \varrho\, \theta \,\Big(\sum_{n\geq 0} B_n(\theta)\varrho^n\Big) \text{ with } B_n(\theta) \text{ some functions.}$$

and

Anisotropy on the diffusion: $\quad -\mu_x \Delta_x u - \mu_z \partial_z^2 u$ with $\mu_x \neq \mu_z$ const.

Books on compressible NS eqs (global weak solutions):
P.–L. Lions (1998), E. Feireisl (2004), A. Novotny - I. Straskraba (2004), E. Ferireisl - A. Novotny (2009), P. Plotnikov - J. Sokolowski (2012), E. Feireisl - M. Pokorny (2014 - Notes on web).

Based on joint work with:
P.–E. JABIN (Maryland USA)

What is known actually on global weak solutions with constant viscosities?

Barotropic case:

$$[CNS] \quad \begin{array}{c} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{array}$$

with a given law $s \mapsto P(s)$, $\mu > 0$ and $\lambda + 2\mu/d > 0$.

The case $P(s) = a\, s^\gamma$ with $a > 0$:

- P.–L. Lions (1993–1998): $\gamma \geq 3d/(d+2)$
- E. Feireisl (2001) with co-authors: $\gamma > d/2$
- Note the recent work: P. Plotnikov-W. Weigant (2015): $d = 2$ and $\gamma = 1$

Some important non-monotone cases

- E. Feireisl (2002)
- B. Ducomet, E. Feireisl, H. Petzeltova, I. Straskarba (2004)

Hypothesis on P with $P'(\rho) \geq C^{-1} \varrho^{\gamma - 1} - C$ for all $\varrho \in [0, +\infty)$.

Case $P(\varrho) = a\varrho^\gamma$ (Estimates):

Energy estimates:

$$\sup_{t\in[0,T]} \Big(\frac{1}{2}\int_{\Pi^d} \varrho|u|^2 + \frac{1}{\gamma-1}\int_{\Pi^d} \varrho^\gamma\Big) + \mu\int_0^T\int_{\Pi^d} |\nabla u|^2 + (\lambda+\mu)\int_0^T\int_{\Pi^d} |\mathrm{div}\, u|^2$$

$$\le \frac{1}{2}\int_{\Pi^d} \frac{|m_0|^2}{\varrho_0} + \frac{1}{\gamma-1}\int_{\Pi^d} \varrho_0^\gamma$$

Extra integrability on the density (Bogovskii operator):

$$\int_{\Pi^d} \varrho^p = \int_{\Pi^d} \varrho^{\gamma+\theta} \le C < +\infty$$

with

$$\theta \le 2\gamma/d - 1.$$

Remark. We have ϱ square integrable namely $p \ge 2$ if $\gamma \ge 3d/(d+2)$ (P.–L. Lions constraint)

Compactness to pass to the limit in ϱu and $\varrho u \otimes u$ mostly relies on

- compactness (negative sobolev space) on $\varrho_k u_k$: Aubin-Lions-Simon Lemma
- convergence in norm to have compactness on $\sqrt{\varrho_k} u_k$ in $L^2((0,T) \times \Pi^d)$

The main difficulty in the proof: passage to the limit in ϱ_k^γ in weak formulation
How to get compactness on ϱ in Lebesgue spaces?

The main step where the monotonicity is required (case $\gamma \geq 3d/(d+2)$)

$$\partial_t(\varrho \ln \varrho) + \mathrm{div}(\varrho \ln \varrho u) + \varrho \mathrm{div} u = 0.$$

noticing that

$$s \mapsto s \ln s$$

is a strictly convex function and

$$s \mapsto p(s)$$

is an increasing function.

Goal: show that

$$\overline{\varrho \ln \varrho} = \varrho \ln \varrho$$

$\Longrightarrow$ commutation between stricly convex function and weak limit

$$\partial_t(\varrho \ln \varrho) + \mathrm{div}(s\varrho \ln \varrho u) + \varrho \mathrm{div}\, u = 0.$$

This uses the property (effective flux property): weak compactness

$$\overline{\rho\, \mathrm{div} u} - \frac{\overline{P(\rho)\rho}}{\lambda + 2\mu} = \rho\, \mathrm{div} u - \frac{\overline{P(\rho)}\rho}{\lambda + 2\mu}$$

which gives

$$\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u = \frac{\overline{P(\rho)\rho} - \overline{P(\rho)}\rho}{\lambda + 2\mu} \qquad \Longrightarrow \text{ appropriate sign due to monotonicity}$$

For more general γ, use a clever troncature procedure: see E. Feireisl.

In the anisotropic case

$$-\mu_x \Delta_x u - \mu_z \partial_z^2 u - \lambda \nabla \mathrm{div} u \text{ with } \mu_x \neq \mu_z \text{ const}$$

Then

$$\overline{\varrho \mathrm{div} u} - \varrho \mathrm{div} u \leq \frac{\overline{\varrho} \overline{A_\mu \varrho^\gamma} - \overline{\varrho A_\mu \varrho^\gamma}}{\mu_x + \lambda}$$

where $A_\mu = a_\mu (\Delta - (\mu_x - \mu_z)\partial_z^2)^{-1} \partial_z^2$ with $a_\mu = (\mu_x - \mu_z)$.

No *a priori* sign on the right-hand side: Non-local effects.

$\Longrightarrow$ difficulty: Possible mixing phenomena (small/large value of density)

See discussions in D.B., B. Desjardins, D. Gérard-Varet (2004).

The first compressible Navier-Stokes system under consideration

Consider the following barotropic system in periodic box:

$$[CNS] \quad \begin{array}{c} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{array}$$

with the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1} \varrho^\gamma - C \leq P(\varrho) \leq C \varrho^\gamma + C$$

and for all $s \geq 0$, we only assume

$$|P'(s)| \leq s^{\widetilde{\gamma} - 1}$$

for some $\widetilde{\gamma} > 1$.

Mathematical result

Theorem. Let (ϱ_0, u_0) such that

$$E(\varrho_0, u_0) = \int_{\Pi^d} \frac{|m^0|^2}{2\varrho_0} + \varrho_0 e(\varrho_0) < +\infty$$

with $e(s) = \int_0^s P(\tau)/\tau^2 d\tau$. Let P satisfying the previous hypothesis with

$$\gamma > (\max(2, \widetilde{\gamma}) + 1)\, d/(d+2)$$

then there exists a global weak solution to the compressible Barotropic Navier-Stokes equations (CNS).

Remark:

- If $\widetilde{\gamma} = \gamma$ then $\gamma > 3d/(d+2)$.
- Truncated procedure as introduced by E. Feireisl could give $\gamma > d/2$.
- Importance of such pressure: biology, solar events........

The second compressible Navier-Stokes system under consideration

Consider the following barotropic system in periodic box:

$$[ACNS] \quad \begin{aligned} &\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mathrm{div}(A(t)\nabla u) - (\lambda + \mu)\nabla \mathrm{div} u + \nabla P(\varrho) = \mathbf{0}, \end{aligned}$$

with the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1}\rho^{\gamma-1} - C \leq P'(\rho) \leq C\rho^{\gamma-1} + C$$

and a $d \times d$ matrix $A = \mu \mathrm{Id} + \delta A(t)$ with time dependent smooth coefficient.

Remarks:

- Case usually encountered in geophysics: $-\nu_x \Delta_x u - \nu_z \partial_z^2 u$ (see Handbook R. Temam and M. Ziane).
- We can consider: $-\mathrm{div}(A(t)D(u)) + \lambda \nabla \mathrm{div} u$.
- Incompressible flows - weak sol.: anisotropy no problem if not degenerate.
- Compressible feature: Possible "density mixing"due to non-local operator.

Mathematical result

Theorem. Let (ϱ_0, u_0) such that

$$E(\varrho_0, u_0) = \int_{\Pi^d} \frac{|m^0|^2}{2\varrho_0} + \varrho_0 e(\varrho_0) < +\infty$$

with $e(s) = \int_0^s P(\tau)/\tau^2 d\tau$. Let P satisfying the monotonicity assumption and assume that

$$\gamma > \frac{d}{2}\Big[\big(1 + \frac{1}{d}\big) + \sqrt{1 + \frac{1}{d^2}}\Big].$$

There exists a universal constant $C_\star > 0$ such that if

$$\|\delta A\|_\infty \le C_\star(2\mu + \lambda).$$

then there exists a global weak solution to the compressible Barotropic Navier-Stokes equations (CNS).

Remark. Seems a straightforward perturbation result......
BUT it is trickier than the non-monotone pressure case due to non-local terms!!

How it works on a more simple case?

Let us consider the following system

$$[CS] \quad \begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ \mathrm{div}\, u = P(\varrho) + Q \end{aligned}$$

with a given pressure law $s \mapsto P(s)$: System encountered in biology for instance.

We assume the pressure P locally Lipschitz on $[0, +\infty)$, with $P(0) = 0$ and

$$C^{-1}\varrho^\gamma - C \le P(\varrho) \le C\varrho^\gamma + C$$

and for all $s \ge 0$, we only assume

$$|P'(s)| \le s^{\gamma-1}.$$

Compactness on the density?

A compactness Lemma

Let ϱ_k bounded in $L^p((0,T)\times \Pi^d)$ (with $1\le p<+\infty$) and

$$\partial_t \varrho_k \in L^q(0,T;W^{-1,q}(\Pi^d))$$

with $q>1$. Let K_h positive, bounded functions s.t.

$$\forall \eta>0, \qquad \sup_h \int_{|x|\ge\eta} K_h(x)\,dx<+\infty$$

and

$$\|K_h\|_{L^1(\Pi^d)} \to +\infty \text{ when } h\to +0$$

If

$$\limsup_k \limsup_t \Big[\frac{1}{\|Kh\|_{L^1}}\int_{\Pi^d} K_h(x-y)|\varrho_k(t,x)-\varrho_k(t,y)|^p\,dxdy\Big]\to 0, \qquad \text{as } h\to 0$$

Then ϱ_k compact in $L^p((0,T)\times\Pi^d)$.

A compactness Lemma

Some references:

- J. Bourgain, H. Brézis, P. Mironescu: Functional spaces (2001)
- A.C. Ponce: Functional spaces (2004)
- F. Ben Belgacem, P.–E. Jabin: Nonlinear continuity equations (2013)

The problem:

Weak solutions:
No Sobolev regularity propagation on ϱ for compressible Navier-Stokes Eqs.

The frame:

- Weak regularity on the velocity field
- Vacuum state for the density.

The idea:

- Introduce some appropriate weights w_k in the quantity to be controlled
  Precise the rate of convergence in terms of h.
- Derive appropriate properties on these weights
  Go back to the definition without weights without too much lost in h.

Introduce weights: first idea

In the sequel, we write: $\operatorname{div} u_k = P(\varrho_k) + Q_k$
with Q_k compact in k (with a corresponding $\epsilon_k(h)$).

1) Introduce:

$$R_h(t) = \int_{\Pi^d} K_h(x-y)|\varrho_k(t,x) - \varrho_k(t,y)|\, w(t,x)w(t,y)dxdy$$

with w solution of

$$\partial_t w + u_k \cdot \nabla w + \lambda D w = 0$$

with D an appropriate positive damping term linked to (ϱ_k, u_k).

Choose appropriate damping terms in D:
$\Longrightarrow$ to control the propagation of the quantity R_h in time explicitly in h

2) Show some properties of the weights when D is chosen:

For instance:

$$0 \leq w \leq 1, \qquad \int \varrho_k |\log w|^{\theta} < +\infty$$

with some $\theta > 0$.

Remove the weights using their properties to apply the compactness lemma

Let w be solution of

$$\partial_t w + u_k \cdot \nabla w = -\lambda(M|\nabla u_k| + \varrho_k^\gamma)\, w, \qquad w|_{t=0} = 1$$

where Mf is the maximal function of f namely

$$Mf(x) = \sup_{r\leq 1} \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x+z)\, dz.$$

Let us look at propagation of the quantity for the simple system

$$R_h(t) = \int_{\Pi^d} K_h(x-y)|\varrho_k(t,x) - \varrho_k(t,y)|w(t,x)w(t,y)\,dxdy.$$

We get

$$\frac{d}{dt}R_h(t) = \int_{\Pi^{2d}} \nabla K_h(x-y)\cdot(u_k(t,x)-u_k(t,y))\,|\varrho_k(t,x)-\varrho_k(t,y)|\,w(t,x)\,w(t,y)$$
$$-\frac{1}{2}\int_{\Pi^{2d}} K_h(x-y)\,(\mathrm{div}\,u_k(t,x)-\mathrm{div}\,u_k(t,y))\,(\varrho_k(x)+\varrho_k(y))\,s_k\,w(t,x)\,w(t,y)$$
$$+\int_{\Pi^{2d}} K_h(x-y)\,|\varrho_k(t,x)-\varrho_k(t,y)|\,(\partial_t w(t,x)+u_k(t,x)\cdot\nabla w(t,x))\,w(t,y)$$
$$+\,symmetric.$$

$$\int_{\Pi^{2d}} \nabla K_h(x-y)\cdot(u_k(t,x)-u_k(t,y))\,|\rho_k(t,x)-\rho_k(t,y)|\,w(t,x)\,w(t,y)$$

$$\leq C\int_{\Pi^{2d}} K_h(x-y)(M|\nabla u_k|(t,x)+M|\nabla u_k|(t,y))w(t,x)w(t,y)$$

Thanks to

$$|u_k(t,x)-u_k(t,y)|\leq C|x-y|(M|\nabla u_k|(t,x)+M|\nabla u_k(t,y)|)$$

and

$$|\nabla K_h(x-y)||x-y|\leq C\,K_h(x-y).$$

$$\int_{\Pi^{2d}} K_h(x-y)\,(\mathrm{div}\, u_k(t,x) - \mathrm{div}\, u_k(yt,))\,(\varrho_k(t,x) + \varrho_k(t,y))\, s_k\, w(t,x)\, w(t,y)$$

$$\leq \|K_h\|_{L^1}\,(\varepsilon_k(h))^{1-1/p}$$

$$+ \int_{\Pi^{2d}} K_h(x-y)\,(P(\rho_k(t,x) - P(\varrho_k(t,y))\,(\varrho_k(t,x) + \varrho_k(t,y))\, s_k\, w(t,x)\, w(t,y)$$

Using locally Lipschitz property of P and the control on $|P'(s)| \leq Cs^{\gamma-1}$.

$$\int_{\Pi^{2d}} K_h(x-y)\,(P(\varrho_k(t,x) - P(\varrho_k(t,y))\,(\varrho_k(t,x) + \varrho_k(t,y))\, s_k\, w(t,x)\, w(t,y)$$

$$\leq C \int_{\Pi^{2d}} K_h(x-y)(\varrho_k^\gamma(t,x) + \varrho_k^\gamma(t,y))|\varrho_k(t,x) - \varrho_k(t,y)|\, w(t,x)\, w(t,y)$$

Thus

$$\frac{d}{dt}R(t) \leq C \int_{\Pi^{2d}} K_h(x-y)(M|\nabla u_k|(t,x) + \varrho_k(t,x)^\gamma)|\varrho(t,x) - \varrho(t,y)|w(t,x)w(t,y)$$

$$- \lambda \int_{\Pi^{2d}} K_h(x-y)\,|\varrho_k(t,x) - \varrho_k(t,y)|\,Pw(t,x)\,w(t,y)$$

$$+ \, symmetric$$

$$+ \|K_h\|_{L^1}(\varepsilon_k(h))^{1-1/p}$$

Therefore assuming λ large enough in the weight definition we get

$$\frac{d}{dt}R(t) \leq \|K_h\|_{L^1}(\varepsilon_k(h))^{1-1/p}$$

Property on the weight w:

$$\int_{\Pi^d} |\log w(t,x)|\varrho_k(t,x) \leq \int_0^T \int_{\Pi^d} P(\rho_k(t,x))\varrho_k(t,x) < +\infty$$

$\Longrightarrow$ hypothesis $p \geq \gamma + 1$ needed. This will be relax later-on!

Remind p index of the integrability of ϱ through Bogovski type estimates

Control the size where $w(t,x)$ is small by the size where $\rho(t,x)$ is small !!

Do not control the size where $w(t, y)$ small when in front of $\varrho_k(t, x)$......
Remark $w(t, y)$ may be small when $\varrho(t, x)$ is large......
Therefore no compactness because no control close to vacuum !!!

Remark. If transport equation considered with compactness properties on $\mathrm{div}\, u_k$ then in many respect: Equivalent of the method of G. Crippa and C. De Lellis at the PDE level instead of ODE level: No weight needed.

See paper by F. Ben Belgacem and P.–E. Jabin:
Nice results on non-linear continuity Eq.

Let us propose a better candidate for quantity R_h!!

Assume

$$\frac{1}{\|K_h\|_{L^1}}\int_{\Pi^d} K_h(x-y)|\varrho_k(t,x)-\varrho_k(t,y)|(w(t,x)+w(t,y))\,dxdy = o(h).$$

and

$$\int_{\Pi^d} \rho_k|\log w| \le C < +\infty$$

Then

$$\int_{\Pi^d} K_h(x-y)|\varrho_k(t,x)-\varrho(t,y)|\,dxdy$$

$$\le \frac{1}{\eta'}\int_{\Pi^d} 1_{\{w(t,x)\ge\eta' \text{ or } w(t,y)\ge\eta'\}} K_h(x-y)|\varrho_k(t,x)-\varrho_k(t,y)|(w(t,x)+w(t,y))\,dxdy$$

$$+2\frac{1}{|\log\eta'|^{1/2}}\int_{\Pi^d} 1_{\{w(t,x)\le\eta' \text{ and } w(t,y)\le\eta'\}} K_h(x-y)\rho_k(t,x)|\log w|^{1/2}.$$

Use that $\varrho \in L^p$ with $p>2$ and optimize η' in terms of h to conclude by the compactness Lemma.

For compressible Navier–Stokes equations:

More complicated (see D.B., P.–E. Jabin : arXiv:1507.04629)

Thank you for your attention!