Hierachy of fluid models and environmental problem.

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Part 1

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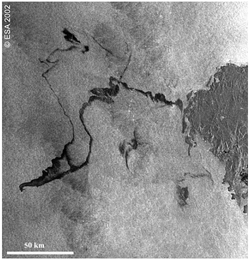
Examples

Powder-snow avalanche



Spreading of pollutant in water

Main objectives

- Understand the structure of the systems used in modelization
- Go as closed as possible to weakest regularity (Energy regularity).
- Understand continuous level to hope to enrich the discrete level.

Mixture system

Consider the following system in periodic box:

$$[M-NS]\quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho)\operatorname{div}\mathbf{u}) + \nabla\Pi = \mathbf{0}, \\ &\operatorname{div}\mathbf{u} = -2\kappa\Delta\varphi(\varrho). \end{aligned}$$

where $D(\mathbf{u}) = (\nabla u + \nabla^t u)/2$ or equivalently

$$\begin{aligned} &\partial_t \varrho + \nabla\varrho \cdot (\mathbf{u} + 2\kappa\nabla\varphi(\varrho)) - 2\kappa\operatorname{div}(\varrho\nabla\varphi(\varrho)) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho)\operatorname{div}\mathbf{u}) + \nabla\Pi = \mathbf{0}, \\ &\operatorname{div}\mathbf{u} = -2\kappa\Delta\varphi(\varrho). \end{aligned}$$

Note here κ const

Physical literature

Such system:

- 1) Low mach number limit from
  Heat-conducting compressible Navier-Stokes eq. with large heat release.
  See the book by P.–L. Lions.
- 2) Formally obtained as mixture equations with Fick law to close the system.
  See the book by Rajagopal and Tao.

Some special cases:

- 1) For $\mu(\varrho) = \log(\varrho)$ (i.e. $\varphi(\varrho) = -1/\varrho$) we recover combustion model.
  See works by Embid, Majda, Lions, Laffitte, Dellacherie, Penel...
- 2) For $\mu(\varrho) = \mathrm{const}$ (i.e. $\varphi(\varrho) = \log \varrho$) we recover pollutant model.
  See works by Graffi, Straughan, Antonsev, Kazhikhov, Monakov...

$\kappa = 0 \implies$ Non-homogeneous incompressible Navier–Stokes equations.

$$[NH-INS] \quad \begin{aligned} &\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho) D(\mathbf{u})) + \nabla \Pi = \mathbf{0}, \\ &\mathrm{div}\,\mathbf{u} = 0. \end{aligned}$$

Global well posedness: A. Kazhikhov '77, J. Simon '87, P.–L. Lions '98.

Mathematical literature on the mixture system

- Local strong solutions

  Beirão Da Veiga '82, Secchi '82,
  Danchin & Liao '12 (in critical Besov spaces).

- Global in time solutions

  Kazhikov & Smagulov '77: Modified conv. term, constraint on c_0
  existence of generalized solution which is unique in 2d,
  Lions '98: 2d weak solutions ($\varphi = -1/\varrho$), small perturb. const. ρ_0,
  Secchi '88: 2d unique solution for small c_0
  Danchin & Liao '12: Small perturb. const. ρ + small initial velocity.

- No smallness assumption

  B., Essoufi & Sy '07, for special relation

  $$\varphi'(s) = \mu'(s)/s, \quad \kappa = 1 \qquad \Longrightarrow \quad \text{Kazhikhov-Smagulov type system}$$

  Cai, Liao & Sun '12: Uniqueness in 2d,
  Liao '14: Global strong solution in 2d, critical Besov spaces.

Numerical literature

- J. Etienne, E. Hopfinger, P. Saramito.
  Numerical simulations of high density ratio lock-exchange flows.
  No change of variable.
  Finite element + characteristic method with mesh refinements.
- C. Acary-Robert, D. Bresch, D. Dutykh.
  Numerical simulation of powder-snow avalanche interaction with obstacle.
  Numerical test using Open-Foam,
  change of variable + relation between μ and φ
  Discussion around a new entropy encountered in a theoretical paper.
- C. Calgaro, E. Creusé, T. Goudon.
  Simulation of Mixture Flows: Pollution Spreading and Avalanches.
  Change of variable + get ride of high-order terms
  (Kazhikhov-Smagulov type system).
  Numerical schemes: hybrid Finite Volume/Finite Element method.
  Test and comparison.

Goal of this part on this powder snow avalanches system:
A two-velociy hydrodynamics in this model

The case $\mu'(s) = s\varphi'(s)$:

$\Longrightarrow$ A non-linear hypocoercivity property!
$\Longrightarrow$ A two-velocity hydrodynamic in the spirit of H. Brenner but.......
.... with two different velocities:
not volume and mass velocities as in H. Brenner's work
That means not $\mathbf{u}$ and $\mathbf{u} + 2\kappa\nabla\varphi(\rho)$ but two others specified later on.
$\Longrightarrow$ Global existence of weak solutions for a wide range of coefficient.
$\Longrightarrow$ An answer to an open question in P.–L. Lions's book.
$\Longrightarrow$ An interesting numerical scheme
(work in progress with P. Noble, J.–P. Vila).

The case $\mu'(s) \neq s\varphi'(s)$:

A conclusion under some inequalities constraints.
$\Longrightarrow$ An answer to an other open question in P.–L. Lions's book.

Special case where φ and μ are related: Two velocity hydrodynamics

Let us remark

$$\int \nabla \Pi_1 \cdot \mathbf{u} = 2\,\kappa \int \Pi_1 \Delta\varphi(\varrho)$$

and

$$\int \nabla \Pi_1 \cdot (\mathbf{u} + 2\nabla\varphi(\varrho)) = -2\,(1-\kappa) \int \Pi_1 \Delta\varphi(\varrho)$$

Thus

$$\int \nabla \Pi_1 \cdot ((1-\kappa)\mathbf{u}) + \int \nabla \Pi_1 \cdot (\kappa(\mathbf{u} + 2\nabla\varphi(\varrho))) = 0$$

Momentum equation on $\mathbf{u}$:

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho)D(\mathbf{u})) = -\nabla\Pi_1$$

Momentum equation on $\mathbf{v} = \mathbf{u} + 2\nabla\varphi(\varrho)$:

$$\partial_t(\varrho\mathbf{v}) + \mathrm{div}(\varrho\mathbf{v}\otimes\mathbf{u}) - 2\,\mathrm{div}(\mu(\varrho)A(\mathbf{v})) - 2\nabla\Big((\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{u}\Big) = -\nabla\Pi_1$$

where $A(\mathbf{v}) = (\nabla\mathbf{v} - \nabla^t\mathbf{v})/2$.

Additional entropy equality

Testing Eq on $\mathbf{u}$ by $(1-\kappa)\mathbf{u}$ and Eq on $\mathbf{v}$ by $\kappa\mathbf{v}$ and adding we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \varrho\left((1-\kappa)\frac{|\mathbf{u}|^2}{2}+\kappa\frac{|\mathbf{v}|^2}{2}\right)\,\mathrm{d}x$$
$$+2(1-\kappa)\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x+2\kappa\int_\Omega \mu(\varrho)|A(\mathbf{u})|^2\,\mathrm{d}x$$
$$+2(1-\kappa)\kappa^2\int_\Omega(\mu'(\varrho)\varrho-\mu(\varrho))|2\Delta\varphi|^2\,\mathrm{d}x=0,$$

which generalizes the "B-D" entropy to the M-NS system.

Two-velocity hydrodynamic: joint velocity and drift velocity

Remark that:

$$(1-\kappa)|\mathbf{u}|^2 + \kappa|\mathbf{u} + 2\nabla\varphi|^2 = |\mathbf{w}|^2 + (1-\kappa)\kappa|2\nabla\varphi|^2.$$

$\Longrightarrow$ See two-velocity hydrodynamics papers by S.C. Shugrin and S. Gavrilyuk
Defining a new velocity vector field (joint velocity)

$$\mathbf{w} = \mathbf{u} + \kappa\nabla\varphi(\varrho), \quad \text{we see that} \quad \operatorname{div}\mathbf{w} = 0$$

Note that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ is called the drift velocity.

Appropriate unknowns: $\mathbf{w}$ and $\sqrt{(1-\kappa)\kappa}\mathbf{v_1}$.

If $\kappa = 1$, then we get the following system on $(\varrho, \mathbf{v})$:

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{v}) - 2\Delta\mu(\varrho) = 0, \\ [KS] \quad &\partial_t (\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) A(\mathbf{u})) + \nabla \Pi_1 = \mathbf{0}, \\ &\operatorname{div} \mathbf{v} = 0 \end{aligned}$$

with $\mathbf{u} = \mathbf{v} - 2\nabla\varphi(\varrho)$ (Note that $\mathbf{v}$ is divergence free).

$\Longrightarrow$ Kazhikhov-Smagulov type system

Global well posedness without asking any size constraint on the initial density !!

Proved by D.B., E. Hassan Essoufi, M. Sy '07

With the κ-entropy type estimate: More general results!!

Special case where φ and μ are related

For $0 < T < \infty$, $\Omega = \mathbb{T}^3$ and the low Mach number system

$$[M-NS] \quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi = \mathbf{0}, \\ &\operatorname{div} \mathbf{u} = -2\kappa \Delta \varphi(\varrho), \end{aligned}$$

with $\varphi'(s) = \mu'(s)/s$, $0 < \kappa < 1$ we have:

T1: For the initial conditions satisfying

$$\sqrt{(1-\kappa)\kappa \varrho^0} \in H^1(\Omega), \quad 0 < r \le \varrho^0 \le R < \infty, \quad \mathbf{u}^0 + 2\kappa \nabla \varphi(\rho^0) \in H,$$

$\mu(\varrho)$ such that $\mu(\varrho) \in C^1([r,R])$, $\mu'(\varrho) > 0$, $\mu \geq c > 0$ on $[r,R]$, and

$$\left(\frac{1-d}{d} \mu(\varrho) + \mu'(\varrho)\varrho \right) \geq c > 0.$$

There exists a global in time weak solution* to [M-NS].

T2: For $\kappa \to 0$ this solution converges to the weak solution of the non-homogenous incompressible N-S equations; for $\kappa \to 1$ (and $\kappa \varrho_\kappa^0 \in H^1(\Omega)$) it converges to the weak solutions of the Kazhikhov-Smagulov system.

D.B., V. Giovangigli, E. Zatorska '14

General case

For $T < \infty$, $\Omega = \mathbb{T}^3$ and the General low Mach number system

$$[M-NS-G] \quad \begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2 \operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla \Pi &= \mathbf{0}, \\ \operatorname{div} \mathbf{u} &= -2\Delta \tilde{\varphi}(\varrho), \end{aligned}$$

with $\tilde{\varphi}'(s) = \tilde{\mu}'(s)/s$, we have:

T3: Under previous assumptions on the data and, for $\mu(\cdot) \in C^1([r,R])$, $\mu'(\cdot) > 0$, $\mu \geq c > 0$ on $[r,R]$ and $\tilde{\varphi}(\cdot) \in C^1([r,R])$ and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ related by

$$c \leq \min_{\varrho \in [r,R]} \left(\mu(\varrho) - \tilde{\mu}(\varrho)\right),$$

$$\max_{\varrho \in [r,R]} \frac{(\mu(\varrho) - \tilde{\mu}(\varrho) - \xi \tilde{\mu}(\varrho))^2}{2\left(\mu(\varrho) - \tilde{\mu}(\varrho)\right)} \leq \xi \min_{\varrho \in [r,R]} \left(\tilde{\mu}'(\varrho)\varrho + \frac{1-d}{d} \tilde{\mu}(\varrho) \right).$$

for some positive constants c, ξ. There exists global weak solution to [M-NS-G].

D.B., V. Giovangigli, E. Zatorska '14

Remarks

- For $\mu(\varrho) = \varrho^{\alpha}$ the exponent α from T1 is

$$\alpha > 1 - \frac{1}{d}.$$

  In particular, it does not depend on κ.

- Assume that all assumptions of T3 are satisfied and $\mu(\varrho)$, $\tilde{\mu}(\varrho)$ are replaced by

$$\mu(\varrho) = \varrho, \quad \tilde{\mu}(\varrho) = \log \varrho \quad (\textit{i.e.} \;\; \tilde{\varphi}(\varrho) = -1/\varrho).$$

  Then there exist a non-empty interval $[\tilde{r}, \tilde{R}]$ such that if

$$0 < \tilde{r} \leq \varrho^{0} \leq \tilde{R} < 0,$$

  then the weak solution to [M-NS-G] exists globally in time, which corresponds to the dense gas approximation:

S. Chapman and T.G. Cowling:
The mathematical theory of non-uniform gases, 1970.

$\Longrightarrow$ Generalization of P.–L. Lions's result to the 3d case!

Construction of solution

We consider the augmented regularized system with three unknowns $(\varrho, \mathbf{w}, \mathbf{v_1})$:

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{w}) - 2\kappa \Delta \mu(\varrho) = 0,$$

$$\partial_t\,(\varrho \mathbf{w}) + \mathrm{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{w}) - 2(1-\kappa)\,\mathrm{div}(\mu(\varrho) D(\mathbf{w}))$$
$$- 2\kappa\,\mathrm{div}(\mu(\varrho) A(\mathbf{w})) + \nabla \Pi_1 + \varepsilon \Delta^2 \mathbf{w} = -2\kappa(1-\kappa)\,\mathrm{div}(\mu(\varrho)\nabla \mathbf{v_1}),$$

$$\partial_t(\varrho \mathbf{v_1}) + \mathrm{div}((\varrho \mathbf{w} - 2\kappa \nabla \mu(\varrho)) \otimes \mathbf{v_1}) - 2\kappa\,\mathrm{div}(\mu(\varrho)\nabla \mathbf{v_1})$$
$$- 2\kappa \nabla((\mu'(\varrho)\varrho - \mu(\varrho))\,\mathrm{div}\,\mathbf{v_1}) = -2\,\mathrm{div}(\mu(\varrho)\nabla^t \mathbf{w}),$$
$$\mathrm{div}\,\mathbf{w} = 0.$$

Of course, we have to prove that $\mathbf{v_1} = 2\nabla\varphi(\varrho)$ to solve the initial system.
To do that Important property: $\mathrm{div}\,\mathbf{w} = 0.$

Augmented system in other topics: numerical schemes (D. Jamet *et al.*)

For compressible NS equations (see later-on): an extra integrability needed
This is the term $-\varepsilon\,\mathrm{div}\left(|\nabla \mathbf{w}|^2 \nabla \mathbf{w}\right)$:
An hyper diffusive term introduced by O.A. Ladhyzenskaya.

Extensions to density dependent viscosities compressible NS equations

$$[CNS] \quad \begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ &\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - 2\operatorname{div}(\mu(\varrho) D(\mathbf{u})) - \nabla(\lambda(\varrho) \operatorname{div} \mathbf{u}) + \nabla P(\varrho) = \mathbf{0} \end{aligned}$$

with $\lambda(\varrho) = 2(\mu'(\varrho)\varrho - \mu(\varrho))$ (algebraic relation found by D.B., B. Desjardins).

$P(\rho) = \rho^2/2, \qquad \mu(\varrho) = \mu\varrho \implies$ Viscous shallow-water type system

Let us introduce an arbitrary coefficient κ such that $0 < \kappa < 1$.

Extensions to density dependent viscosities compressible NS equations

For this compressible barotropic system we have generalized κ- entropy:

$$\int_\Omega \varrho \left(\frac{|\mathbf{w}|^2}{2} + (1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho)|^2}{2}\right)(T)\,\mathrm{d}x + \int_\Omega \varrho e(\varrho)\,\mathrm{d}x$$

$$+\,2(1-\kappa)\int_0^T\!\!\int_\Omega \mu(\varrho)|D(\mathbf{u})|^2\,\mathrm{d}x\,\mathrm{d}t + 2(1-\kappa)\int_0^T\!\!\int_\Omega (\mu'(\varrho)\varrho-\mu(\varrho))(\mathrm{div}\,\mathbf{u})^2\,\mathrm{d}x\,\mathrm{d}t$$

$$+\,2\kappa\int_0^T\!\!\int_\Omega \mu(\varrho)|A(\mathbf{w})|^2\,\mathrm{d}x\,\mathrm{d}t + 2\kappa\int_0^T\!\!\int_\Omega \frac{\mu'(\varrho)p'(\varrho)}{\varrho}|\nabla\varrho|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_\Omega \varrho_0\left(\frac{|\mathbf{w}_0|^2}{2} + (1-\kappa)\kappa\frac{|2\nabla\varphi(\varrho_0)|^2}{2}\right)\,\mathrm{d}x + \int_\Omega \varrho_0 e(\varrho_0)\,\mathrm{d}x,$$

where we have introduced $e(\varrho)$ defined as

$$\frac{\varrho^2\mathrm{d}e(\varrho)}{\mathrm{d}\varrho} = p(\varrho).$$

κ-entropy may be used to construction of κ-entropy solutions with a simple construction scheme for the compressible system with extra terms (singular pressure or drag terms).

D.B., B. Desjardins, E. Zatorska '14

Extensions to density dependent viscosities compressible NS equations

- Global weak solution of the CNS depending on κ through the κ-entropy
- Non-linear extension of hypocoercivity property known for linearized CNS.
- Interesting framework for numerics:
  In progress with P. Noble and J.-P. Vila

Euler-Korteweg systems (joint work with F. Couderc, P. Noble, J.–P. Vila)

$$[E-K] \quad \begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) &= \mathrm{div}\mathbf{K} \end{aligned}$$

with

$$\mathbf{K} = \Big(\rho \mathrm{div}(K(\rho)\nabla\rho) + \frac{1}{2}(K(\rho) - \rho K'(\rho))|\nabla\rho|^2\Big)\mathrm{Id} - K(\rho)\nabla\rho \otimes \nabla\rho$$

where $K(\rho)$ is the capillary coefficient. Note that

$$\mathbf{K} = \rho(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\, ds) = \mathrm{div}\Big(F(\rho)\nabla\nabla\varphi(\rho)\Big) - \nabla\Big((F(\rho) - F'(\rho)\rho)\Delta\varphi(\rho)\Big)$$

with $\sqrt{\rho}\varphi'(\rho) = \sqrt{K(\rho)}$), $F'(\rho) = \sqrt{F(\rho)\rho}$.

$\Longrightarrow$ extended formulation of the Euler-Korteweg system with $w = \nabla\varphi(\rho)$.

$\Longrightarrow$ Stable schemes under hyperbolic CFL condition.

Numerical simulations

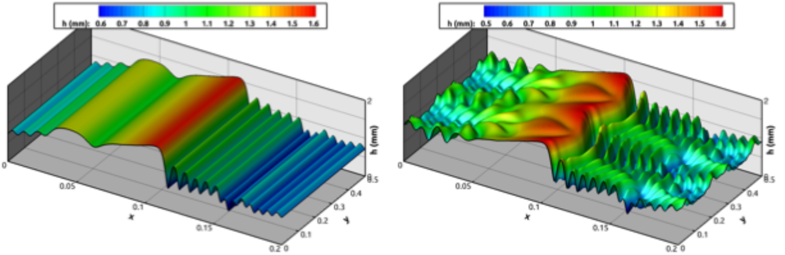


Figure 1. Numerical simulation of a roll-wave in presence of surface tension. On the left: one dimensional roll-wave without transverse perturbations. On the right: a two-dimensional roll-wave

Relative κ entropy (joint work with P. Noble, J.–P. Vila)

A useful tool to measure distance between quantities!!

Idea: Assume $e(u,\rho) = e_1(u) + e_2(\rho)$, calculate

$$E(u,\rho) := e(u,\rho) - e(U,r) - \nabla e_1(u)\cdot(u-U) - e_2'(r)\cdot(\rho - r)$$

If global strict-convexity then control of

$$|u-U|^2 + |\rho - r|^2$$

See for instance:
C. Dafermos, R. Di Perna, H.T. Yau, Y. Brenier, C. Bardos, F. Otto, A. Tzavaras, L. St Raymond........

Let

$$E(\rho, \mathbf{v}, \mathbf{w} \big| r, V, W) = \frac{1}{2} \int_\Omega \varrho(|\mathbf{w}-W|^2+|\mathbf{v}-V|^2)\mathrm{d}x + \int_\Omega (F(\varrho)-F(r)-F'(r)(\varrho-r))\mathrm{d}x$$

We calculate

$$\begin{aligned}
&I = E(\rho, v, w \big| r, V, W)(\tau) - E(\rho, v, w \big| r, V, W)(0) \\
&+2\kappa\mu \int_0^\tau \int_\Omega \varrho[A(\mathbf{v}-V)|^2 + 2\mu \int_0^\tau \int_\Omega \varrho|D(\sqrt{(1-\kappa)}(\mathbf{v}-V) - \sqrt{\kappa}(\mathbf{w}-W))|^2 \\
&+2\kappa\mu \int_0^\tau \int_\Omega \varrho\Big[p'(\varrho)\nabla \log \varrho - p'(r)\nabla \log r\Big] \cdot \Big[\nabla \log \varrho - \nabla \log r\Big]
\end{aligned}$$

for all $\tau \in [0, T]$ and for any pair of test functions

$$r \in C^1([0, T] \times \overline{\Omega}), \qquad r > 0, \qquad V, W \in C^1([0, T] \times \overline{\Omega}).$$

$$
\begin{aligned}
I \leq & \int_0^\tau \int_\Omega \varrho \Big(((\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w}) \cdot \nabla W) \cdot (W - \mathbf{w}) \\
& \qquad\qquad + ((\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w}) \cdot \nabla V) \cdot (V - \mathbf{v}) \Big) \\
& + \int_0^\tau \int_\Omega \varrho \Big(\partial_t W \cdot (W - \mathbf{w}) + \partial_t V \cdot (V - \mathbf{v}) \Big) + \int_0^\tau \int_\Omega \partial_t F'(r)(r - \varrho) \\
& - \int_0^\tau \int_\Omega \nabla F'(r) \cdot \Big[\varrho(\mathbf{v} - \sqrt{\frac{\kappa}{(1-\kappa)}}\mathbf{w}) - r(V - \sqrt{\frac{\kappa}{(1-\kappa)}} W) \Big] \\
& + \int_0^\tau \int_\Omega (p(r) - p(\varrho)) \operatorname{div}(V - \sqrt{\frac{\kappa}{(1-\kappa)}} W) \\
& - \kappa \int_0^\tau \int_\Omega p'(\varrho) \nabla \varrho \cdot [2\mu \frac{\nabla r}{r} - \frac{1}{\sqrt{(1-\kappa)\kappa}} W] \\
& + 2\mu \int_0^\tau \int_\Omega \varrho \Big(D(\sqrt{(1-\kappa)} V) - \nabla(\sqrt{\kappa} W) \Big) : \qquad\qquad (1) \\
& \qquad\qquad \Big(D(\sqrt{(1-\kappa)}(V - \mathbf{v})) - \nabla(\sqrt{\kappa}(W - \mathbf{w})) \Big) \\
& + 2\kappa\mu \int_0^\tau \int_\Omega \varrho A(V) : A(V - \mathbf{v}) + 2\kappa\mu \int_0^\tau \int_\Omega \frac{\varrho}{r} p'(r) \nabla r \cdot (\frac{\nabla r}{r} - \frac{\nabla \varrho}{\varrho}) \\
& + 2\sqrt{\kappa(1-\kappa)}\mu \int_0^\tau \int_\Omega \varrho \Big[A(W) : A(\mathbf{v} - V) - A(\mathbf{w} - W) : A(V) \Big]
\end{aligned}
$$

Using that

$$
\begin{aligned}
&\varrho[p'(\varrho)\nabla \log \varrho - p'(r)\nabla \log r] \cdot [\nabla \log \varrho - \nabla \log r] = \\
&\varrho p'(\varrho)|\nabla \log \varrho - \nabla \log r|^2 + \nabla[p(\varrho) - p(r) - p'(r)(\varrho - r)] \cdot \nabla \log r \\
&-\big[\varrho(p'(\varrho) - p'(r)) - p''(r)(\varrho - r)r\big]\big|\nabla \log r\big|^2,
\end{aligned}
$$

Let (r, V, W) strong solution with $r > 0$ then

$$
E(\rho, v, w/r, V, W)(\tau) - E(\rho, v, w/r, V, W)(0)
$$

$$
\leq \int_0^\tau \Big[A(r, V, W)(E(\rho, v, w/r, V, W)\Big]
$$

where

$$
A(r, V, W)(= \|\nabla V\|_{L^\infty(\Omega)} + \|\nabla W\|_{L^\infty(\Omega)} + \|\nabla \log r\|^2_{L^\infty(\Omega)} + \|\Delta \log \rho\|_{L^\infty(\Omega)})
$$

$\Longrightarrow$ Weak strong uniqueness.

$\Longrightarrow$ useful to compare continuous to discrete solution.

An other application of the relative entropy:

Definition. The pair $(\overline{\varrho}, \overline{u})$ is a dissipative solution of the compressible Euler equations if and only if $(\overline{\varrho}, \overline{u})$ satisfies the relative energy inequality

$$E(\overline{\varrho}, \overline{u}, 0 \big| r, U, 0)(t) \leq E(\overline{\varrho}, \overline{u}, 0 \big| r, U, 0)(0) \exp\Big[c_0(r) \int_0^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)} d\tau\Big]$$

$$+ \int_0^t \exp\Big[c_0(r) \int_s^t \|\mathrm{div}\, U(\tau)\|_{L^\infty(\Omega)}\Big] \int_\Omega \varrho E(r, U) \cdot (U - \overline{u})\, dxds$$

for all smooth test functions (r, U) defined on $[0, T] \times \overline{\Omega})$ so that r is bounded above and below away from zero and (r, w) solves

$$\partial_t r + \mathrm{div}(rU) = 0,$$
$$\partial_t U + U \cdot \nabla U + \nabla F'(r) = E(r, U)$$

for some residual $E(r, U)$.

For constant viscosity: See C. Bardos and T. Nguyen.

For incompressible Euler: See P.–L. Lions' book.

For degenerate viscosity, using the κ-relative entropy
Convergence of global κ-entropy solutions of NS to dissipative solution of Euler

Relative entropy may be used for singular limit behavior:

- Low Mach Number limit
- High-rotating fluid
- Inviscid limit
- Low Weissenberg effect
- ....

Today:

- Low Mach number limit with large heat release
  (Examples: Combustion, pollutant, avalanches)
- Compressible Navier-Stokes with degenerate viscosities
  (Example: Shallow-water)
- Effect of bathymetry or stratification in the low mach number limit system
  (Anelastic limit examples: lake equation, Durran model for atmopshere).
- Compressible Euler with dispersive term
  (shallow-water with surface tension, Quantum Euler).

Tomorrow:

- Compressible Navier-Stokes equations:
  Non-monotone pressure, anisotropy in stress tensor
  Virial pressure (stellar), biology, eddy viscosity (geophysics)
- Multi-fluid system derivation.
  Baer-Nunziato model ? Justification of relaxation terms..
  Aerated flows, nuclear industry, champagne (Stéphanie.....)

Thank you for your attention!