Active Matter: Collective Motion of Interacting Self-Propelled Particles.

Simple generic models of active matter and their continuum coarse-grained approximations

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Outline

- **Brief overview of experiments**
- **Vicsek Models**
- **Continuum Models**
- **Novel Material Properties**

Active matter

- Definition: Energy is spent *locally* to produce directed, persistent, non-random motion

- Examples abound: in biology (animals, cells, motor proteins…) but not only (micro- and nano-swimmers, ‘smart’ colloids, robots…)

- Largely unexplored, novel collective properties

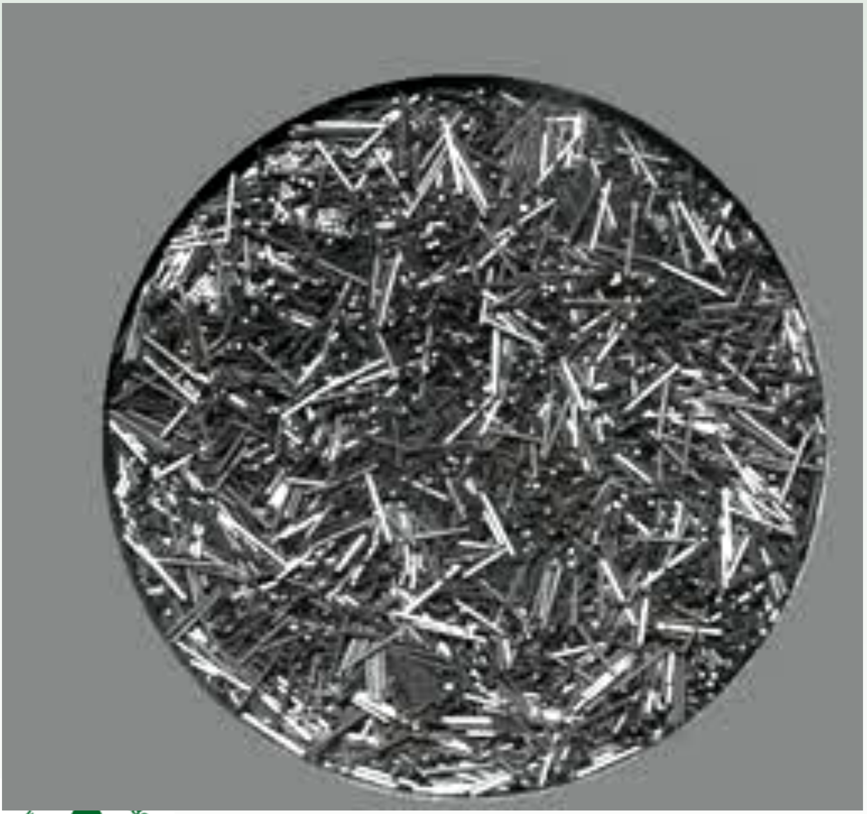
- “Swarm intelligence”, self-organized dynamical structures, new materials…

Most convincing experiments

- Shaken granular particles, rolling colloids
- Microtubule motility assay
- Bacterial suspensions
- And, of course, bird flocks, fish schools, animal swarms

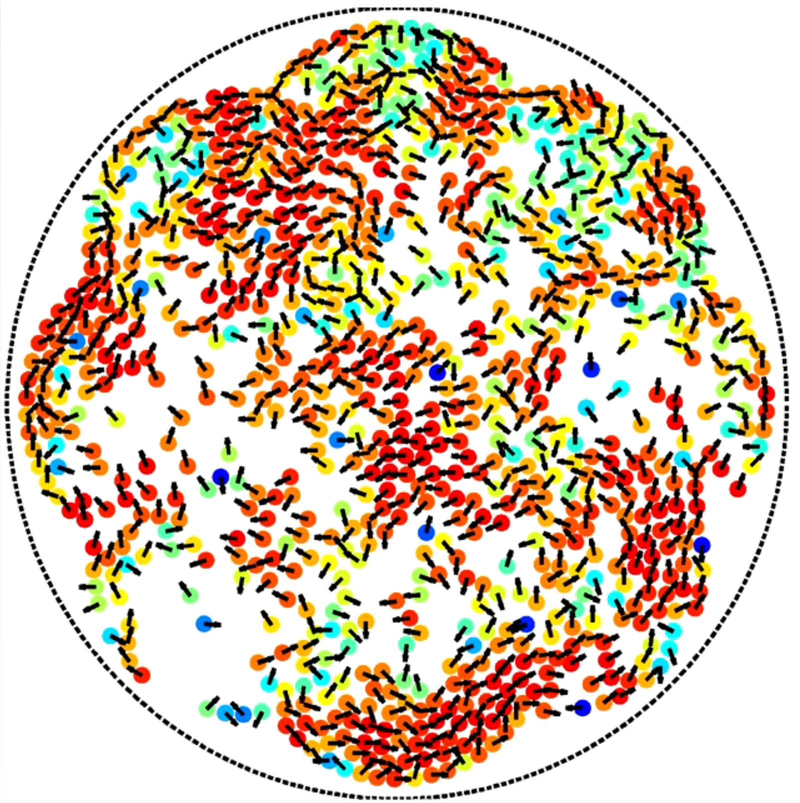
Shaken Granular Particles

- Vortices in vertically vibrated rods



Blair and Kudrolli

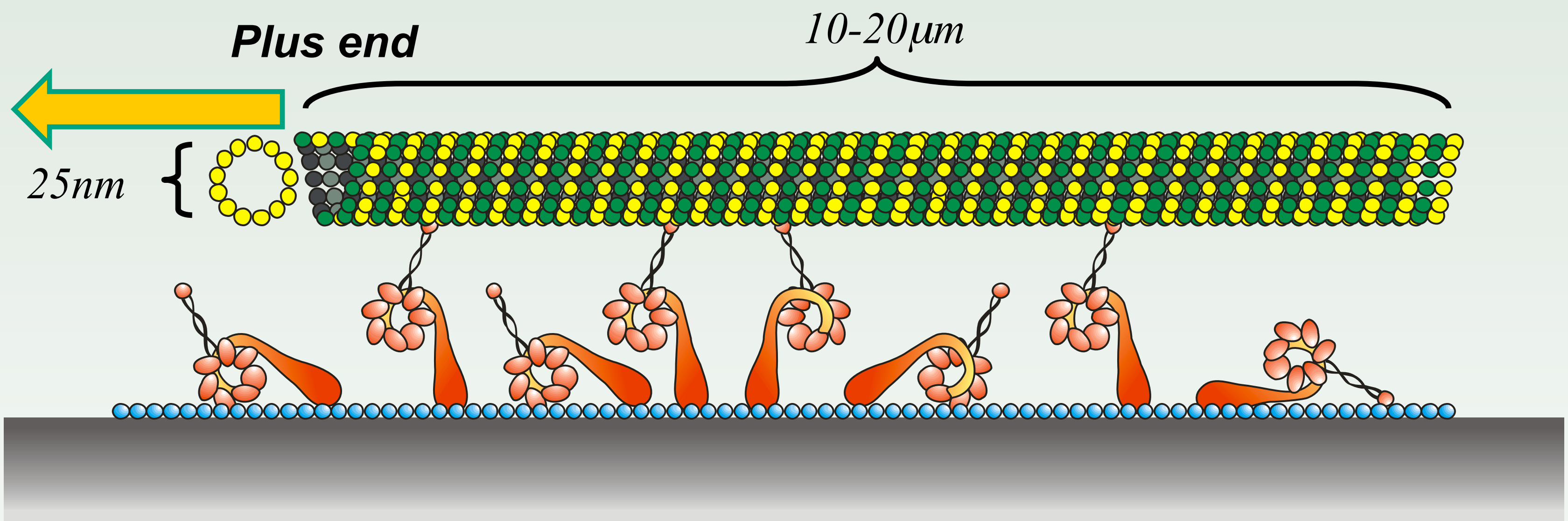
- Swarming of vibrated polar disks



Deseigne et al

In vitro motility assay: dyneins + microtubules

(Sumino et al.)

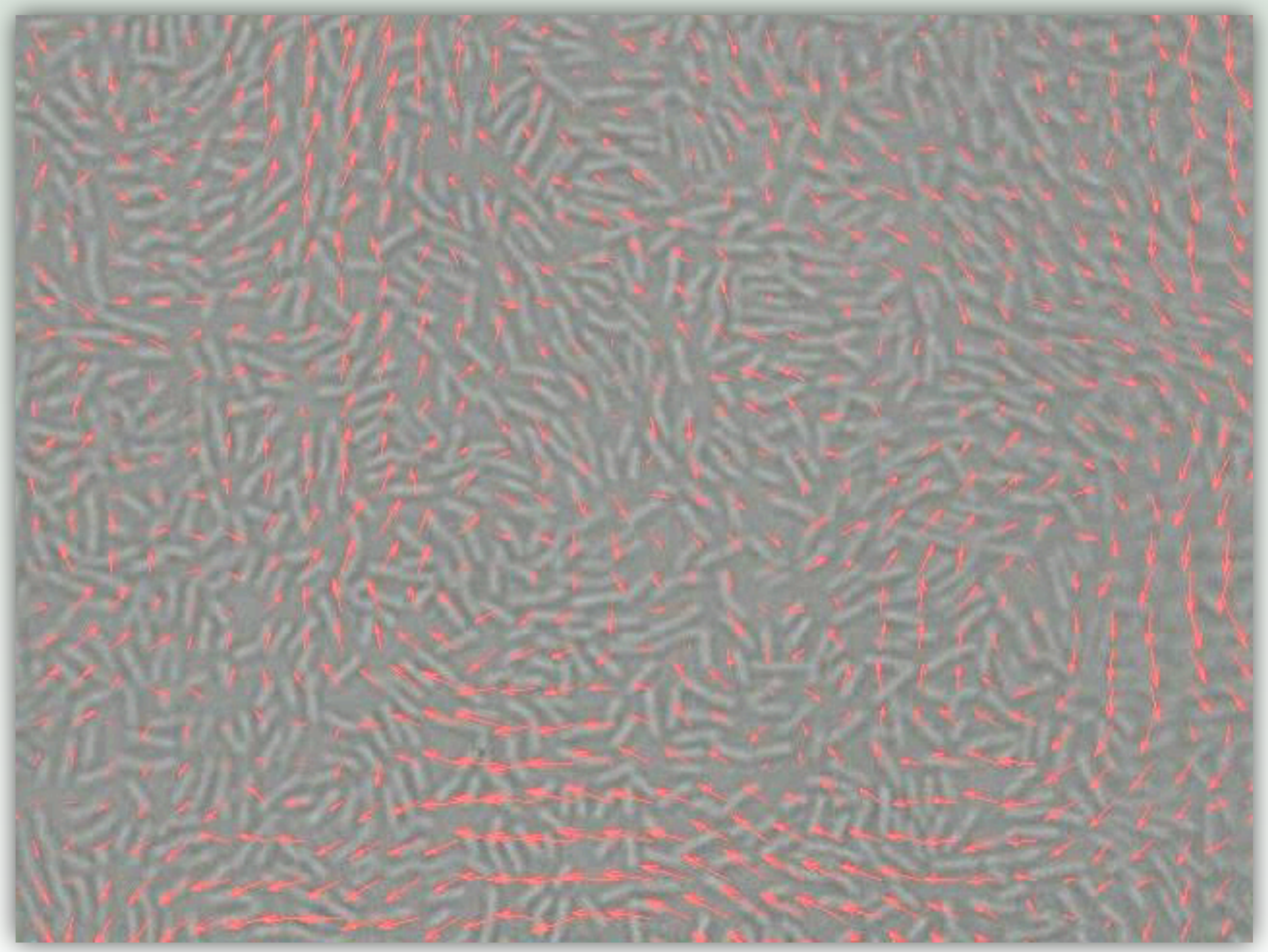


- *Dynein-c motor proteins, grafted on a substrate, move stabilized microtubules*
- *with high density of motors (1000/μm^2), smooth, constant-speed motion of single MT*

Bacterial Suspension

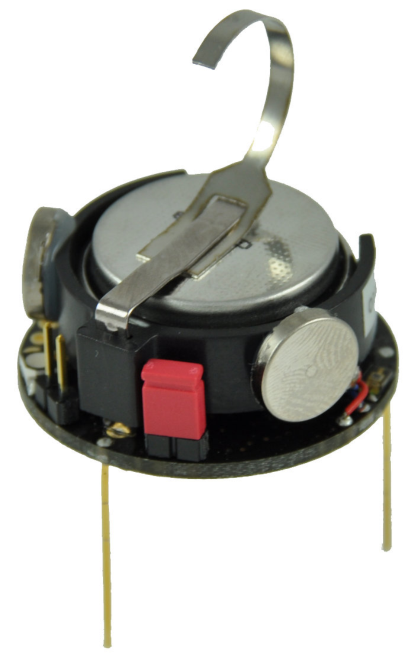
Inelastic collisions

Bacterial Turbulence



Sokolov, Goldstein, Kessler, I.A PRL (2007)

Swarm Intelligence



- A low-cost scalable robot system demonstrating collective behavior
- http://www.eecs.harvard.edu/ssr/projects/progSA/kilobot.html
- Cost – 50$ per robot (but you may get a discount)
- Includes differential drive locomotion, on-board computer, neighbor-neighbor communication

Swarm Intellegence

Kilobot project, Harvard University

Vicsek-style models:

- Constant-speed point particles
- Discrete time update algorithm
- local alignment within a certain distance distance
- In competition with noise
- 2 main parameters: global density and noise strength

Vicsek-style models:

3 possible classes depending on symmetry:

- *Polar particles with ferromagnetic alignment (original VM)-birds, swimming bacteria*
- *Apolar nematic particles with nematic alignment ("active nematics")- suspensions of microtubules and molecular motors, myxobacteria*
- *Polar particles with nematic alignment ("self-propelled rods") – gliding assay of microtubules*

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The Original Vicsek Model

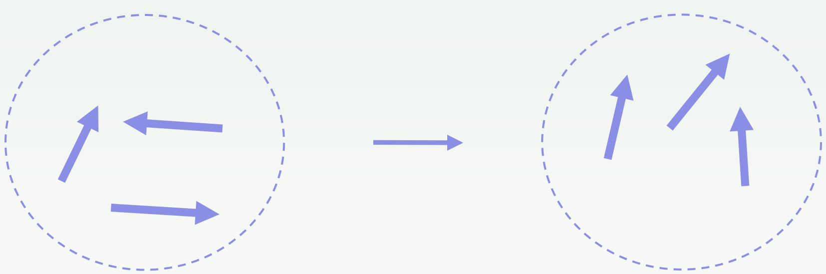
- Driven overdamped (no inertia effects) dynamics (velocity rather than acceleration proportional to applied force)

•Strictly local interaction range

•Alignment according to average direction of the neighbors

•Simple update algorithm for the position/orientation of particles
•Not necessarily reproduce observed phenomenology

1. Polar orienting interaction in a noisy environment



2. Streaming: motion along the polar direction

The Original (Polar) Vicsek Model (angular noise)

- Particles with position x_i, velocity v_i and orientation θ_i

- Velocity v_i points in the direction θ_i $\mathbf{v}_i = v_0(\cos(\theta), \sin(\theta)), v_0$ =const

- At each time step particle assumes the average direction of motion in its local neighborhood S(i) (e.g. a circle of radius R)

- Δt time step, ξ random variable from uniform distribution $[-\eta, \eta]$

$$\theta_i(t+\Delta t) = \left\langle \theta(t) \right\rangle_{S(i)} + \xi$$

$$\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t, \mathbf{v}_i = v_0(\cos(\theta), \sin(\theta))$$

Vicsek et al, PRL 1995

How to understand "average direction"

- We should not average the angles θ_i
- We should average unit vectors of orientation $\boldsymbol{\tau}=(\cos(\theta),\sin(\theta))$

$$\tau_x + i\tau_y = e^{i\theta}$$

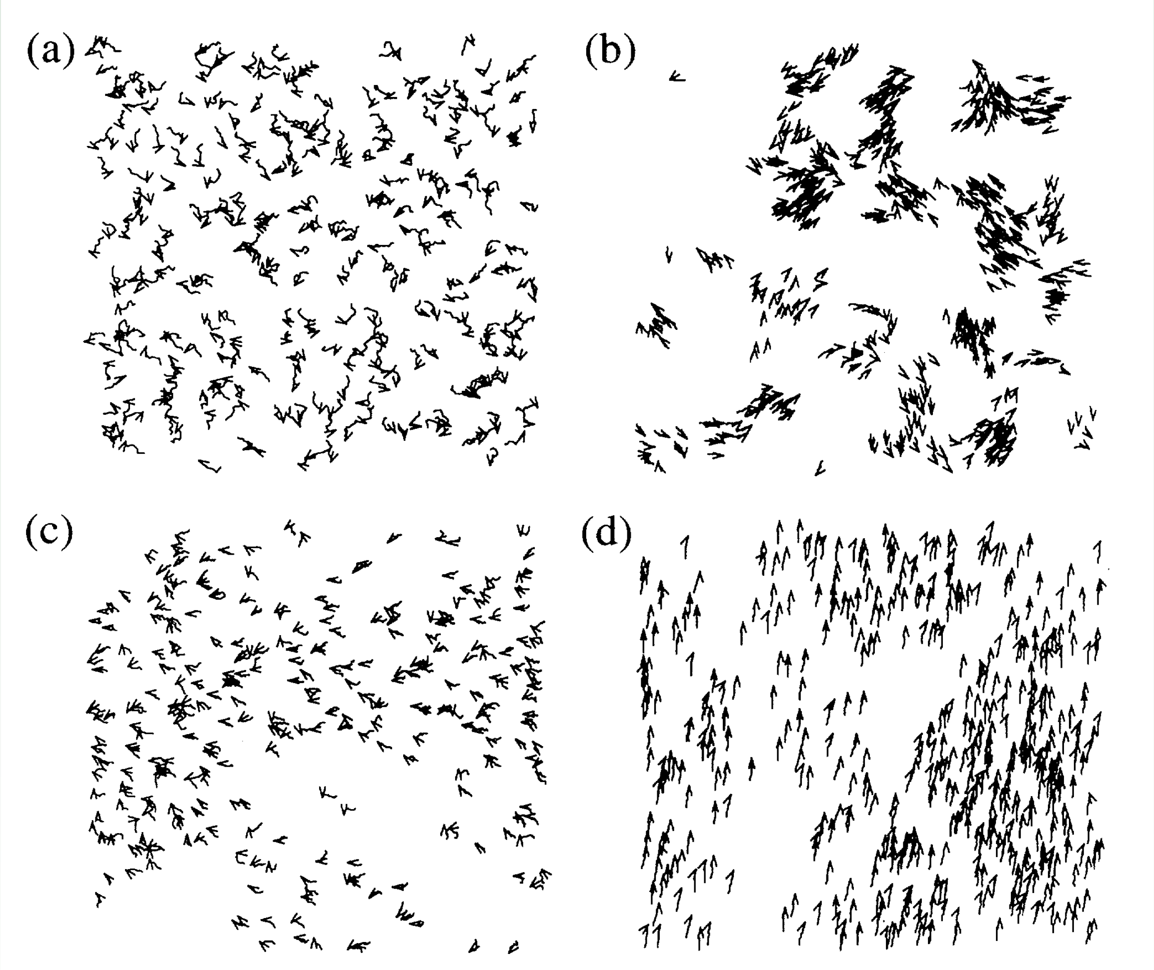
$$\left\langle \theta(t) \right\rangle_{S(i)} = \arg\left[\frac{1}{N}\sum_j e^{i\theta_j}\right]$$

- Summation is taken over all j particles within radius R of a particle i

Vicsek et al, PRL 1995

Results from the original Vicsek paper

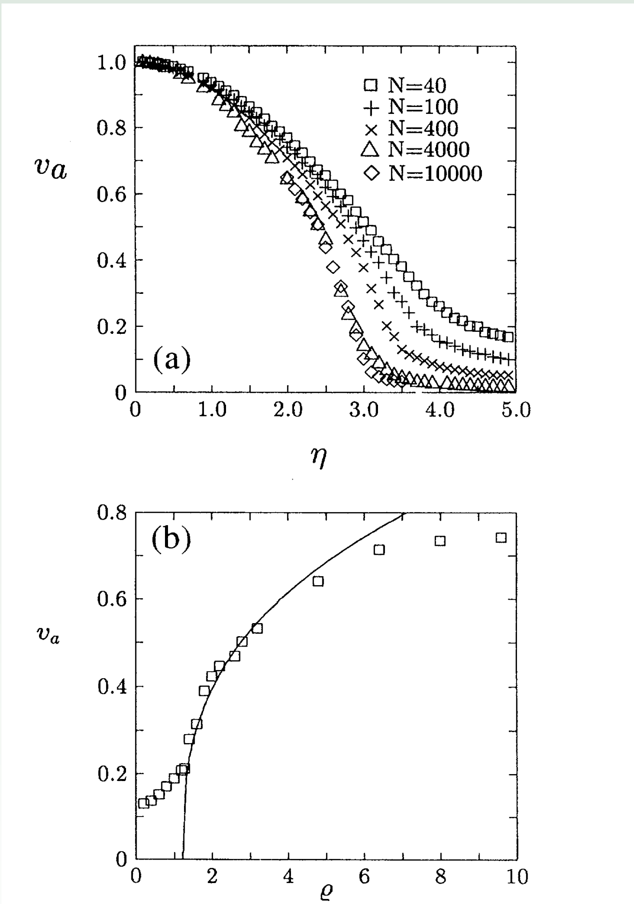
Phases for different density and noise



Vicsek et al, PRL 1995

Results from the original Vicsek paper

Dependence of the OP on the noise and density



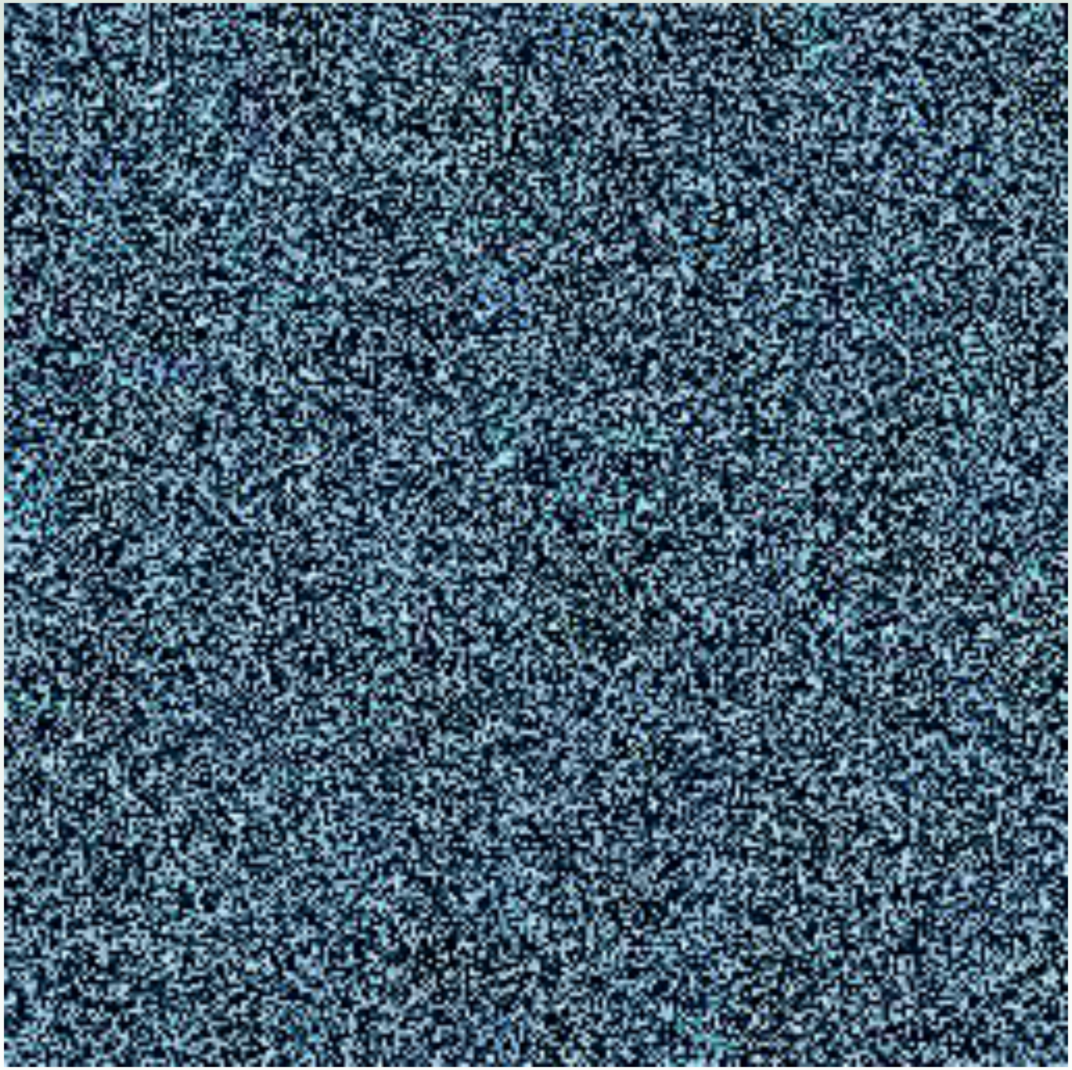
Conjecture on continuous phase transition with the scaling exponents β=0.45 & δ=0.35

$$v_a \sim [\eta_c(\rho) - \eta]^\beta \quad \text{and} \quad v_a \sim [\rho - \rho_c(\eta)]^\delta ,$$

Vicsek et al, PRL 1995

Simulations of the Vicsek model: Vicsek bands

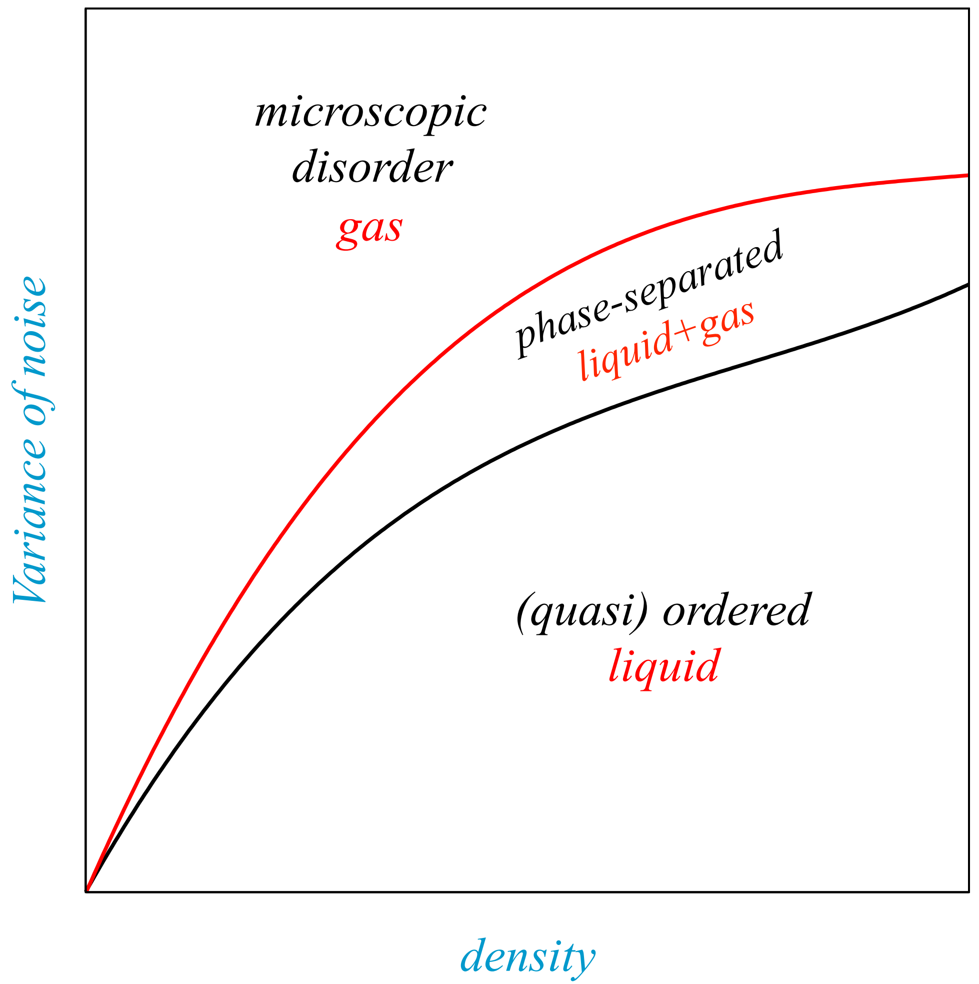
1,000,000 particles



Chate and Gregoire,PRL 2004

Common features at microscopic level:

- Disordered gas phase at low density/strong noise
- (Quasi-) ordered liquid phase at high density/low noise, with superdiffusion
- In between: phase-separated inhomogeneous phase with dense and ordered regions



Vicsek model with the vectorial noise

$$\theta_j^{i+1} = \arg\left[\sum_k e^{i\theta_k} + \eta N_j e^{i\xi_j}\right]$$

N_j – current # of neighbors

ξ_j – delta-correlated noise $\xi \in [-\pi, \pi]$

η-noise amplitude

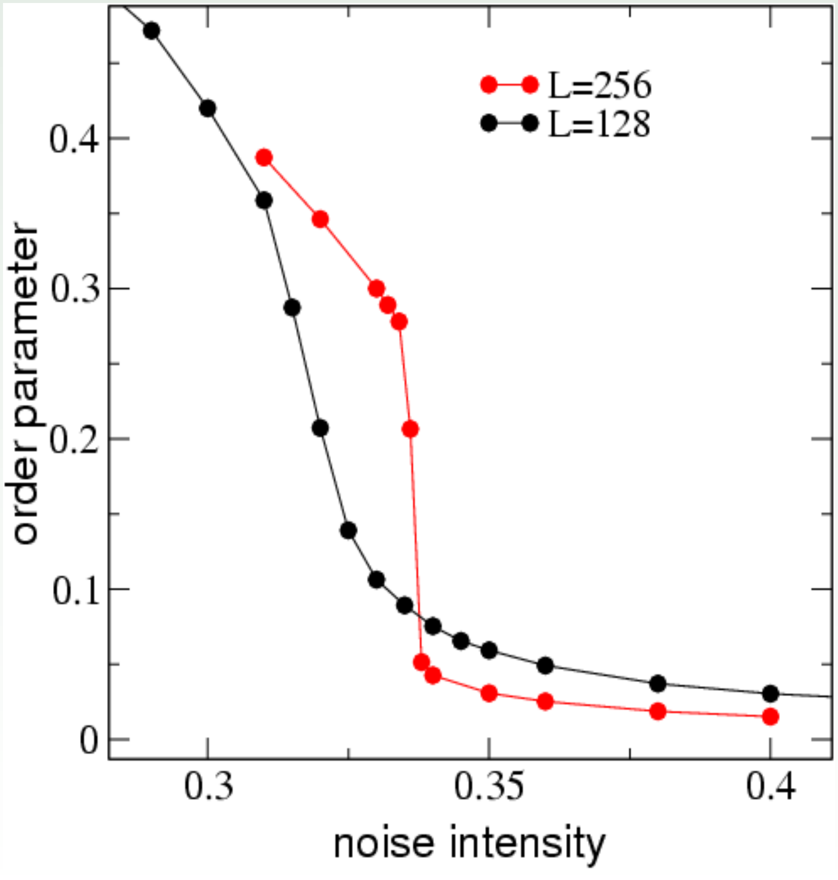
Chate and Gregoire, PRL 2004

Discontinuons transition

Order parameter:

$$\varphi(t) = \frac{1}{N}\left|\sum_{i=1}^{N} v_i(t)\right|$$

at large size, discontinuous transition



Discontinuons transition in the orignal Vicsek Model (angular noise)

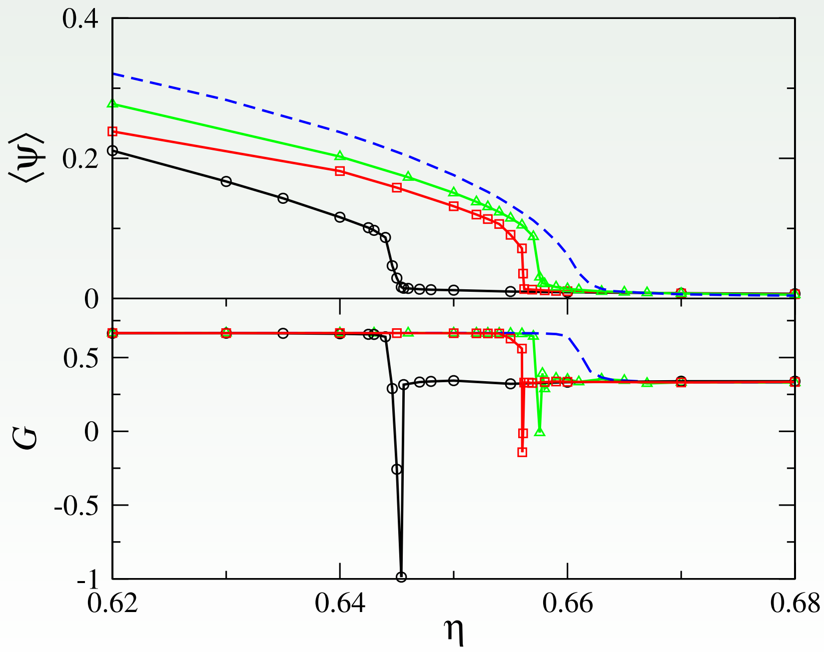
Order parameter:

$$\phi(t) = \frac{1}{N}\left|\sum_{i=1}^{N} v_i(t)\right|$$

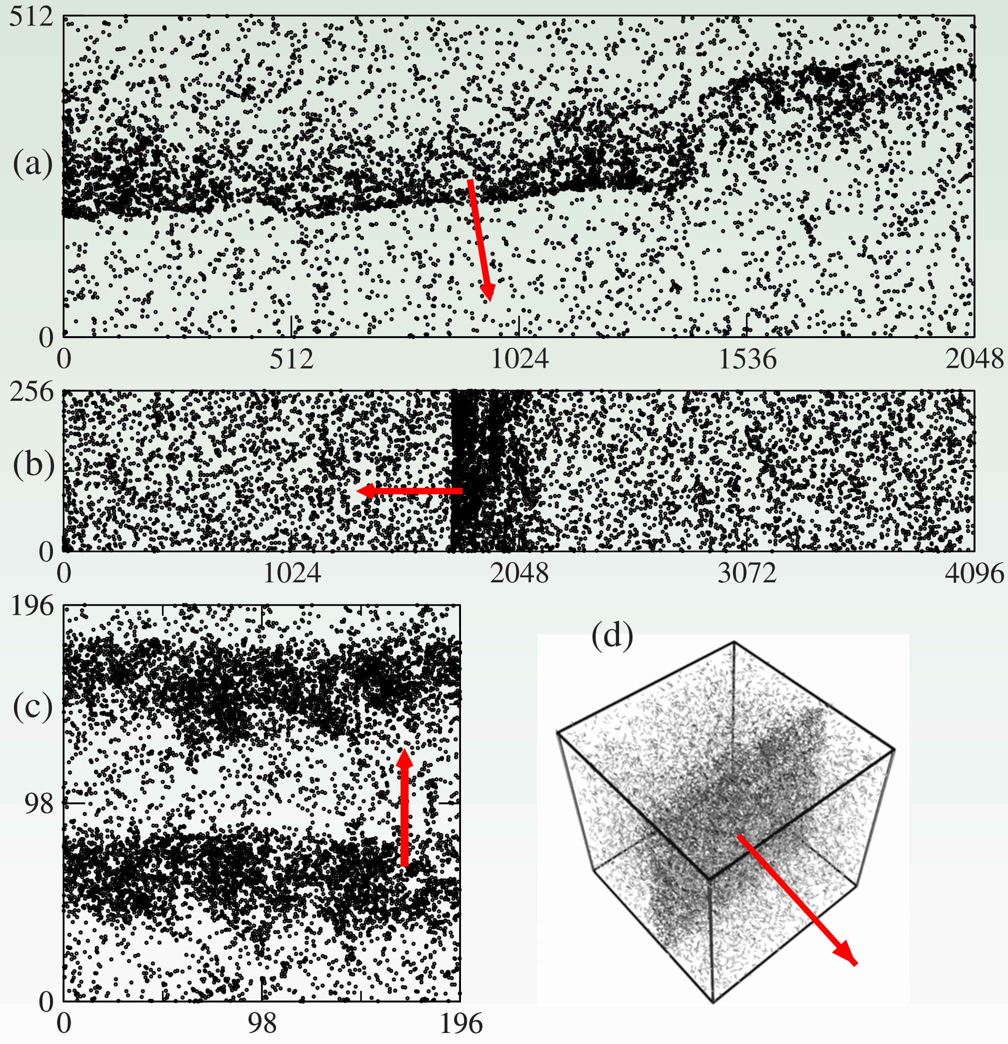
Binder cumulant

$$G = 1 - \langle \phi^4 \rangle / 3\langle \phi^2 \rangle^2$$

at large size, discontinuous transition

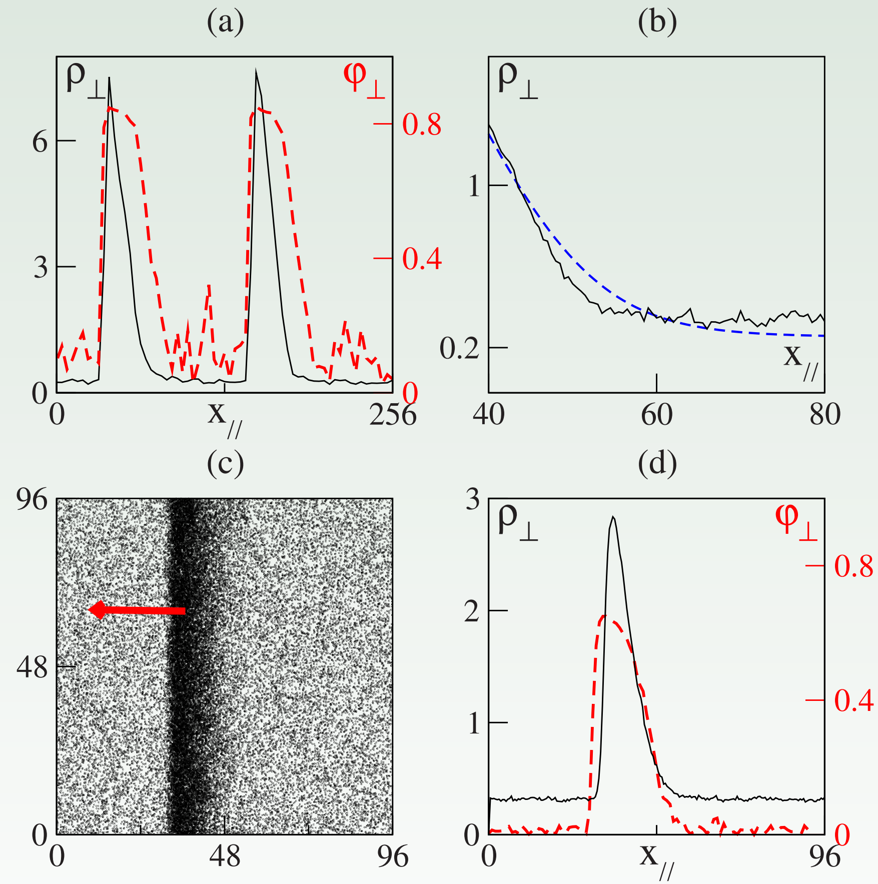


Vicsek bands (solitons) in 2D and 3D



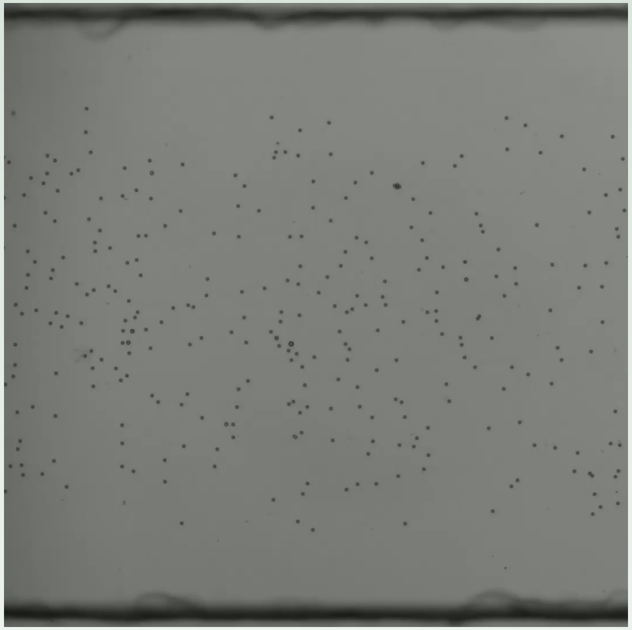
Chate t al, PRE 2008

Density profiles

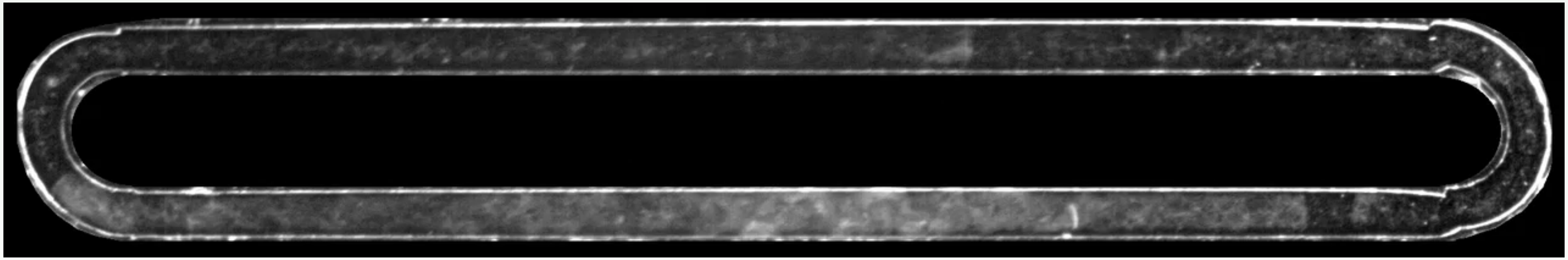


Chate et al, PRE 2008

Vicsek bands in experiments



Rolling colloids: Quincke effect



Bricard et al, Nature, 2013

Flocking of birds

Starlings

Vicsek Model with local attraction

Formation of the dense cohesive flocks due to local attraction between the particles

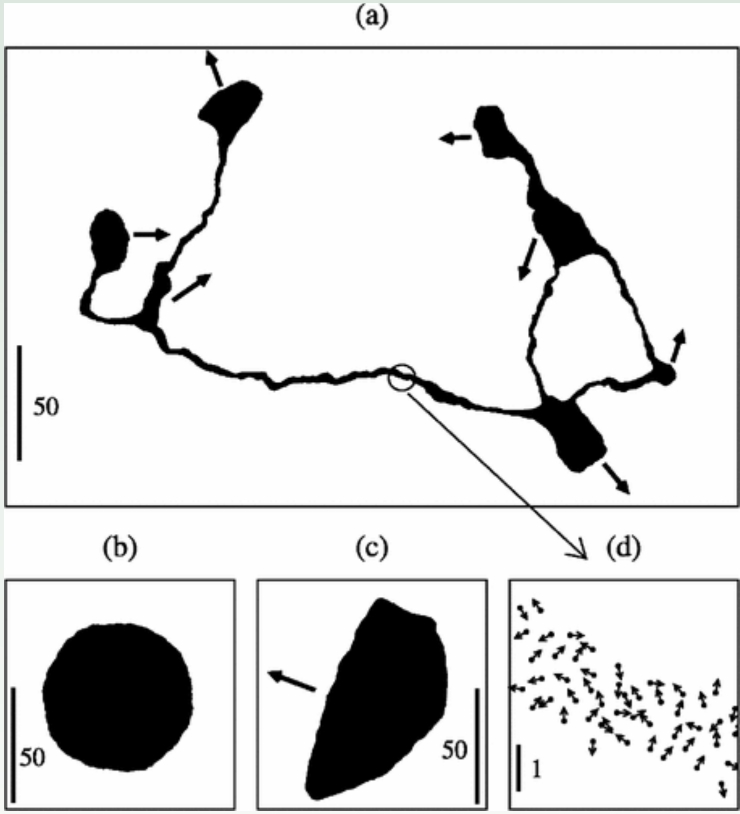
$$\theta_j^{i+1} = \arg\left[\alpha \sum_k e^{i\theta_k} + \beta \sum_k f_{jk} e^{i\theta_{kj}} + \eta N_j e^{i\xi_j}\right]$$

N_j – current # of neighbors

α, β – strength of aligment and adhesion

f -repulsive for very small r and atractive for larger r

$$f_{jk} = \begin{cases} -\infty & \text{if } r_{jk} < r_c \\ \frac{1}{4}\frac{r_{jk}-r_e}{r_a-r_e} & \text{if } r_c < r_{jk} < r_a \\ 1 & \text{if } r_a < r_{jk} < r_0, \end{cases}$$



Gregoire and Chate, PRL 2004

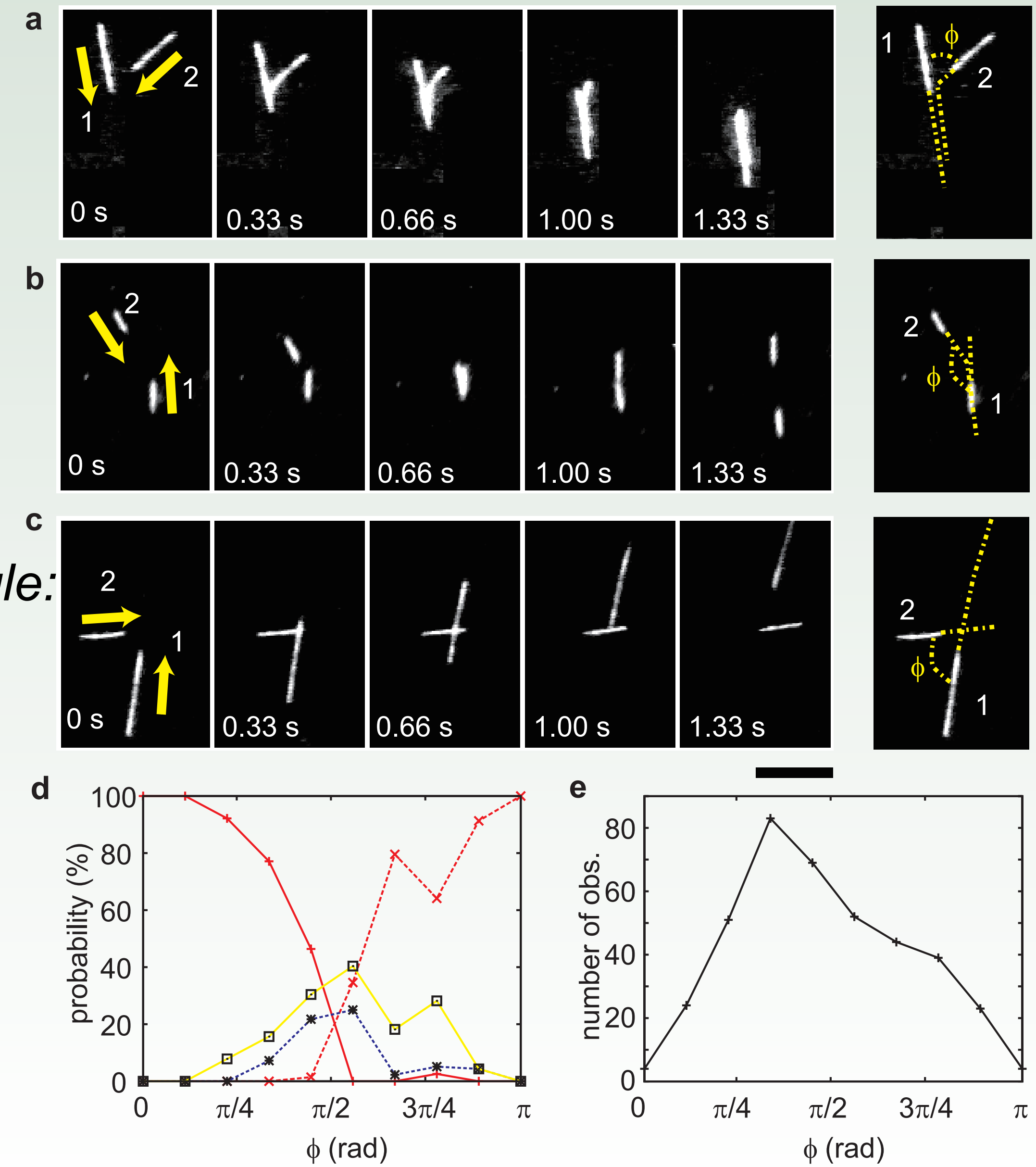
Near-perfect nematic (non-polar) alignment via collisions

Acute incoming angle:
Complete alignment

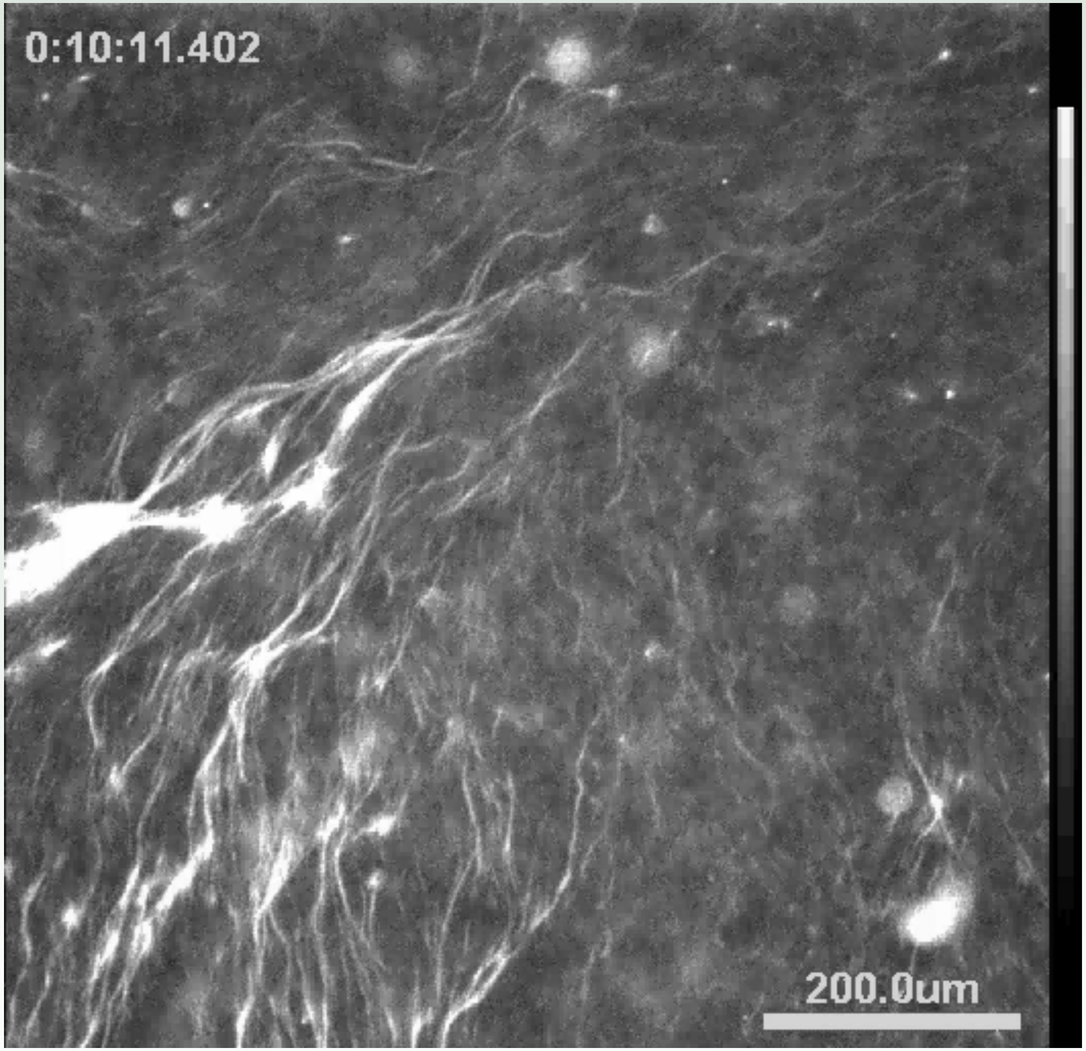
Obtuse incoming angle:
Complete anti-alignment

(Near-) right incoming angle:
Crossing (or stopping)

Statistics over some 400 binary collisions

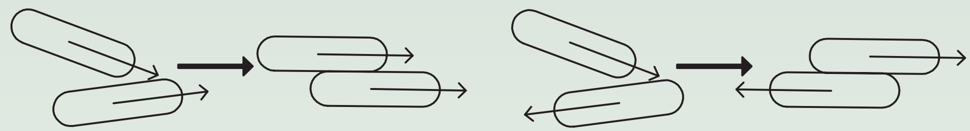


Formation of Giant Vortices



Vicsek Model with apolar interaction (apolar rods)

Reversal of particle direction does not change outcome of interaction



Modified average

$$\left\langle \theta(t) \right\rangle_{S(i)} = \arg\left[\frac{1}{N_i} \sum_j \mathrm{sign}[\cos(\theta_j - \theta_i)] e^{i\theta_j} \right]$$

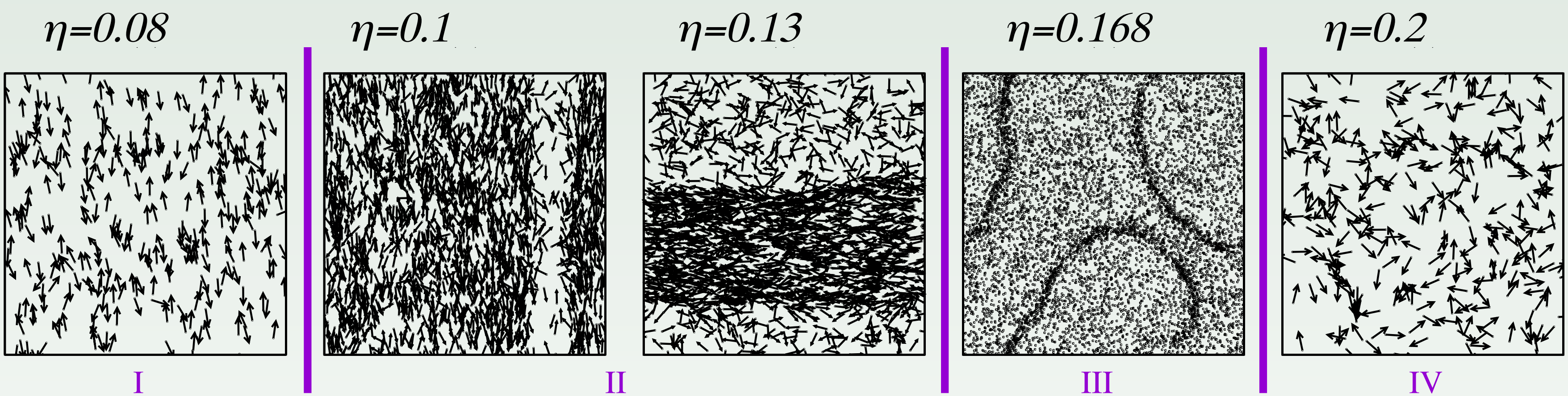
Modified Model

$$\theta_j^{t+1} = \arg\left[\sum_{k\sim j} \mathrm{sign}[\cos(\theta_k^t - \theta_j^t)] e^{i\theta_k^t} \right] + \eta \xi_j^t$$

$$\mathbf{r}_j^{t+1} = \mathbf{r}_j^t + v_0 e^{i\theta_k^{t+1}},$$

Ginelli et al, PRL 2010

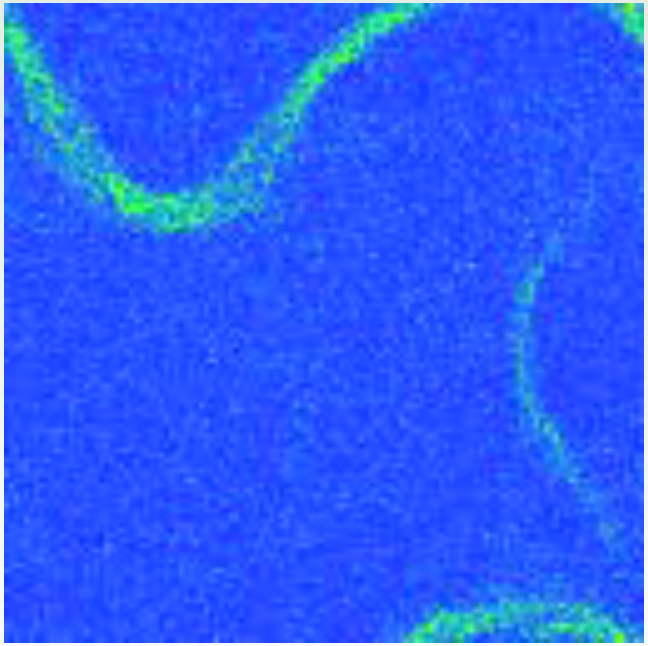
Various phases for different noise levels



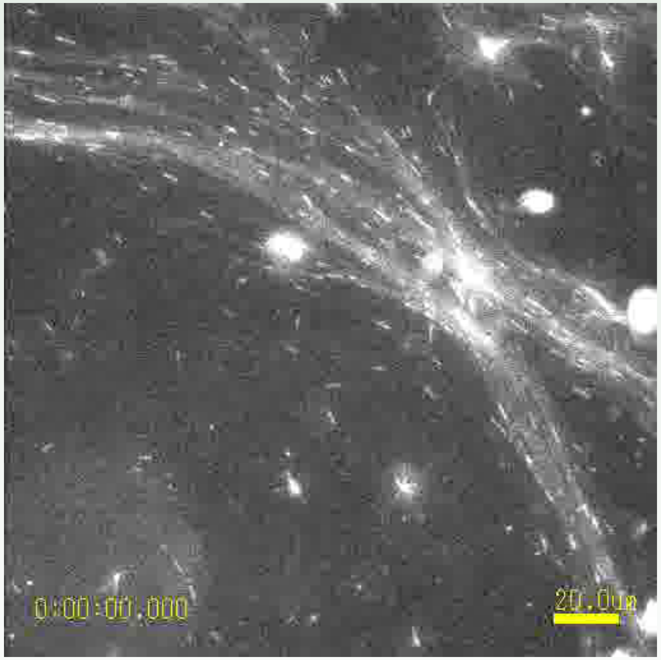
Ginelli et al, PRL 2010

Vicsek Model with the apolar interaction

Formation of bands



Collective motion of millions of microtubules



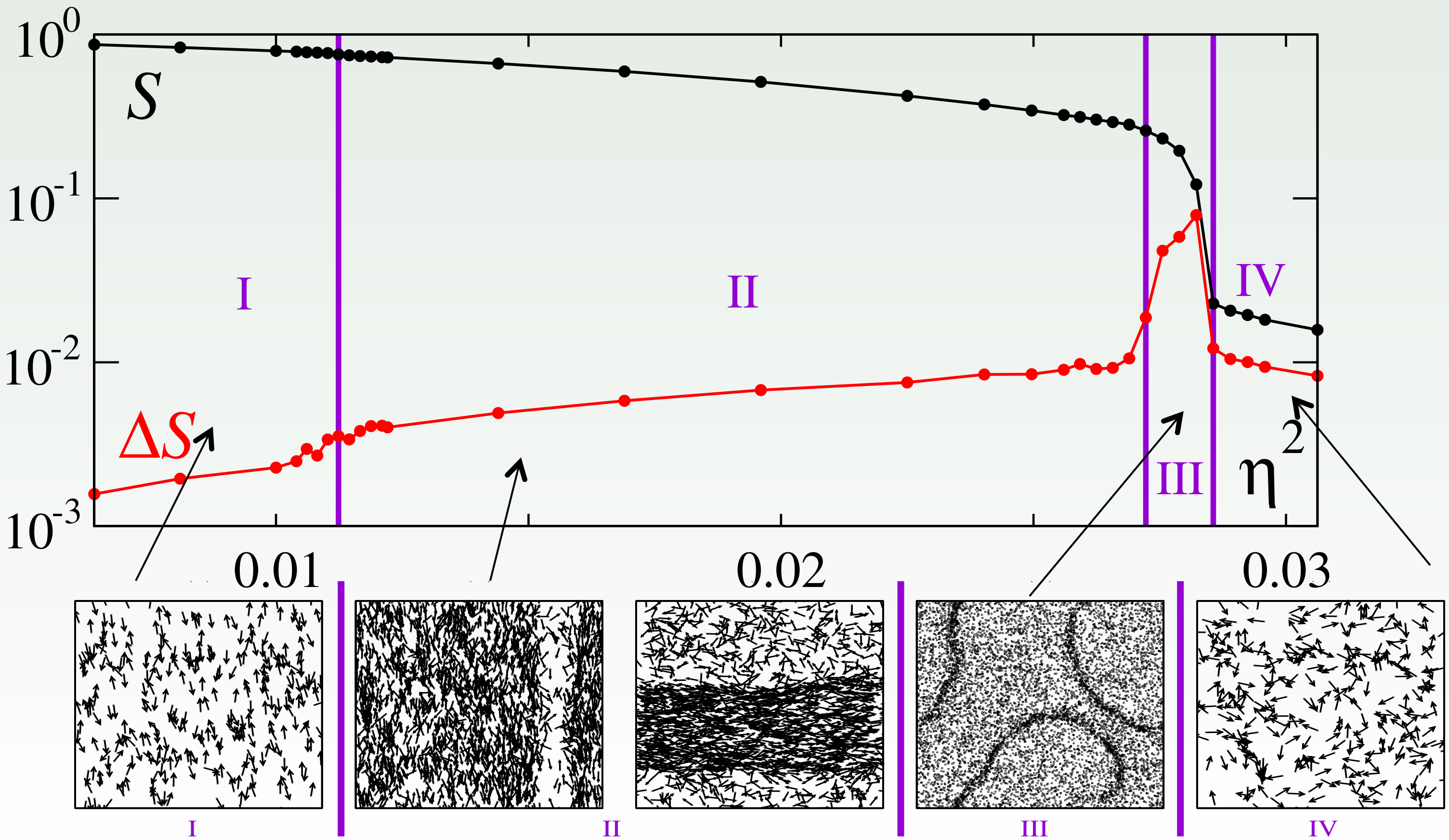
Ginelli et al, PRL 2010

Discontinuons   transition in the apolar model

Nematic (apolar) Order parameter:

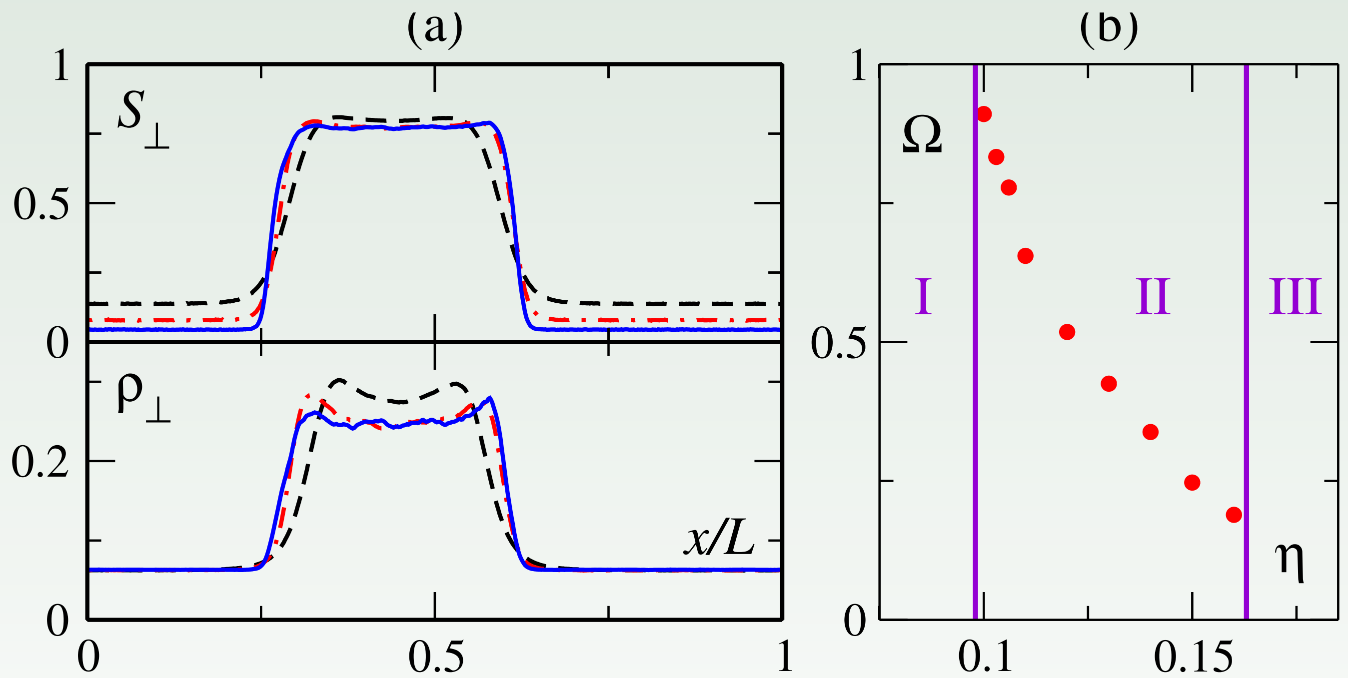
$$S(t) = \frac{1}{N}\left|\sum_{j=1}^{N}\exp(2i\theta_j)\right|$$

at large size, discontinuous transition



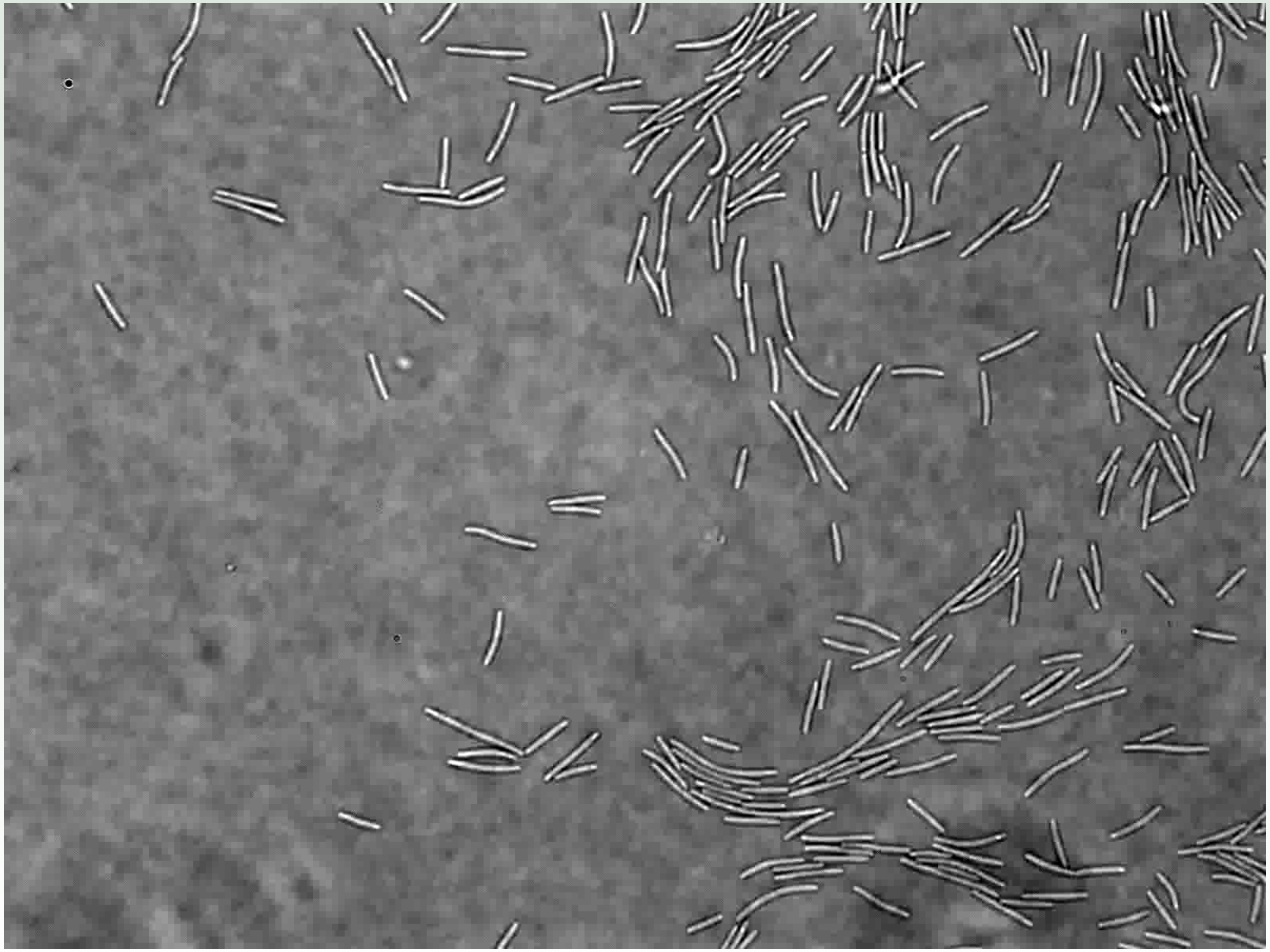
Bands in the apolar model

Ω – surface fraction



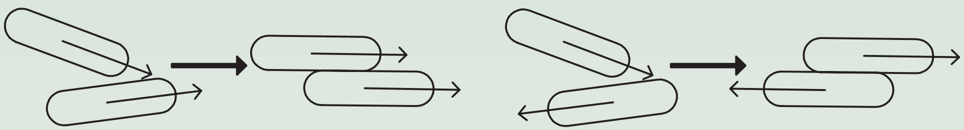
Myxobacteria gliding on the agar surface

Speed – 2-4 μ/min
Direction reversal – 8 min



Vicsek Model for the active nematic (apolar reversing rods)

Sings + or minus are chosen randomly, particles move along the director **n**



$$\theta_j^{i+1} = \frac{1}{2}\arg\left[\frac{1}{N_j}\sum_k e^{i2\theta_k}\right] + \eta\xi_j$$

$$\mathbf{x}_j^{i+1} = \mathbf{x}_j^{i} \pm v_0\mathbf{n}_j^{i}$$

N_j – current # of neighbors

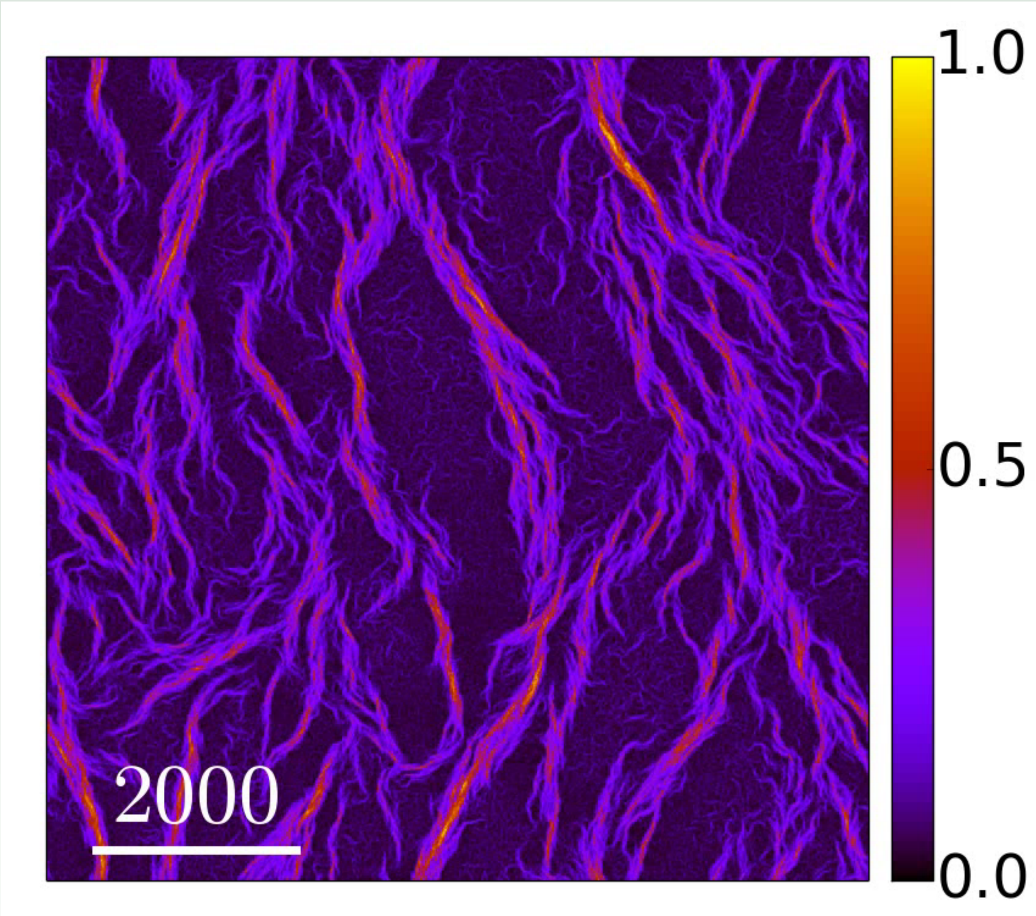
ξ_j – delta-correlated noise $\xi \in [-\pi/2, \pi/2]$

η-noise amplitude

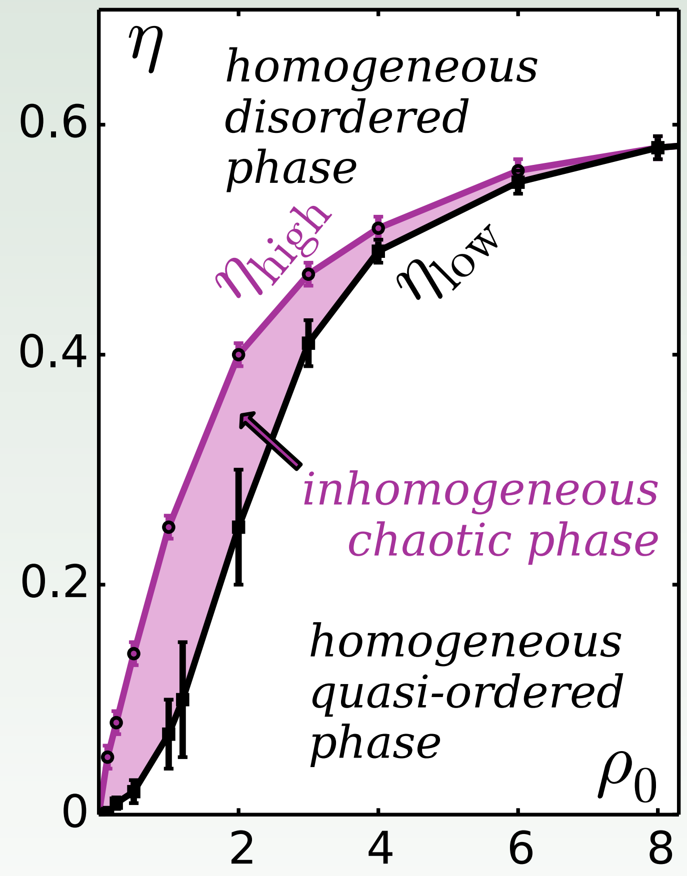
$\mathbf{n}$=(cos(θ), sin(θ))-director

Ngo et al, PRL 2014

Chaotic Dynamics of the bands



Phase diagram



Ngo et al, PRL 2014

ODE version of the Vicsek model

$$\dot{\mathbf{x}}_i = v_0 \mathbf{e}(\theta_i)$$

$$\dot{\theta}_i = -\gamma \frac{\partial U}{\partial \theta_i}(\mathbf{x}_i, \theta_i) + \eta(t)$$

$$U(\mathbf{x}_i, \theta_i) = - \sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \cos(m(\theta_i - \theta_j))$$

m=1 – polar case
m=2 – apolar case

Peruani, Deutsch, Bar, EPJ 2008

Connection with the Vicsek model: large γ limit

$$\dot{\theta}_i = -\gamma \sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(\theta_i - \theta_j) + \eta(t)$$

$$\gamma \sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(\theta_i - \theta_j) = \eta(t)$$

$$\theta_i \approx \arctan\left[\frac{\sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(\theta_j)}{\sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \cos(\theta_j)}\right] + \tilde{\eta}(t)$$

Peruani, Deutsch, Bar, EPJ 2008

Connection with the Kuramoto model for globally coupled oscillators: large r_0 limit

$$\frac{d\theta_i}{dt} = \omega_i + \zeta_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$$\langle \zeta_i(t) \rangle = 0$$

$$\langle \zeta_i(t)\zeta_j(t') \rangle = 2D\delta_{ij}\delta(t - t')$$

ω – *frequencies*

Kuramoto Y (1984). Chemical Oscillations, Waves, and Turbulence. New York, NY: Springer-Verlag

Meanfield approximation

$$\dot{\theta}_i = -\gamma \sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(\theta_i - \theta_j) + \eta(t)$$

Introduce probability P(θ)
Replace average by neighbors by the average over ensemble

$$\sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(\theta_i - \theta_j) \rightarrow \int_{-\pi}^{\pi} \sin(\theta - \theta')P(\theta')d\theta'$$

Fokker-Plank equation

$$\partial_t P = D\partial_\theta^2 P + \gamma \frac{\partial}{\partial\theta}\left(\int_{-\pi}^{\pi} \sin(\theta - \theta')P(\theta')d\theta' P(\theta)\right)$$

Peruani, Deutsch, Bar, EPJ 2008

Linearized equation

Introduce probability $P(\theta)=1/2\pi+\xi$

Linearized Fokker-Plank equation

$$\partial_t \xi = D\partial_\theta^2 \xi + \frac{\gamma}{2\pi}\frac{\partial}{\partial\theta}\left(\int_{-\pi}^{\pi}\sin(\theta-\theta')\xi(\theta')d\theta'\right)$$

$$= D\partial_\theta^2 \xi + \frac{\gamma}{2\pi}\left(\int_{-\pi}^{\pi}\cos(\theta-\theta')\xi(\theta')d\theta'\right)$$

Peruani, Deutsch, Bar, EPJ 2008

Eigenvalues

Expansion of the linear solution $$\xi \sim \xi_n \exp(in\theta + \lambda_n t)$$

Eigenvalues $$\lambda_n = -Dn^2 + \frac{\gamma}{2}\delta_{1,|n|}$$

Instability possible for $$n = \pm 1$$

Peruani, Deutsch, Bar, EPJ 2008

Meanfield approximation for apolar case

$$\dot{\theta}_i = -\gamma \sum_{|\mathbf{x}_i - \mathbf{x}_i| - r_0} \sin(2(\theta_i - \theta_j)) + \eta(t)$$

Fokker-Plank equation

$$\partial_t P = D\partial_\theta^2 P + \gamma \frac{\partial}{\partial \theta} \left(\int_{-\pi}^{\pi} \sin(2(\theta - \theta'))P(\theta')d\theta' P(\theta) \right)$$

Peruani, Deutsch, Bar, EPJ 2008

Linearized equation

Introduce probability P(θ)=1/2π+ ξ

Linearized Fokker-Plank equation

$$\partial_t \xi = D\partial_\theta^2 \xi + \frac{\gamma}{2\pi}\frac{\partial}{\partial\theta}\left(\int_{-\pi}^{\pi} \sin(2(\theta-\theta'))\xi(\theta')d\theta'\right)$$
$$= D\partial_\theta^2 \xi + \frac{\gamma}{\pi}\left(\int_{-\pi}^{\pi} \cos(2(\theta-\theta'))\xi(\theta')d\theta'\right)$$

Peruani, Deutsch, Bar, EPJ 2008

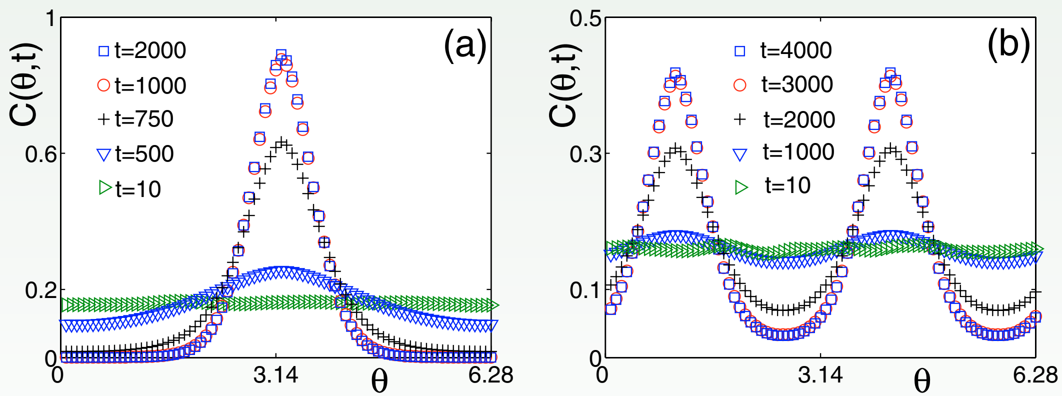
Eigenvalues

Expansion of the linear solution $$\xi \sim \xi_n \exp(in\theta + \lambda_n t)$$

Eigenvalues $$\lambda_n = -Dn^2 + \gamma\delta_{2,|n|}$$

Instability possible for $$n = \pm 2$$

polar *apolar*



Peruani, Deutsch, Bar, EPJ 2008

Exact steady-state solution to the FP equation

Fokker-Plank equation

$$\partial_t P = D\partial_\theta^2 P + \gamma \frac{\partial}{\partial\theta}\left(\int_{-\pi}^{\pi} \sin(\theta-\theta')P(\theta')d\theta' P(\theta)\right)$$

After integration over θ

$$D\partial_\theta P + \gamma \int_{-\pi}^{\pi} \sin(\theta-\theta')P(\theta')d\theta' P(\theta) = const$$

Introducing τ_x, τ_y

$$D\partial_\theta P + \gamma\left(\sin(\theta)\tau_x - \cos(\theta)\tau_y\right)P(\theta) = const$$

$$\tau_x = \int_{-\pi}^{\pi} \cos(\theta)P(\theta)d\theta, \tau_y = \int_{-\pi}^{\pi} \sin(\theta)P(\theta)d\theta$$

We can set τ_y *=0, const = 0*

$$D\partial_\theta P + \gamma \sin(\theta)\tau_x P(\theta) = 0$$

Exact steady-state solution to the FP equation

$$D\partial_\theta P + \gamma \sin(\theta)\tau_x P(\theta) = 0$$

Explicit solution, C = const

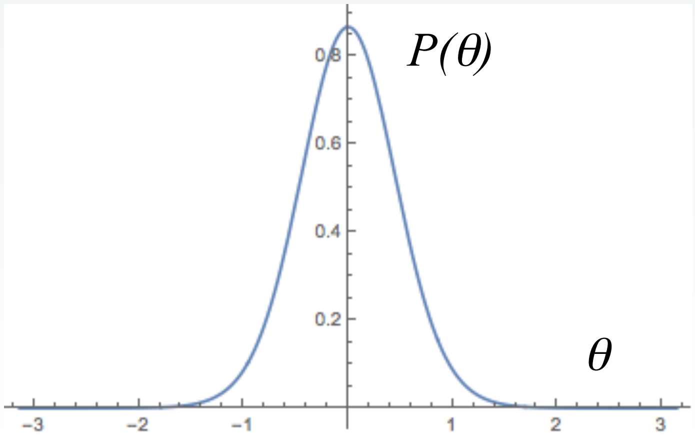
$$P = C \exp\left(\frac{\gamma\tau_x}{D}\cos(\theta)\right)$$

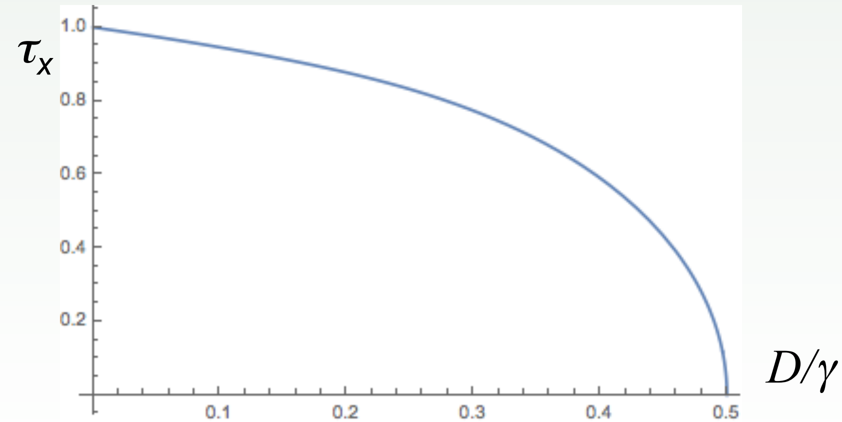
C is determined from the normalization condition

$$\int_{-\pi}^{\pi} P(\theta) d\theta = 1 \to C = \frac{1}{2\pi I_0(\gamma\tau_x/D)}$$

τ_x is determined by integrating P with cos(θ)

$$P(\theta) = \frac{\exp\left(\frac{\gamma\tau_x}{D}\cos(\theta)\right)}{2\pi I_0(\gamma\tau_x/D)} \qquad \tau_x = \frac{I_1(\gamma\tau_x/D)}{I_0(\gamma\tau_x/D)}$$





Phenomenological Toner-Tu theory

VOLUME 75, NUMBER 23 PHYSICAL REVIEW LETTERS 4 DECEMBER 1995

Long-Range Order in a Two-Dimensional Dynamical *XY* Model: How Birds Fly Together

John Toner[1,2] and Yuhai Tu[1]

[1]*IBM T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598*

[2]*Department of Physics, University of Oregon, Eugene, Oregon 97403-1274**

(Received 9 June 1995)

We propose a nonequilibrium continuum dynamical model for the collective motion of large groups of biological organisms (e.g., flocks of birds, slime molds, etc.) Our model becomes highly nontrivial, and different from the equilibrium model, for $d < d_c = 4$; nonetheless, we are able to determine its scaling exponents *exactly* in $d = 2$ and show that, unlike equilibrium systems, our model exhibits a broken continuous symmetry even in $d = 2$. Our model describes a large universality class of microscopic rules, including those recently simulated by Vicsek *et al.*

Phenomenological Toner-Tu theory

$$\partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} + \lambda_2 (\vec{\nabla} \cdot \vec{v}) \vec{v} + \lambda_3 \vec{\nabla} (|\vec{v}|^2)$$

$$= \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} - \vec{\nabla} P + D_B \vec{\nabla} (\vec{\nabla} \cdot \vec{v})$$

$$+ D_T \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f},$$

$$P = P(\rho) = \sum_{n=1}^{\infty} \sigma_n (\rho - \rho_0)^n,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v} \rho) = 0,$$

Phenomenological Toner-Tu theory

β, D_B, D_2, $D_T > 0$
$\alpha > 0$ – ordered state, $\alpha < 0$ disordered state
f- random force

$$\langle f_i(\vec{r},t) f_j(\vec{r'},t')\rangle = \Delta\, \delta_{ij}\delta^d(\vec{r}-\vec{r'})\,\delta(t-t'),$$

Absence of the Galilean Invariance (GI)
If the GI did hold, it would imply
$\lambda_2 = \lambda_3 = 0$, $\lambda_1 = 1$ (compare the NS equation)

Comparison with the microtubules equations

$$\partial_t \vec{v} + \lambda_1(\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2(\vec{\nabla}\cdot\vec{v})\vec{v} + \lambda_3\vec{\nabla}(|\vec{v}|^2)$$

$$= \alpha\vec{v} - \beta|\vec{v}|^2\vec{v} - \vec{\nabla}P + D_B\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

$$+ D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f},$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \left(0.273\rho - 1\right)\boldsymbol{\tau} - 2.18\,|\boldsymbol{\tau}|^2\,\boldsymbol{\tau} + \frac{5\nabla^2\boldsymbol{\tau}}{192} + \frac{\nabla\nabla\cdot\boldsymbol{\tau}}{96} + \frac{B^2\rho\,\nabla^2\boldsymbol{\tau}}{4\pi} +$$

$$+ H\left[\frac{\nabla\rho^2}{16\pi} - \left(\pi - \tfrac{8}{3}\right)\boldsymbol{\tau}\left(\nabla\cdot\boldsymbol{\tau}\right) - \tfrac{8}{3}\left(\boldsymbol{\tau}\nabla\right)\boldsymbol{\tau}\right]$$

From particle models to (deterministic, coarse-grained) continuous theories: the "Boltzmann Ginzburg-Landau" approach

- Start with the simple Boltzmann equation of ideal gases for the probability function *f* (r,θ,t)

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{force}} + \left(\frac{\partial f}{\partial t}\right)_{\text{diff}} + \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}$$

- No external forces but a self-propulsion given by an advection (or diffusion) term

$$\left(\frac{\partial f}{\partial t}\right)_{\text{self}-\text{propulsion}} = v_0\, \mathrm{e}\,(\theta) \cdot \nabla f\,(r, \theta, t)$$

Fourier expansion

- Introduce the angular Fourier expansion

$$
\begin{aligned}
f(r,\theta,t) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}_k(r,t)\, e^{-ik\theta} \\
\hat{f}_k(r,t) &= \int_{-\pi}^{\pi} d\theta\, f(r,\theta,t)\, e^{ik\theta}
\end{aligned}
$$

- The first three modes give the density, the polar, and the nematic order parameters

$$
\rho = \hat{f}_0 \qquad \rho\mathbf{P} = \begin{pmatrix} \mathrm{Re}\hat{f}_1 \\ \mathrm{Im}\hat{f}_1 \end{pmatrix} \qquad \rho\mathbf{Q} = \begin{pmatrix} \mathrm{Re}\hat{f}_2 & \mathrm{Im}\hat{f}_2 \\ \mathrm{Im}\hat{f}_2 & -\mathrm{Re}\hat{f}_2 \end{pmatrix}
$$

- Use complex notations for simplicity, including:

$$
\nabla \equiv \partial_x + i\partial_y\,, \quad \text{and} \quad \nabla^* \equiv \partial_x - i\partial_y
$$

Vicsek polar model

$$\theta_i(t+\Delta t) = \left\langle \theta(t) \right\rangle_{S(i)} + \xi$$

$$\mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t, \mathbf{v}_i = v_0(\cos(\theta), \sin(\theta))$$

In the expansion f_1 is the most unstable and drives the dynamics

$$\begin{aligned} f(r,\theta,t) &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}_k(r,t)\, e^{-ik\theta} \\ \hat{f}_k(r,t) &= \int_{-\pi}^{\pi} d\theta f(r,\theta,t)\, e^{ik\theta} \end{aligned}$$

Hydrodynamic equations in the polar case

Continuity equation $$\partial_t \rho = -\Re\left(\nabla^* f_1\right)$$

"Toner-Tu" equation

$$\partial_t f_1 + \frac{1}{2}\nabla\rho = \left(\mu - \xi\left|f_1\right|^2\right) f_1 + \frac{\nu}{4}\Delta f_1 + \iota f_1^* \nabla f_1 - \chi f_1 \nabla^* f_1$$

$$\mu = \mu'\rho - \mu_0 = \mu'(\rho - \rho_t)$$

With all the transport coefficients depending on local density and noise strength

(in particular linear coefficient μ increases with ρ)

Solitons in the 1D equation (Vicsek bands)

Consider moving localized solution

$$\begin{pmatrix} \rho(x,t) \\ f_1(x,t) \end{pmatrix} = \begin{pmatrix} \rho(x-vt) \\ f_1(x-vt) \end{pmatrix}$$

Continuity equation $\partial_t \rho = -\partial_x f_1$

$$V(\rho - \rho_0) = f_1$$

$$\rho(x \to \pm\infty) \to \rho_0, f_1(x \to \pm\infty) \to 0$$

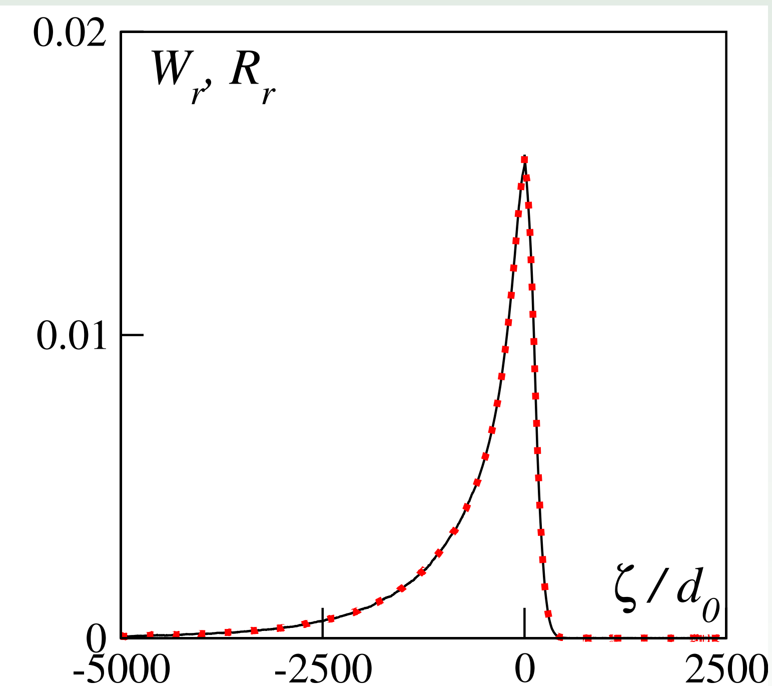
Equation for f_1

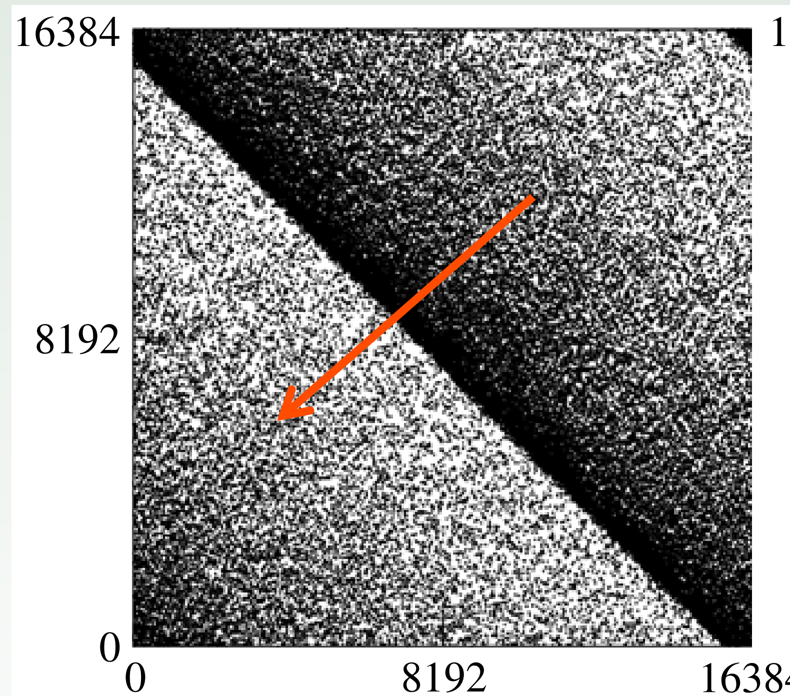
$$-V\partial_x f_1 + \partial_x \rho/2 = (\mu' \rho - \mu_0 - \zeta f_1^2) f_1 + \frac{\nu}{4}\partial_x^2 f_1 + (\iota - \kappa) f_1 \partial_x f_1$$

Solitons in the 1D equation (Vicsek bands)

$$-V\partial_x f_1 + \partial_x \rho/2 = (\mu'\rho - \mu_0 - \zeta f_1^2) f_1 + \frac{\nu}{4}\partial_x^2 f_1 + (\iota - \kappa) f_1 \partial_x f_1$$

$$\rho = f_1/V + \rho_0$$





E. Bertin, M. Droz, G. Gregoire, J. Phys. A, 42, 445001 (2009)

Linear stability in 1D

$$\partial_t \rho = -\partial_x f_1$$

$$\partial_t f_1 + \partial_x \rho/2 = (\mu' \rho - \mu_0 - \zeta f_1^2) f_1 + \frac{\nu}{4} \partial_x^2 f_1 + (\iota - \kappa) f_1 \partial_x f_1$$

Examine state $f_1=0$, $\rho=\rho_0$

$$\delta\rho, \delta f_1 \sim \exp[\lambda t + ikx]$$

$$\lambda_1 = \frac{1}{8}\left(-\nu k^2 + 4(\mu'\rho_0 - \mu_0) + 4\sqrt{-2k^2 + (\nu k^2/4 - (\mu'\rho_0 - \mu_0))^2}\right)$$

Expansion for small k

$$\lambda_1 \approx \mu'\rho_0 - \mu_0 - \left(\frac{\nu}{4} + \frac{1}{2(\mu'\rho_0 - \mu_0)}\right) k^2$$

Examine state $f_1^{\,2}=\mu' \rho_0 - \mu_0$, $\rho=\rho_0$

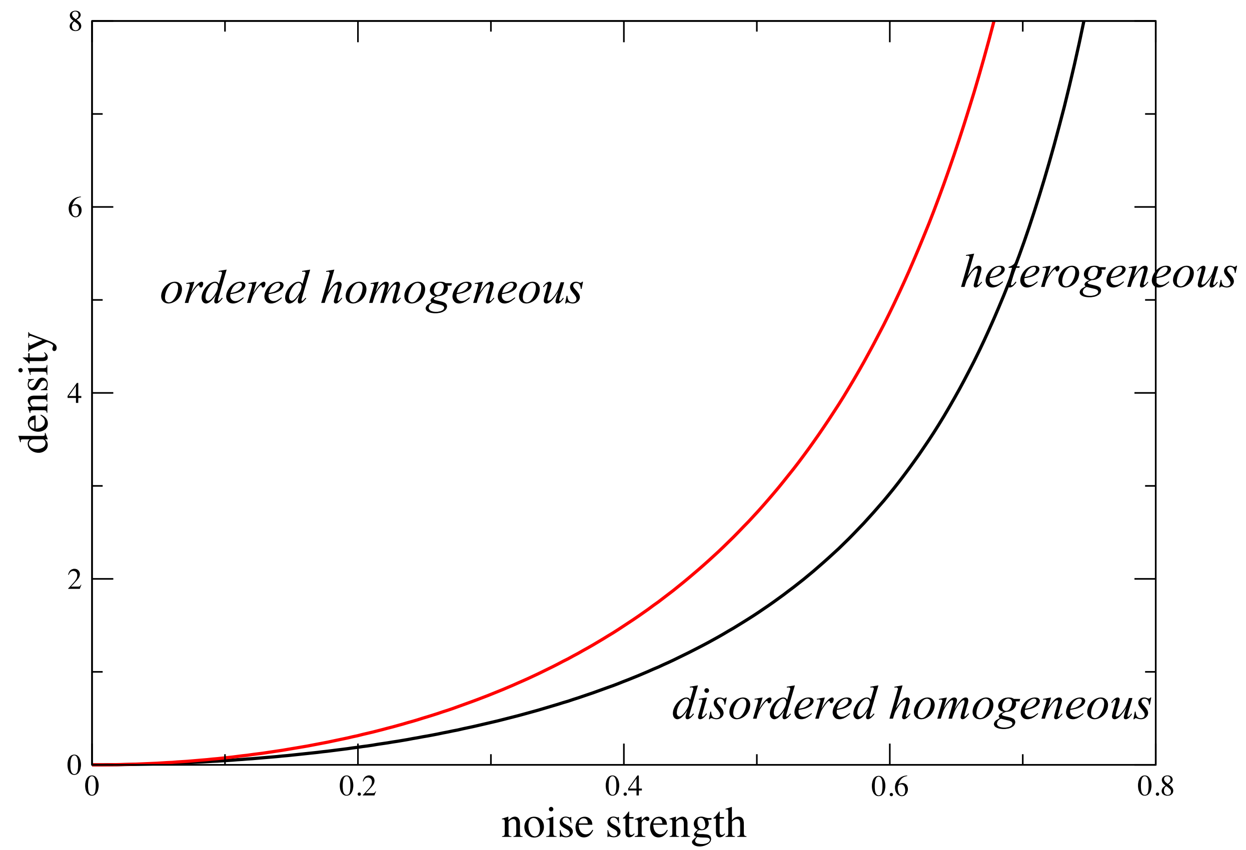
$$\partial_t \rho = -\partial_x f_1$$

$$\partial_t f_1 + \partial_x \rho/2 = (\mu' \rho - \mu_0 - \zeta f_1^2) f_1 + \frac{\nu}{4} \partial_x^2 f_1 + (\iota - \kappa) f_1 \partial_x f_1$$

Expansion for small k

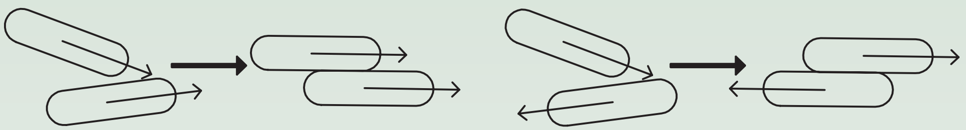
$$\lambda_1 \approx i \frac{\mu'}{f1} k + \frac{(\mu')^2 - 2 f_1^2 + 2\mu'(\iota - \kappa)(\rho_0 - \mu_0)}{f_1^4} k^2$$

Phase diagram



Interacting self-propelled rods

Sings + or minus are chosen randomly, particles move along the director **n**



$$\theta_j^{t+1} = \arg\Big[\sum_{k\sim j} \mathrm{sign}[\cos(\theta_k^t - \theta_j^t)] e^{i\theta_k^t}\Big] + \eta\xi_j^t$$

$$\mathbf{r}_j^{t+1} = \mathbf{r}_j^t + v_0 e^{i\theta_k^{t+1}},$$

In the expansion f_2 is the most unstable and drives the dynamics

$$f(r,\theta,t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}_k(r,t)\, e^{-ik\theta}$$

$$\hat{f}_k(r,t) = \int_{-\pi}^{\pi} d\theta f(r,\theta,t)\, e^{ik\theta}$$

Equations for Interacting self-propelled rods

$$\partial_t \rho = -\Re\left(\nabla^* f_1\right)$$

$$\begin{aligned}\partial_t f_1 = &-\frac{1}{2}(\nabla\rho + \nabla^* f_2) + \frac{\gamma}{2} f_2^* \nabla f_2 \\ &- (\alpha - \beta |f_2|^2) f_1 + \zeta f_1^* f_2,\end{aligned}$$

$$\begin{aligned}\partial_t f_2 = &-\frac{1}{2}\nabla f_1 + \frac{\nu}{4}\nabla^2 f_2 - \frac{\kappa}{2} f_1^* \nabla f_2 - \frac{\chi}{2}\nabla^*(f_1 f_2) \\ &+ (\mu - \xi |f_2|^2) f_2 + \omega f_1^2 + \tau |f_1|^2 f_2,\end{aligned}$$

Equations for Interacting self-propelled rods

$$\partial_t \rho = -\Re\left(\nabla^* f_1\right)$$

$$\partial_t f_1 = -\frac{1}{2}(\nabla\rho + \nabla^* f_2) + \frac{\gamma}{2} f_2^* \nabla f_2 - (\alpha - \beta|f_2|^2) f_1 + \zeta f_1^* f_2,$$

$$\partial_t f_2 = -\frac{1}{2}\nabla f_1 + \frac{\nu}{4}\nabla^2 f_2 - \frac{\kappa}{2} f_1^* \nabla f_2 - \frac{\chi}{2}\nabla^*(f_1 f_2) + (\mu - \xi|f_2|^2) f_2 + \omega f_1^2 + \tau |f_1|^2 f_2,$$

Peshkov, Aranson, Bertin, Chate, Ginelli, PRL 2012

Coefficients

$$\nu = \left[\frac{136}{35\pi}\rho + 1 - \hat{P}_3\right]^{-1} \qquad \omega = \frac{8}{\pi}\left[\frac{1}{6} - \frac{\sqrt{2}-1}{2}\hat{P}_2\right]$$

$$\mu = \frac{8}{\pi}\left[\frac{2\sqrt{2}-1}{3}\hat{P}_2 - \frac{7}{5}\right]\rho - 1 + \hat{P}_2 \qquad \zeta = \frac{8}{5\pi}$$

$$\alpha = \frac{8}{\pi}\left[\frac{1}{3} - \frac{1}{4}\hat{P}_1\right]\rho + 1 - \hat{P}_1 \qquad \chi = \nu\frac{2}{\pi}\left[\frac{4}{5} + \hat{P}_3\right]$$

$$\kappa = \nu\frac{8}{15}\left[\frac{19}{7} - \frac{\sqrt{2}+1}{\pi}\hat{P}_2\right] \qquad \gamma = \nu\frac{4}{3\pi}\left[\hat{P}_1 - \frac{2}{7}\right]$$

$$\tau = \chi\frac{8}{15}\left[\frac{19}{7} - \frac{\sqrt{2}+1}{\pi}\hat{P}_2\right] \qquad \beta = \gamma\frac{2}{\pi}\left[\frac{4}{5} + \hat{P}_3\right]$$

$$\xi = \frac{32}{35\pi}\left[\frac{1}{15} + \hat{P}_4\right]\left[\frac{13}{9} - \frac{6\sqrt{2}+1}{\pi}\hat{P}_2\right] \times \left[\frac{8}{3\pi}\left(\frac{31}{21} + \frac{\hat{P}_4}{5}\right)\rho + 1 - \hat{P}_4\right]^{-1}. \qquad (12)$$

$$\hat{P}_k = \exp[-\tfrac{1}{2}k^2\sigma^2]$$

Exact expression for the Vicsek band

Consider stationary band along x-axis: $f_1=0$, f_2 *is real and depend on y*

- *Continuity equation is trivially satisfied*
- *Equation for* f_1 *integrated to*

$$\rho - f_2 - \frac{1}{2}\gamma f_2^2 = \tilde{\rho} = const$$

- *Equation for* f_2 *yields*

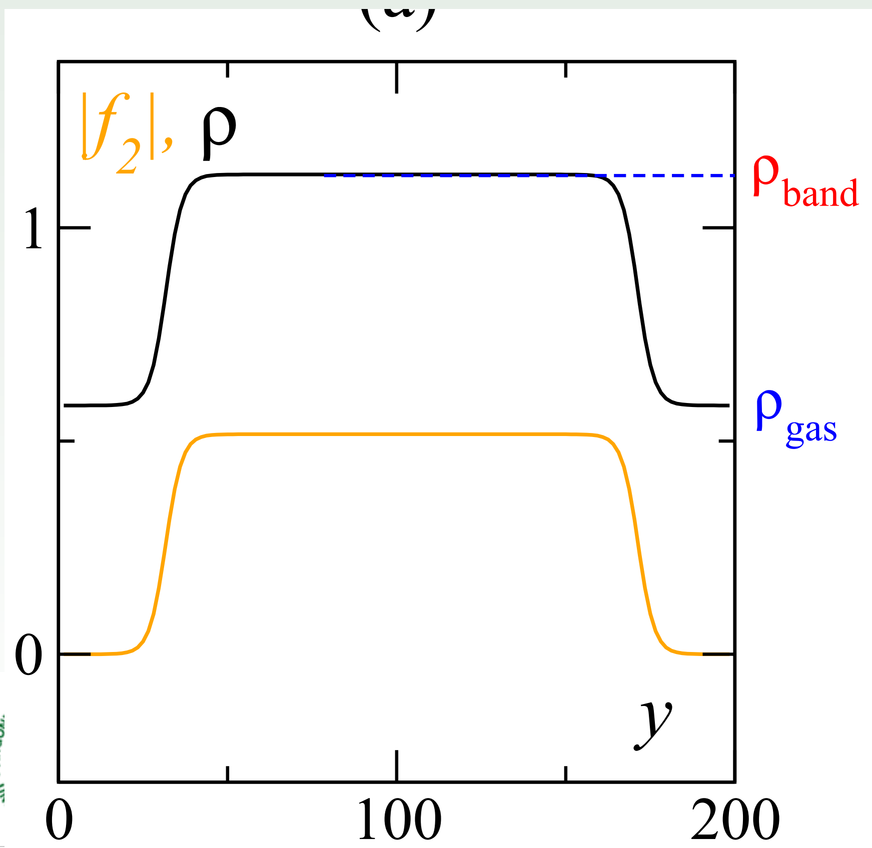
$$\frac{\nu}{4}\partial_y^2 f_2 + \mu' f_2(\tilde{\rho} - \rho_t + f_2) - \left[\xi - \frac{\gamma\mu'}{2}\right] f_2^3 = 0$$

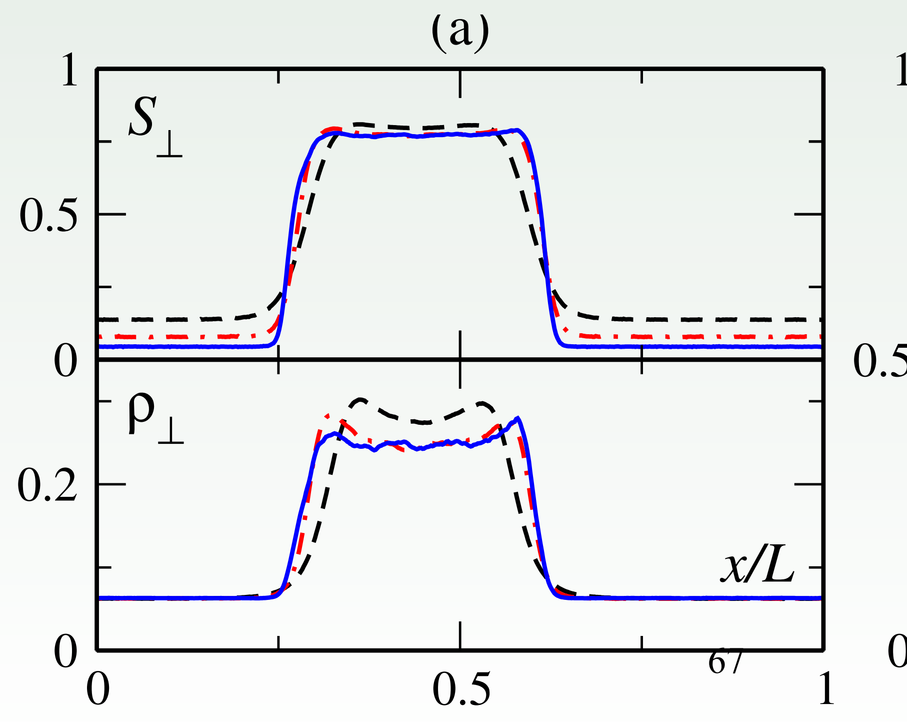
$$\mu = \mu'(\rho - \rho_t)$$

The band

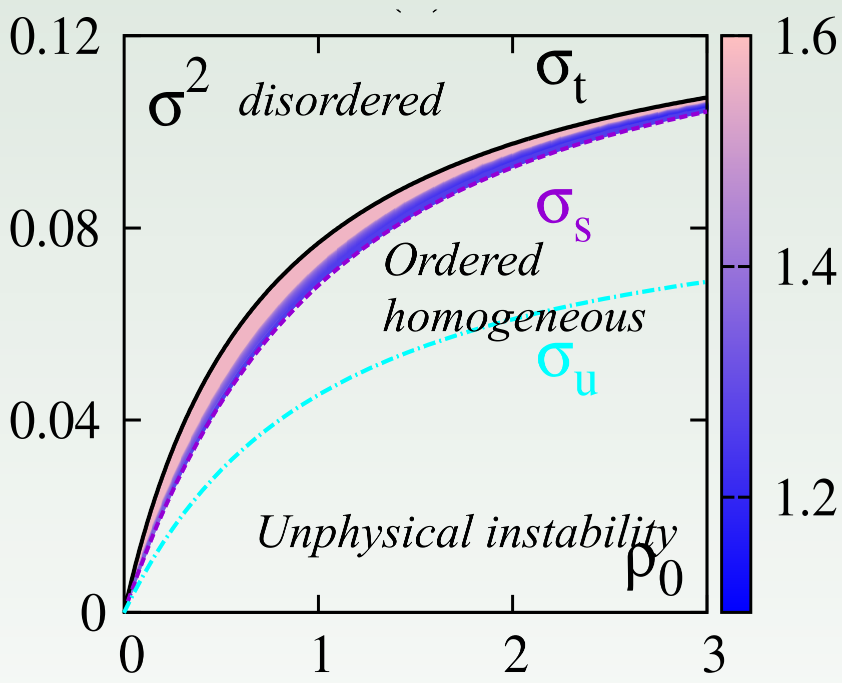
$$f_2(y) = \frac{3(\rho_t - \tilde{\rho})}{1 + a\cosh\left(2y\sqrt{\mu'(\rho_t - \tilde{\rho})/\nu}\right)}$$

$$a = \sqrt{19(\xi - \mu'\gamma/2)(\tilde{\rho} - \rho_t)/2\mu'}$$

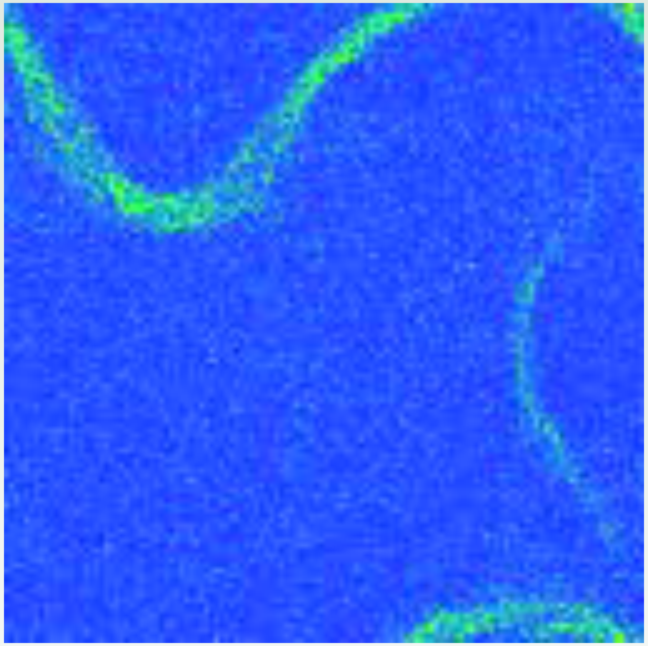


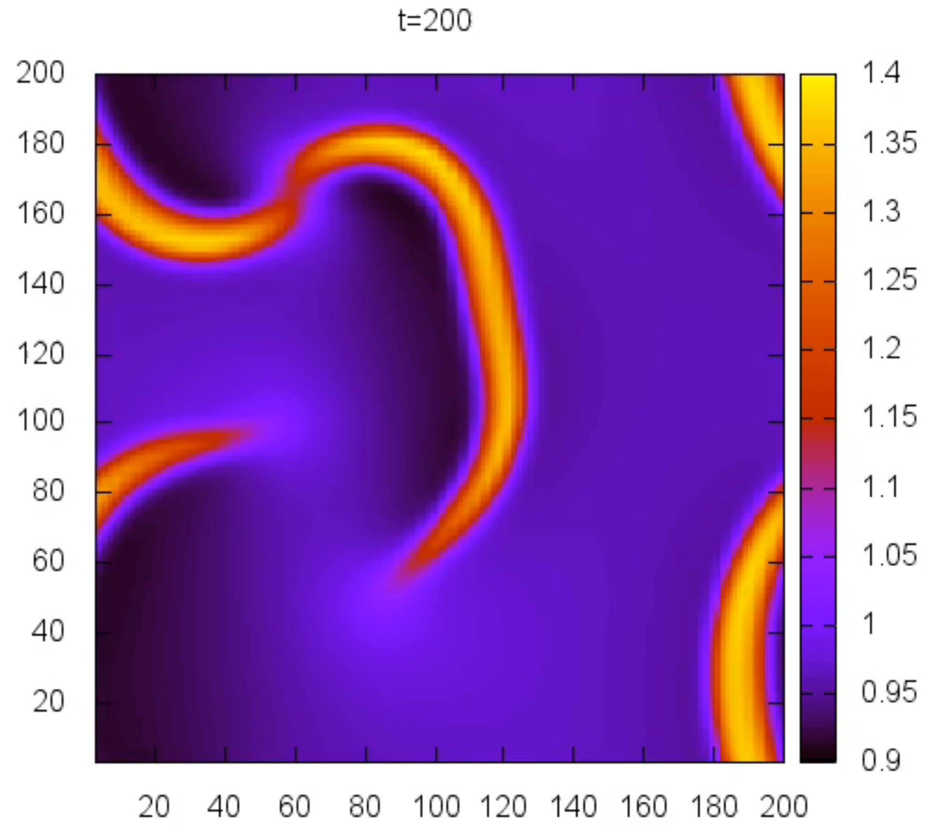


Phase diagram



Comparison with particle model





Active nematic equations

$$\theta_j^{i+1} = \frac{1}{2}\arg\left[\frac{1}{N_j}\sum_k e^{i2\theta_k}\right] + \eta\xi_j$$

$$\mathbf{x}_j^{i+1} = \mathbf{x}_j^i \pm v_0\mathbf{n}_j^i$$

N_j – current number of neighbors

In the expansion f_2 is the most unstable and drives the dynamics, f_1 is enslaved

$$f(\mathbf{x},\theta,t) = \frac{1}{\pi}\sum_{-\infty}^{\infty}\hat{f}(\mathbf{x},t)\exp(-i2k\theta)$$

$$Q = \hat{f}_1(\mathbf{x},t), \rho = \hat{f}_0(\mathbf{x},t)$$

Ngo, Peshkov, Aranson, Bertin, Ginelli, Chate, PRL 2014

Active nematic equations

$$\partial_t \rho = \frac{1}{2}\Delta\rho + \frac{1}{2}\mathrm{Re}(\nabla^{*2} Q),$$

$$\partial_t Q = (\mu(\rho) - \xi|Q|^2)Q + \frac{1}{4}\nabla^2\rho + \frac{1}{2}\Delta Q,$$

$$\mu(\rho) = \mu'(\rho - \rho_t)$$

$$\nabla \equiv \partial_x + i\partial_y, \;\; \nabla^* \equiv \partial_x - i\partial_y, \;\; \text{and} \;\; \Delta \equiv \nabla\nabla^*.$$

Connection to the rods case

$$\partial_t f_1 = -\frac{1}{2}(\nabla\rho + \nabla^* f_2) + \frac{\gamma}{2} f_2^* \nabla f_2 - (\alpha - \beta|f_2|^2) f_1 + \zeta f_1^* f_2,$$

Assume α is large

$$\alpha f_1 = -\frac{1}{2}(\nabla\rho + \nabla^* f_2)$$

Continuity equation

$$\partial_t \rho = -\Re(\nabla^* f_1) = \frac{1}{2\alpha}\Re(\Delta\rho + \nabla^{*2} f_2)$$

Exact band solution

$$Q_0(y) = \frac{3(\rho_t - \rho_{\text{gas}})}{1 + a\cosh\left(\sqrt{4\mu'(\rho_t - \rho_{\text{gas}})}\,y\right)}$$

$$a = \sqrt{1 - 9\xi(\rho_t - \rho_{\text{gas}})/2\mu'}$$

Undulational instability of the band

To examine stability of the band solution with respect to transversal undulations, we rewrite Eqs. (1), (2) (main text) for real and imaginary parts, $f_1 = U + iV$, yielding

$$\partial_t \rho = \frac{1}{2}\Delta\rho + \frac{1}{2}\left((\partial_x^2 - \partial_y^2)U + 2\partial_x\partial_y V\right) \tag{1}$$

$$\partial_t U = (\mu_0 - \xi(U^2 + V^2)U + \mu'\rho U + \frac{1}{4}(\partial_x^2 - \partial_y^2)\rho + \frac{1}{2}\Delta U \tag{2}$$

$$\partial_t V = (\mu_0 - \xi(U^2 + V^2)V + \mu'\rho V + \frac{1}{2}\partial_x\partial_y\rho + \frac{1}{2}\Delta V \tag{3}$$

Here we define $\mu_0 = -\rho_t\mu'$. Band solution (Eq. (4), main text) assumes the form $U = U_0(y), V = 0, \rho = U_0(y) + \rho_{\rm gas} \equiv$ $R_0(y)$. We seek perturbative solution to Eqs. (1)-(3) in the form $U = U_0 + u(y)\exp[ikx + \lambda t], V = v(y)\exp[ikx + \lambda t], \rho =$ $R_0(y) + r(y)\exp[ikx + \lambda t]$, which results in the following linearized system

$$\lambda r = \frac{1}{2}(\partial_y^2 - k^2)r - \frac{1}{2}(k^2 + \partial_y^2)u + ik\partial_y v \qquad (4$$

$$\lambda u = (\mu_0 - 3\xi U_0^2)u + \mu' R_0 u + \mu' U_0 r - \frac{1}{4}(k^2 + \partial_y^2)r + \frac{1}{2}(\partial_y^2 - k^2)u \qquad (5$$

$$\lambda v = (\mu_0 - \xi U_0^2)v + \mu' R_0 v + \frac{1}{2}ik\partial_y r + \frac{1}{2}(\partial_y^2 - k^2)v \qquad (6$$

Undulational instability of the band

Due to translational symmetry, for $k = 0$ there exists a stationary solution ($\lambda = 0$) to the linearized system Eqs. (4)-(6) in the form $v = 0, u = \partial_y U_0(y), r = \partial_y U_0(y)$. Therefore, for $k \to 0$ we can seek perturbative solution as follow (we define $\lambda = \lambda_1 k^2$)

$$\begin{pmatrix} r(y) \\ u(y) \\ v(y) \end{pmatrix} = \begin{pmatrix} \partial_y U_0(y) \\ \partial_y U_0(y) \\ 0 \end{pmatrix} + \begin{pmatrix} k^2 r_1(y) \\ k^2 u_1(y) \\ ikv_1(y) \end{pmatrix} \tag{7}$$

As we will show later, this specific form of the expansion with respect to small parameter k yields nontrivial resul for the eigenvalue λ. After substitution of (7) into Eqs. (4)-(6) we obtain in the first non-trivial order

$$\lambda_1 \partial_y U_0 + \partial_y U_0 = \frac{1}{2}\partial_y^2 r_1 - \frac{1}{2}\partial_y^2 u_1 - \partial_y v_1 \tag{8}$$

$$\lambda_1 \partial_y U_0 + \frac{3}{4}\partial_y U_0 = (\mu_0 - 3\xi U_0^2)u_1 + \mu' R_0 u_1 + \mu' U_0 r_1 - \frac{1}{4}\partial_y^2 r_1 + \frac{1}{2}\partial_y^2 u_1 \tag{9}$$

$$-\frac{1}{2}\partial_y^2 U_0 = (\mu_0 - \xi U_0^2)v_1 + \mu' R_0 v_1 + \frac{1}{2}\partial_y^2 v_1 \tag{10}$$

ne can check that exact solution to Eq. (10) is $v_1 = -2U_0$. So, the remaining equations can be written as

$$(\lambda_1 - 1)\partial_y U_0 = \frac{1}{2}\partial_y^2 r_1 - \frac{1}{2}\partial_y^2 u_1 \tag{11}$$

$$\left(\lambda_1 + \frac{3}{4}\right)\partial_y U_0 = (\mu_0 - 3\xi U_0^2)u_1 + \mu' R_0 u_1 + \mu' U_0 r_1 - \frac{1}{4}\partial_y^2 r_1 + \frac{1}{2}\partial_y^2 u_1 \tag{12}$$

Undulational instability of the band

Using Eq. (17), we derive from Eq. (16) an inhomogeneous second-order differential equation for the function u_1:

$$\left(\frac{3}{2}\lambda_1 + \frac{1}{4}\right)\partial_y U_0 - 2\mu' U_0(\lambda_1 - 1)\int dy\, U_0(y) = (\mu_0 - 3\xi U_0^2)u_1 + \mu' R_0 u_1 + \mu' U_0 u_1 + \frac{1}{4}\partial_y^2 u_1 \tag{18}$$

It is convenient to rewrite formally the above equation as $B(y) = \mathcal{L}u_1(y)$, where $B(y)$ is the l.h.s. of Eq. (18) and $\mathcal{L}$ is the differential operator defined as

$$\mathcal{L} = (\mu_0 - 3\xi U_0^2) + \mu' R_0 + \mu' U_0 + \frac{1}{4}\partial_y^2 \tag{19}$$

The operator $\mathcal{L}$ is self-adjoint with respect to the scalar product of functions defined as $\langle \psi_1, \psi_2 \rangle = \int_{-\infty}^{\infty} dy\, \psi_1(y)\psi_2(y)$, meaning that for any functions ψ_1 and ψ_2, $\langle \mathcal{L}\psi_1, \psi_2 \rangle = \langle \psi_1, \mathcal{L}\psi_2 \rangle$. It is easy to check that $\mathcal{L}$ possesses a localized zero eigenmode $\psi_0(y) = \partial_y U_0(y)$. Therefore, taking the scalar product of Eq. (18) by $\psi_0(y)$, one finds

$$\langle \psi_0, B \rangle = \langle \psi_0, \mathcal{L}u_1 \rangle \tag{20}$$

Thanks to the self-adjoint property, one has $\langle \psi_0, \mathcal{L}u_1 \rangle = \langle \mathcal{L}\psi_0, u_1 \rangle = 0$, eventually yielding $\langle \psi_0, B \rangle = 0$, a condition called solvability condition. In more explicit terms, this condition reads

$$\int_{-\infty}^{\infty} dy\, \partial_y U_0 \left(\left(\frac{3}{2}\lambda_1 + \frac{1}{4}\right)\partial_y U_0 - 2\mu' U_0(\lambda_1 - 1)\int dy\, U_0(y)\right) = 0 \tag{21}$$

Therefore, the band solution is unstable ($\lambda_1 > 0$) if the following condition is fulfilled

$$d = \frac{1}{4}\int_{-\infty}^{\infty} dy(\partial_y U_0)^2 - \mu' \int_{-\infty}^{\infty} dy\, U_0(y)^3 < 0$$

Undulational instability of the band

$$U_0(y) = \frac{c}{1 + a\cosh(by)} \tag{25}$$

where positive parameters a, b, c are of the form

$$c = 3(\rho_t - \rho_{\mathrm{gas}}), \quad a = \sqrt{1 - 9\xi(\rho_t - \rho_{\mathrm{gas}})/2\mu'}, \quad b = 2\sqrt{\mu'(\rho_t - \rho_{\mathrm{gas}})} \tag{26}$$

Evaluation of Eq. (24) using the band solution Eq. (24) gives rise to a relatively simple analytical expression for d, using the relation $\mu' c = 3b^2/4$,

$$d = \frac{1}{4} c^2 b f(a) \tag{27}$$

where the function $f(a)$ is given by

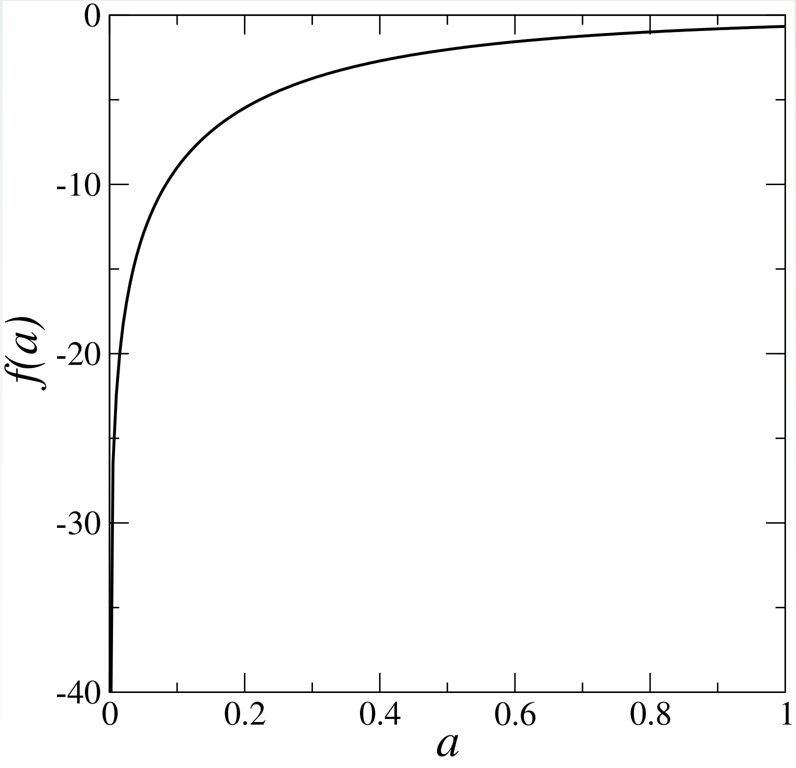
$$f(a) = \int_{-\infty}^{\infty} dx \left(\phi_a'(x)^2 - 3\phi_a(x)^3\right), \qquad \phi_a(x) = \frac{1}{1 + a\cosh x} \tag{28}$$

and ϕ_a' is the derivative of ϕ_a. The integral in Eq. (28) can be computed analytically, leading to

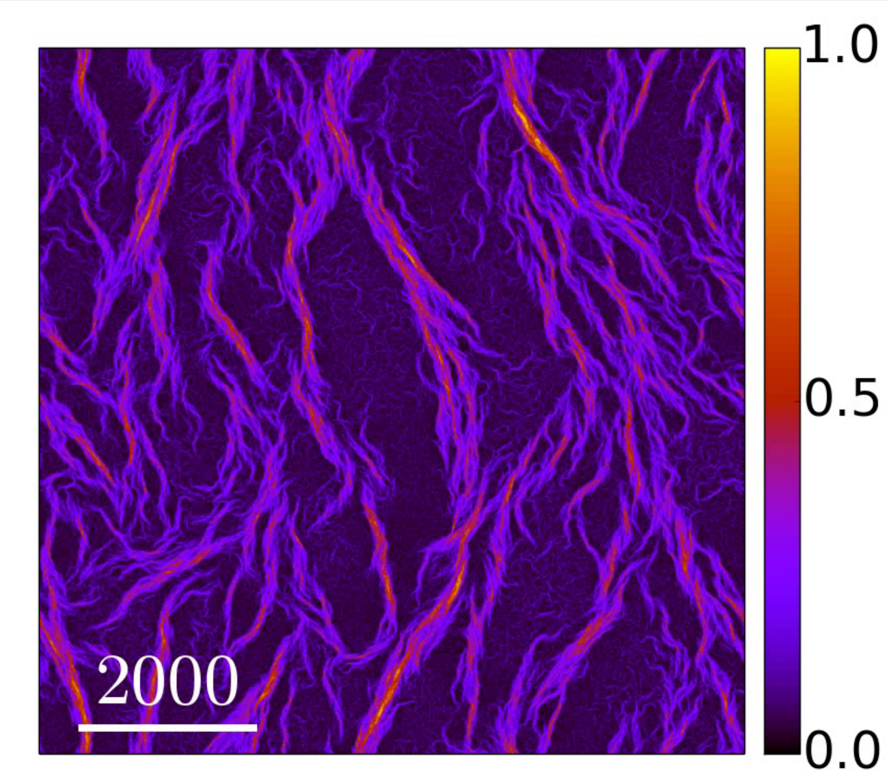
$$f(a) = \frac{2}{3} \frac{\sqrt{1-a^2}\left(a^2+14\right) + \left(6a^2+9\right)\log\left(\frac{a}{\sqrt{1-a^2}+1}\right)}{\left(1-a^2\right)^{5/2}} \tag{29}$$

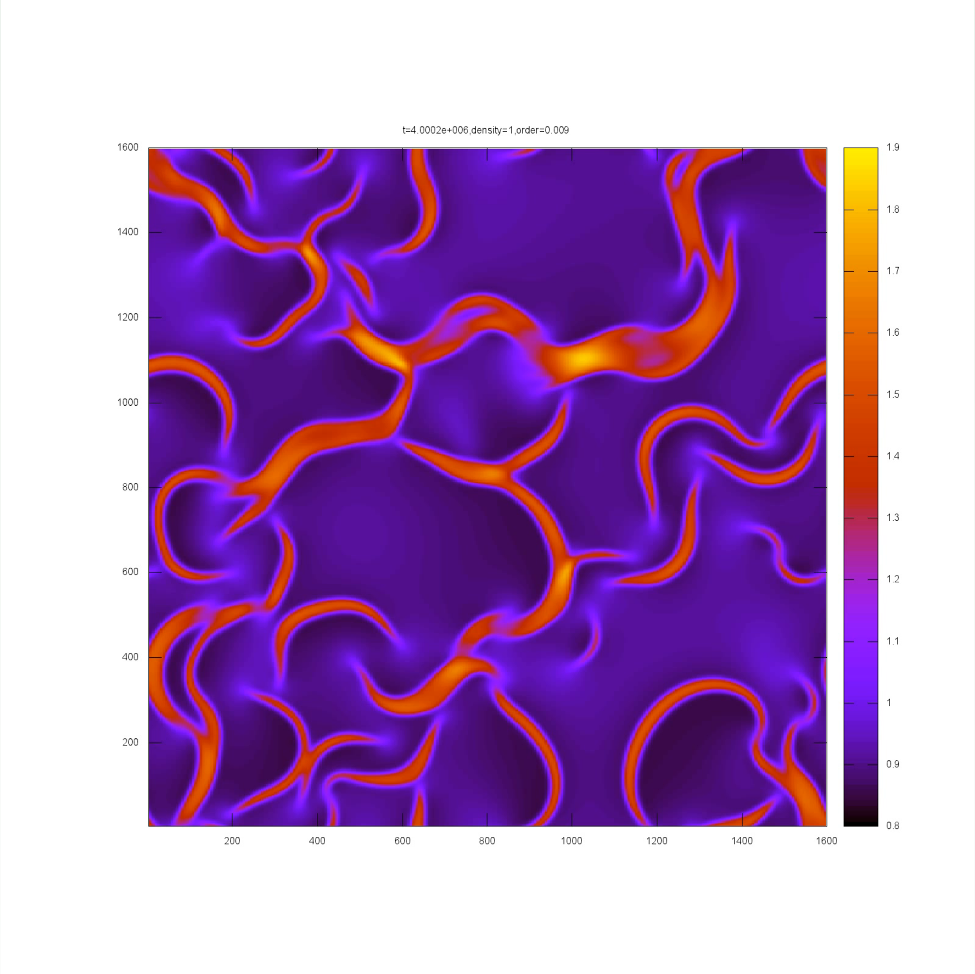
Undulational instability of the band

$$f(a) = \frac{2}{3}\frac{\sqrt{1-a^2}\,(a^2+14) + (6a^2+9)\log\left(\frac{a}{\sqrt{1-a^2}+1}\right)}{(1-a^2)^{5/2}}$$



Continuum and particle simulations





Giant number fluctuations

- Density is well-defined quantity a thermal equilibrium
- For macroscopic system of volume V the containing number of particles N the standard deviation ΔN is

$$\Delta N \sim \sqrt{N}$$

$$\frac{\Delta N}{N} \to 0$$

fluctuations in active systems

- Density fluctuations can be anomalously large (giant fluctuations)

$$\Delta N \sim N^{\alpha}$$

$$\alpha > 1/2$$

- Density is not well-defined
- Activity is the main reason for giant number (or density) fluctuations

How to calculate fluctuations

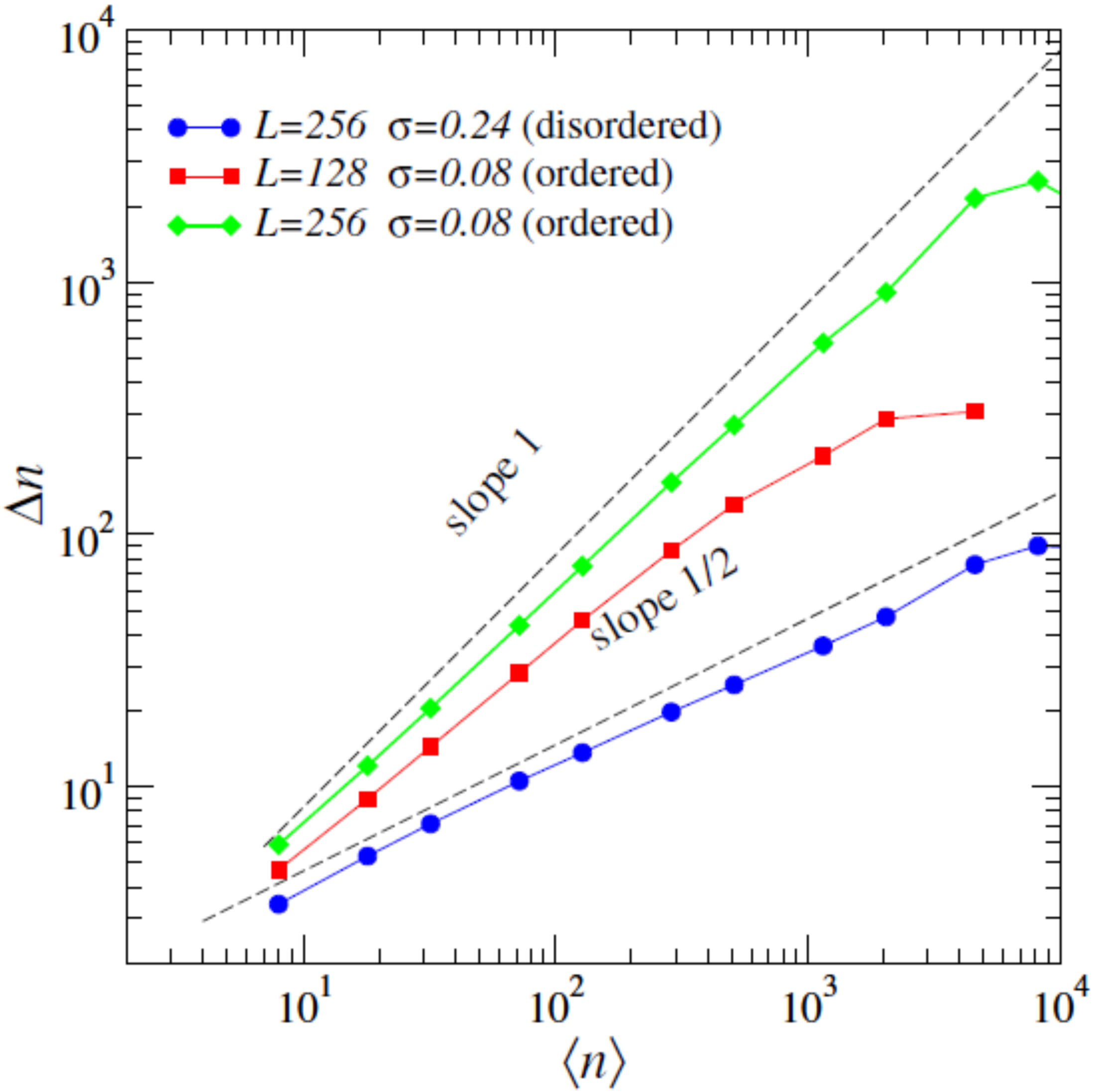
- Divide system in M boxes
- Calculate number of particles in each box
- Calculate average and variance

$$\langle N \rangle = \frac{1}{M}\sum_{i=1}^{M} N_i$$

$$\Delta N = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(N_i - \langle N \rangle\right)^2}$$

- Increase box size
- Average over time

Simulations of Vicsek Model

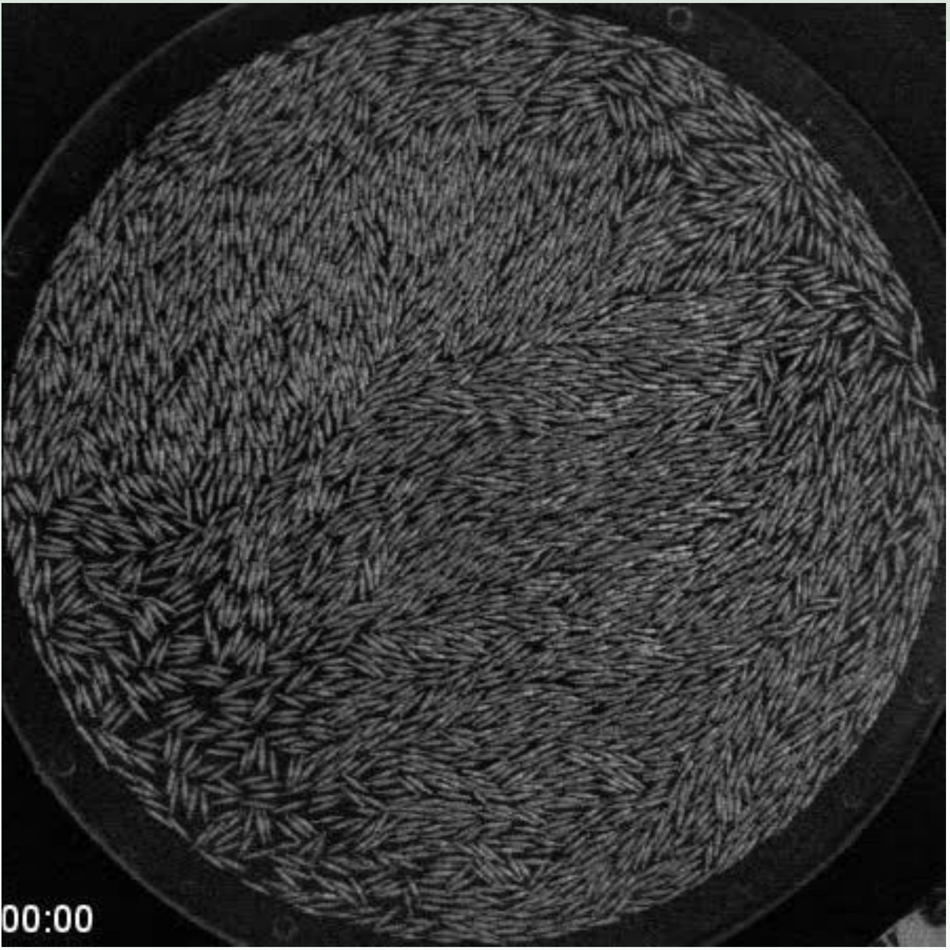


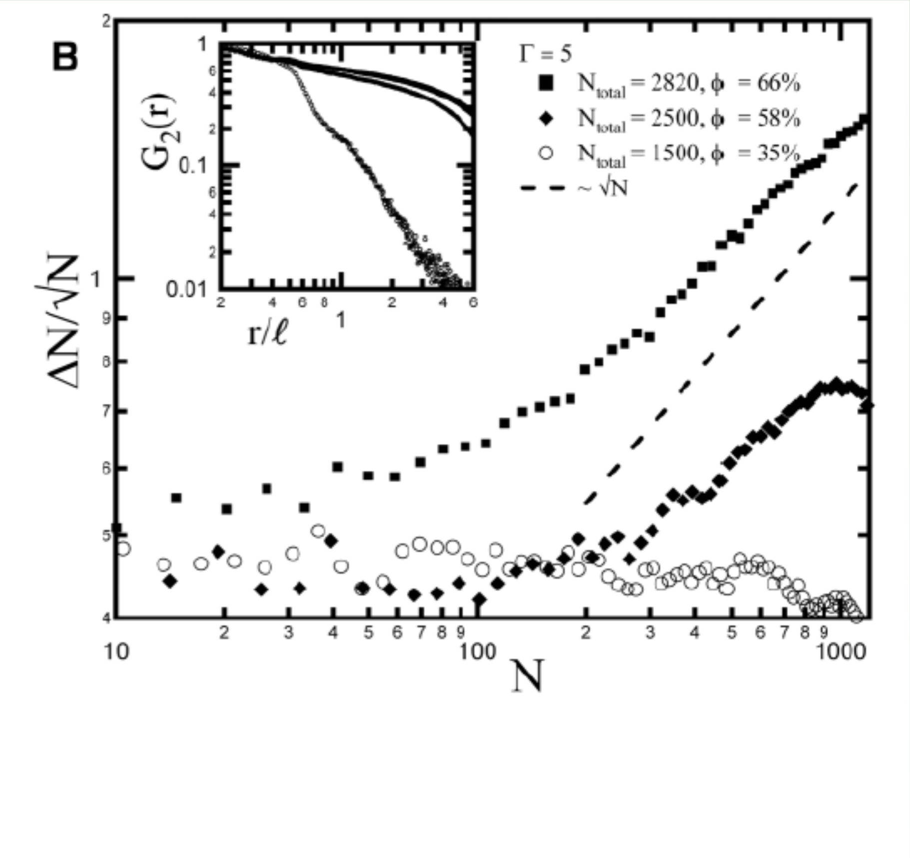
Experimental Verification

- However, experimental evidence is inconclusive
- Apparent large fluctuation of density often emerge due to spatial inhomogeneity and

  and formation of large-scale structures

Shaken granular particles

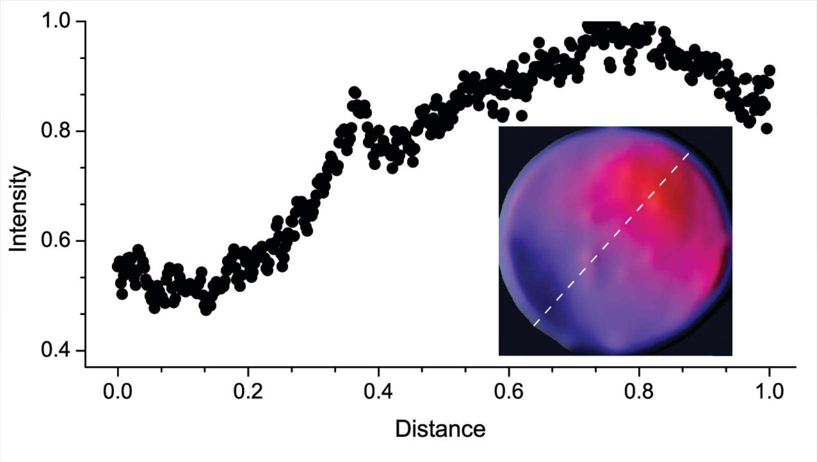




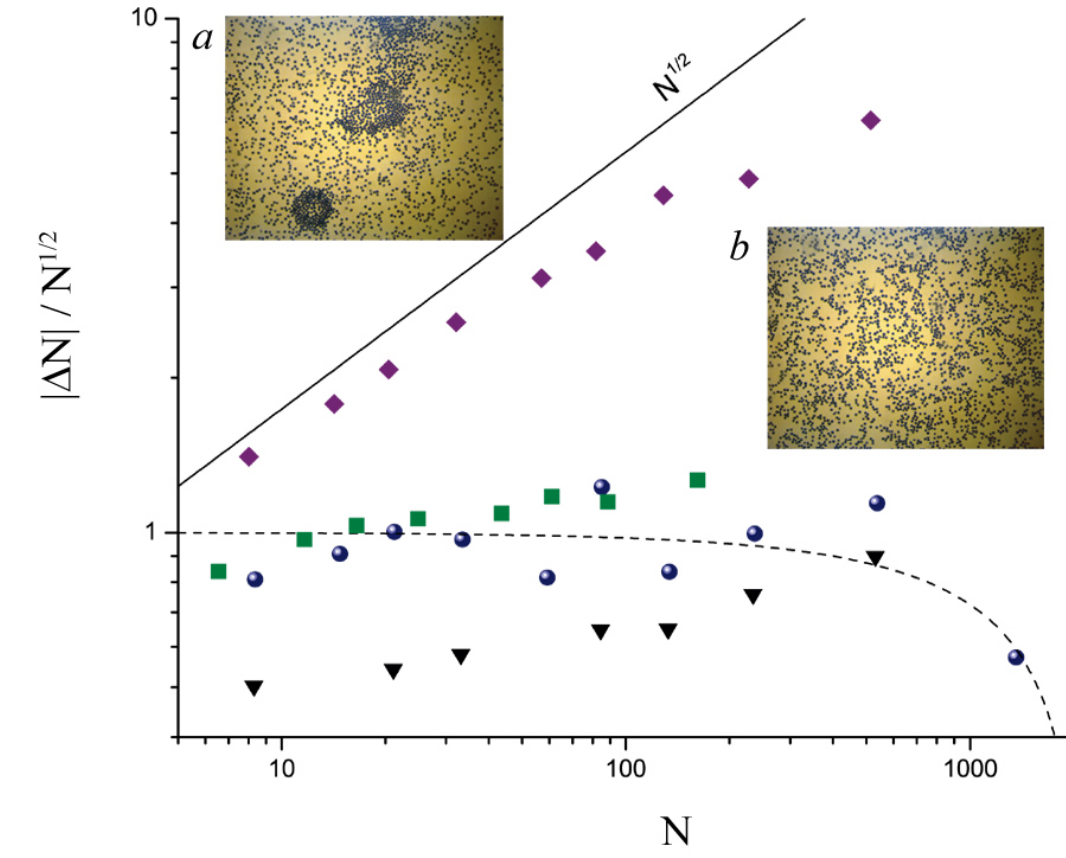
Narayan, Ramaswamy, Menon, Science, 2007

Apparent large fluctuations

Density gradient



Shaken spherical grains



Aranson et al, Science, 2008

Novel material properties

- Reduction of viscosity
- Extraction useful energy from chaotic motion of particles
- Giant number fluctuations

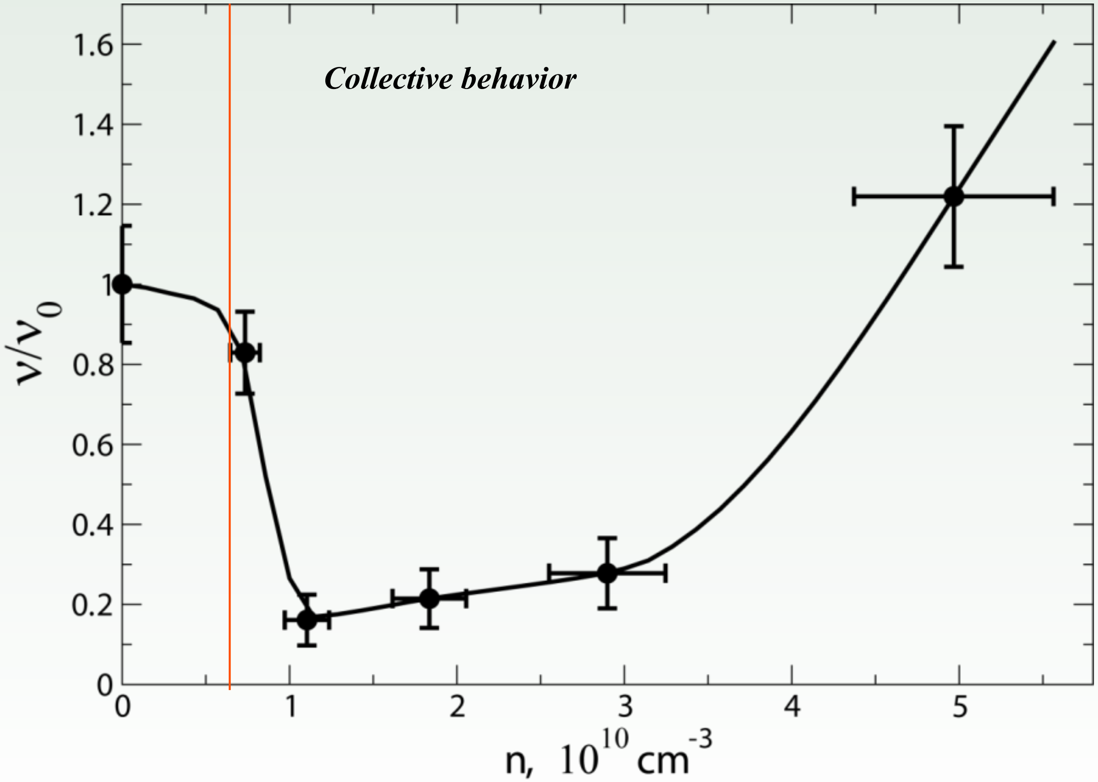
Viscosity of suspensions Einstein Formula

- η_0 viscosity of the suspending liquid
- ϕ volume fraction of solid spherical inclusions
- Viscosity always increases

$$\eta = \eta_0 \left(1 + \frac{5}{2} \phi \right)$$

Viscosity of active suspensions

- Dramatic decrease of the viscosity with the increase of ϕ volume fraction active particles



Extraction of useful energy from chaotic movement