Self-organization of Active Polar Rods: Self-Assembly of Microtubules and Molecular Motors

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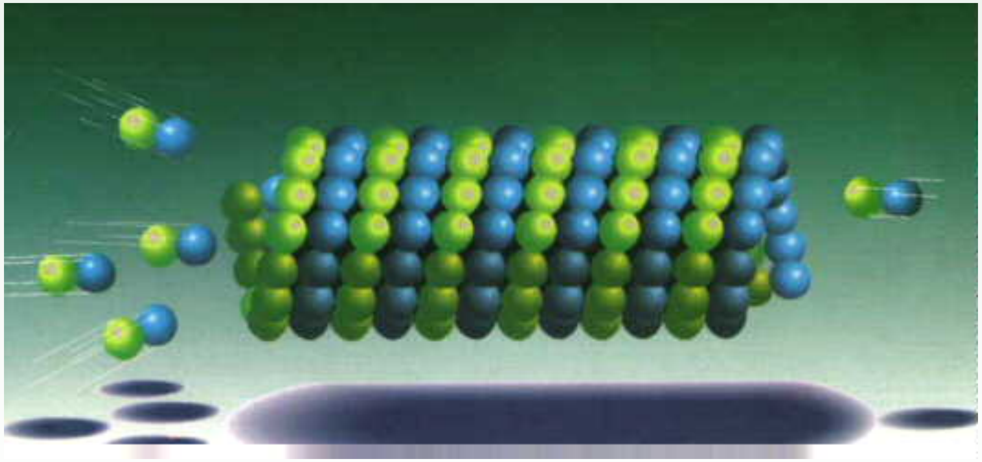
Outline

- *in-vitro* experiments
- Micromechanical calculations
- Maxwell model for polar rods and granular analogy
- Asters and vortices

Purpose: on the example of *in vitro* biological system to demonstrate how continuum equations can be derived from simple interaction rules

Microtubules

- Very long rigid polar hollow rods (length – 5-20 microns, diameter -20 nm, Persistent length – few mm)
- Length varies in time due to polymerization/depolymerization of tubulin
- Multiple function in the cell machinery: cytoskeleton formation, cell division, cell functioning



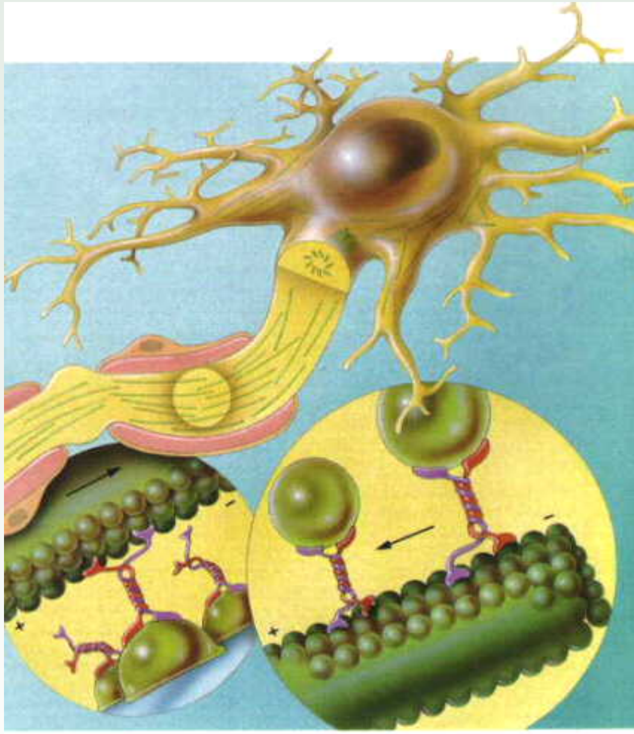
Molecular motors-Associated Proteins

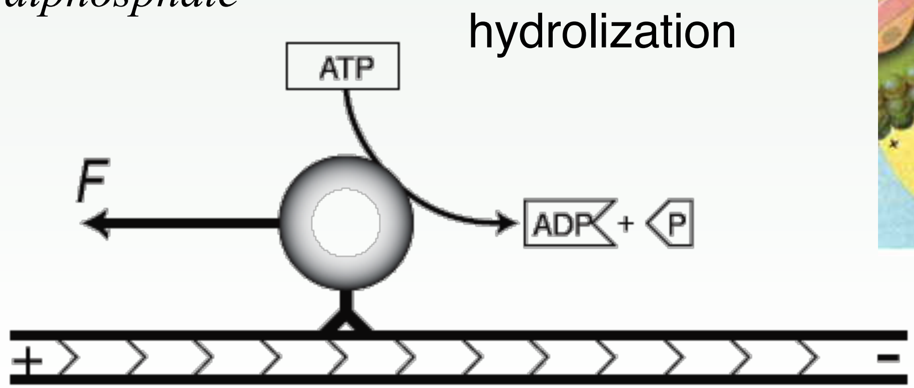
- **Linear motors** (kinesin, dynein, myosin) cytoskeleton formation, transport
- **Rotary motors:** (flagellar motor, F-ATPase) flagella rotation
- **Nucleic acid motors:** (helicase, topoisomerase) – DNA unwinding/translocation

Linear motors clusters:

- Have one head and one tail, but may cluster
- One attached to microtubules (MT)
  Other attached to vesicles, granules, or another MT
- Take energy from hydrolysis of ATP
- Speed ~1μm/s, step length 8 nm, run length ~1μm
- Exert force about 6 pN

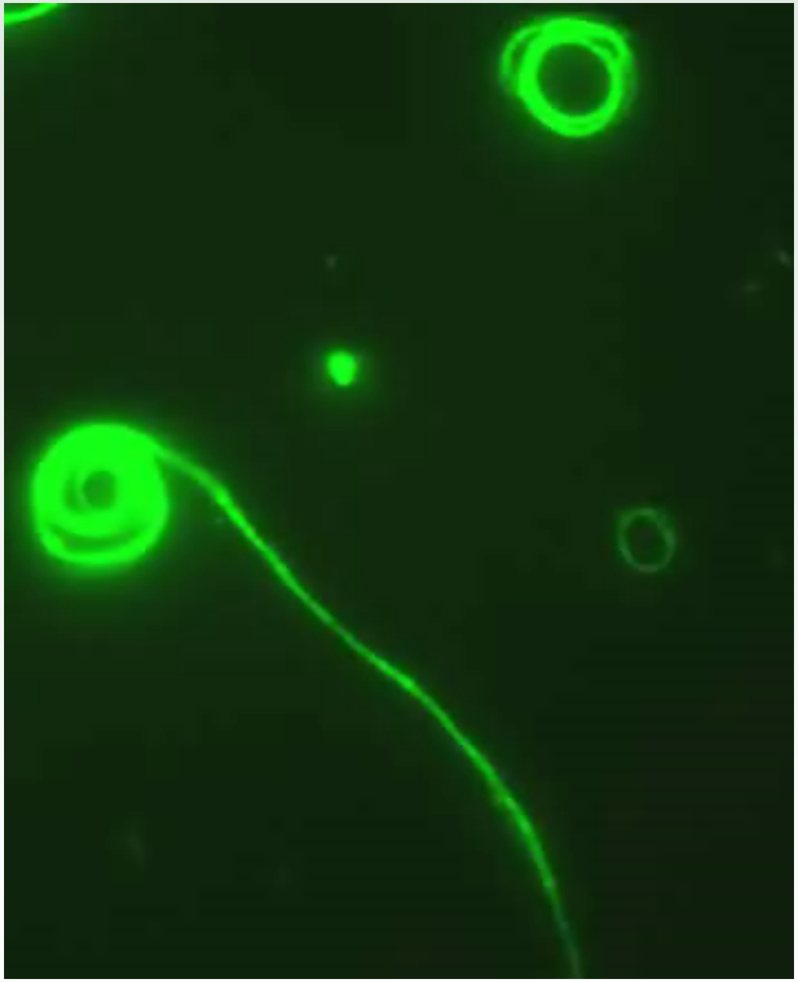
ATP – Adenosine triphosfate
ADP- Adenosine diphosphate





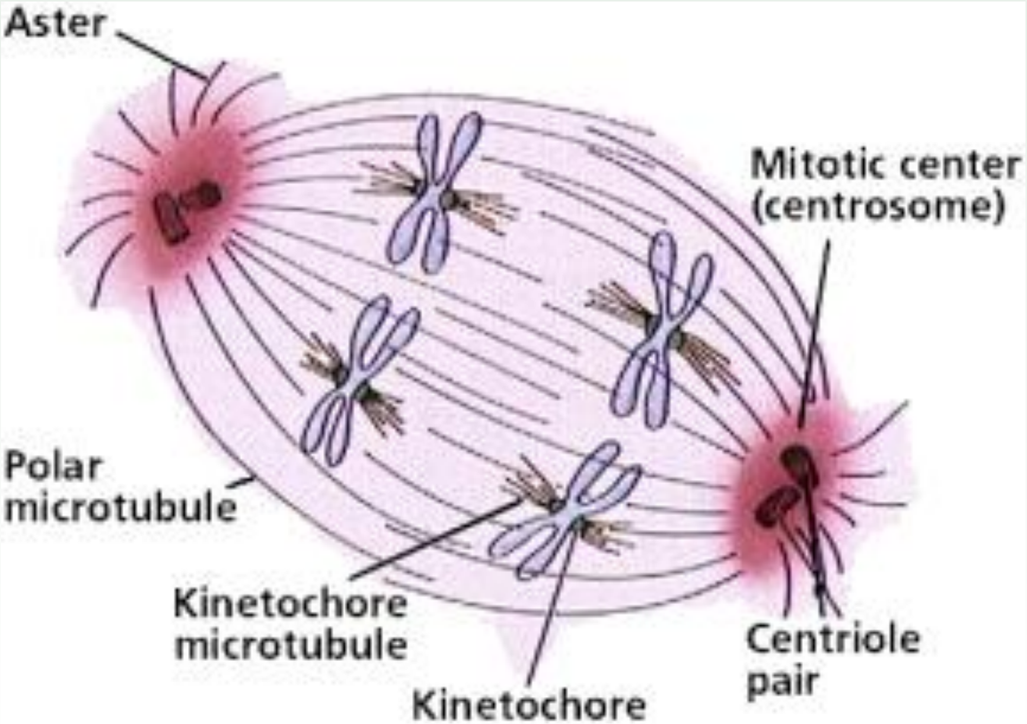
Molecular Motors on the Nanotechnology Workbench

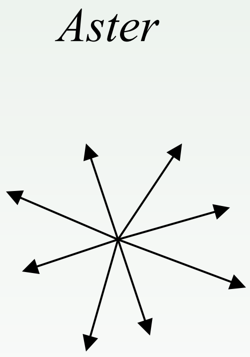
Self-assembly of micro-ring biocomposites



Dividing Cells and Mitotic Spindles

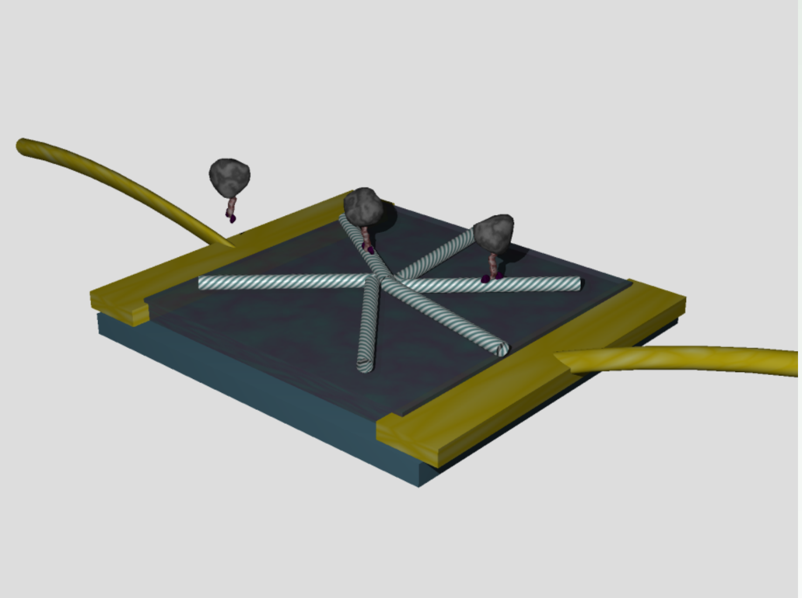
- Microtubules form cytoskeleton of dividing cells
- Separate chromosomes
- Asters: ray-like arrays of microtubules located around centrioles

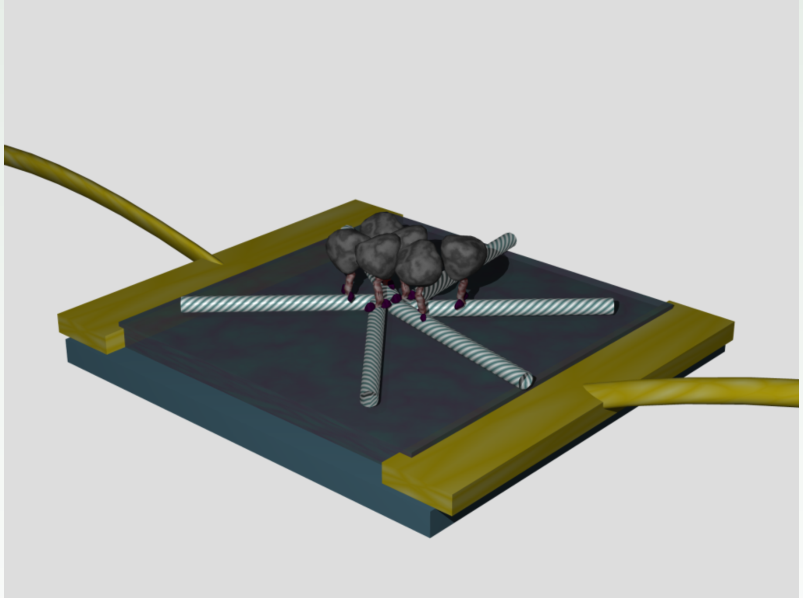




Bio-Inspired Amplification/Recognition

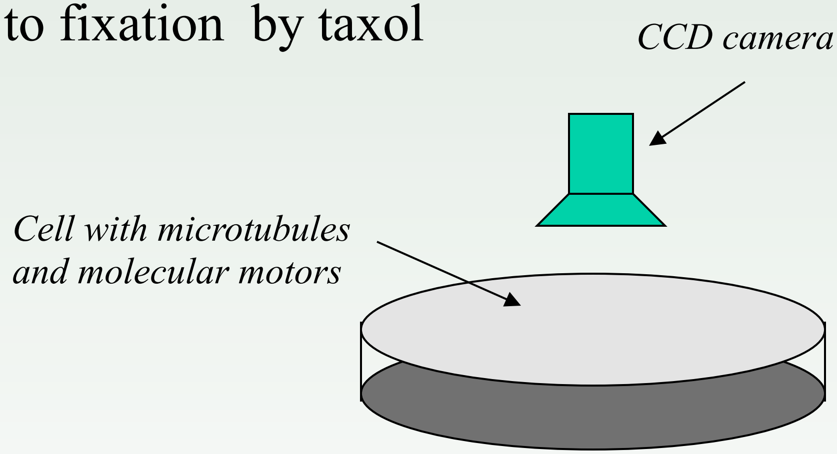
- *Motors bind to functionalized nano-particles (magnetic or fluorescent)*
- *Motors concentrate particles in the centers of self-assembled asters*
- *Particles detected/recognized either optically or magnetically*
- *Intriguing applications for bio-sensors and bio-amplifiers*





in-vitro Self-Assembly of MT and MM

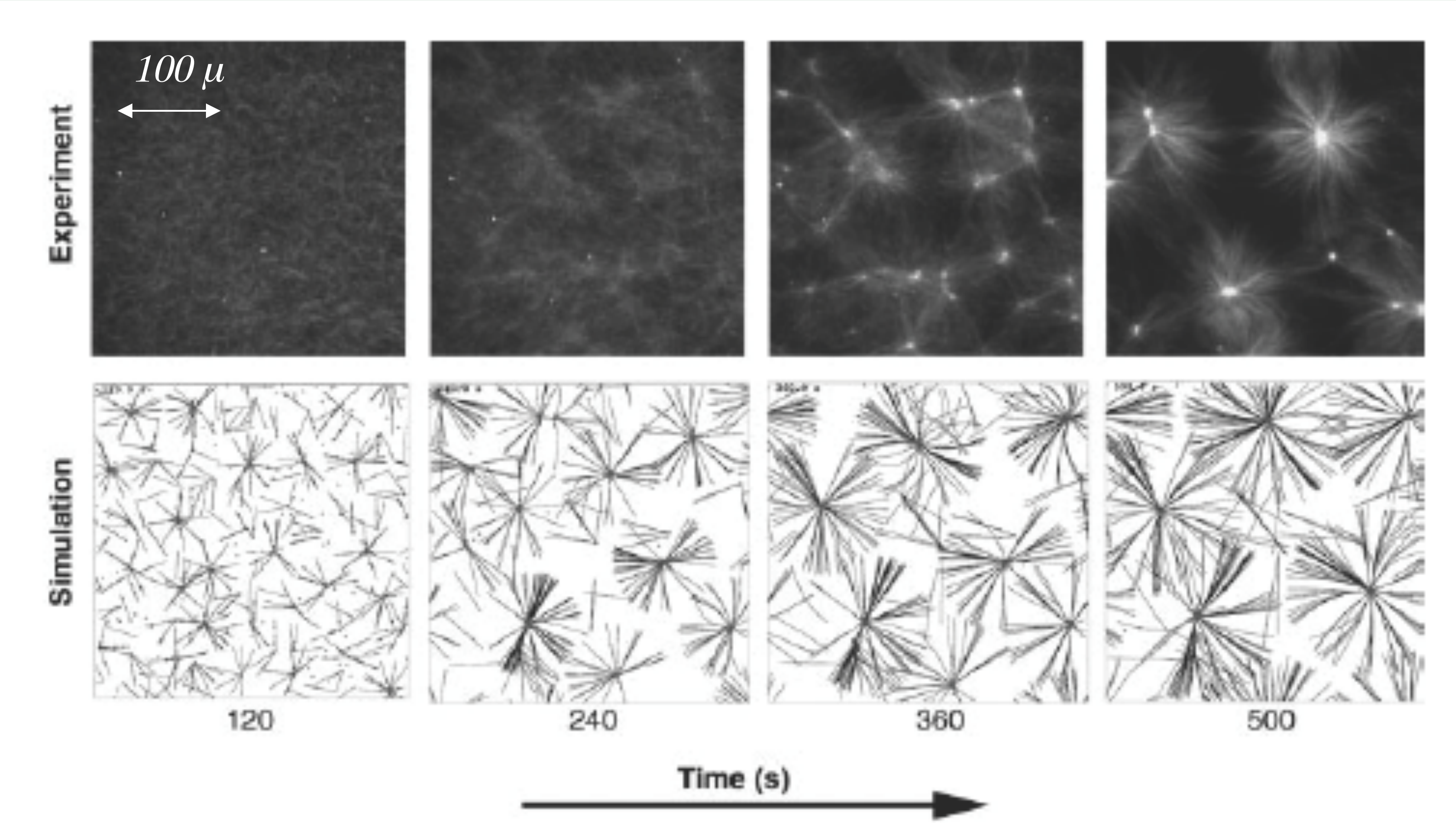
- Simplified system with only few purified components
- Experiments performed in 2D glass container: diameter 100 μm, height 5μm
- Controlled tubulin/motor concentrations and fixed temperature
- MT have fixed length 5μm due to fixation by taxol



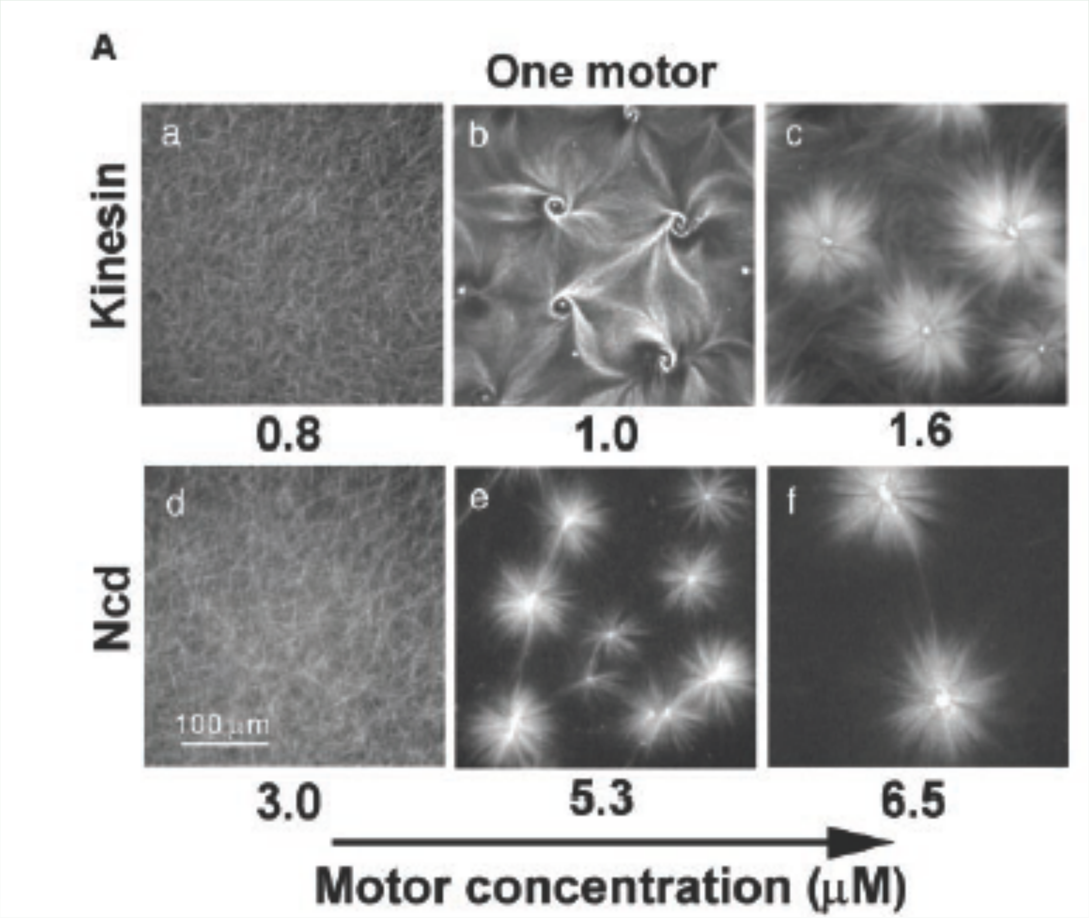
F. Nedelec, T. Surrey, A. Maggs, S. Leibler,
Self-Organization of Microtubules and Motors, Nature, 389 (1997)
T. Surray, F. Nedelec, S. Leibler & E. Karsenti,
Physical Properties Determining Self-Organization of Motors & Microtubules, Science, 292 (2001)

Patterns in MM-MT mixtures

Formation of asters, large kinesin concentration (scale 100 μ)



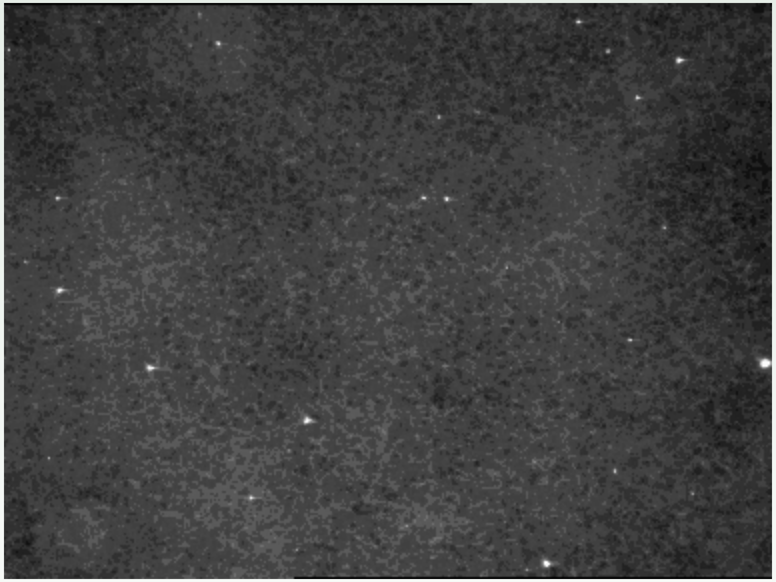
Vortex – Aster Transitions



Ncd – gluththione-S-transferase-nonclaret disjunctional fusion protein
Ncd walks in opposite direction to kinesin

Dynamics of Aster/Vortex Formation

High concentration of motors: asters



Low concentration of motors: vortex

Summary of Experimental Results

- Kinesin: vortices for low density of MM and asters for high density
- Ncd: asters are observed for all MM densities
- Bundles for very high MM density, asters disappear

- Possible difference between kinesin and NCD: kinesin falls off the end of MT, NCD sticks and dwells

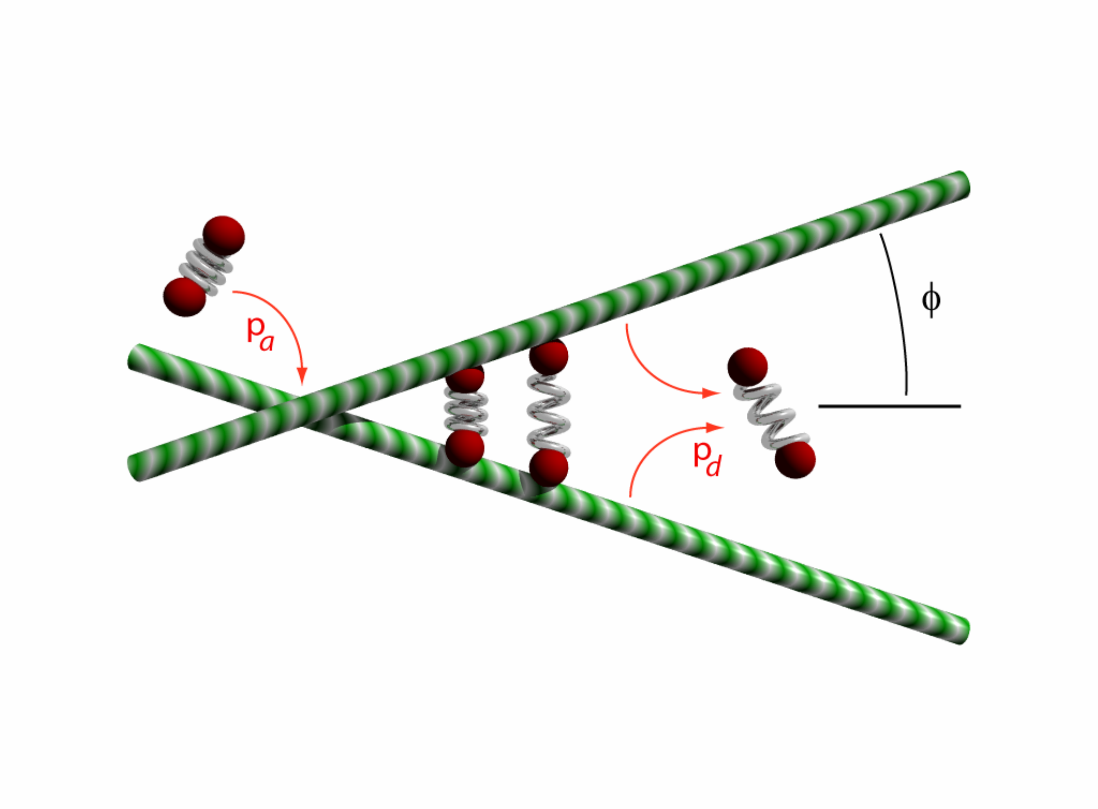
Two competing mechanisms

- Passive process: random reorientation and drift due to thermal fluctuations (compare Brownian motion)

Positions and orientations of microtubules change randomly in time. Due to thermal fluctuations will be no preferred orientation

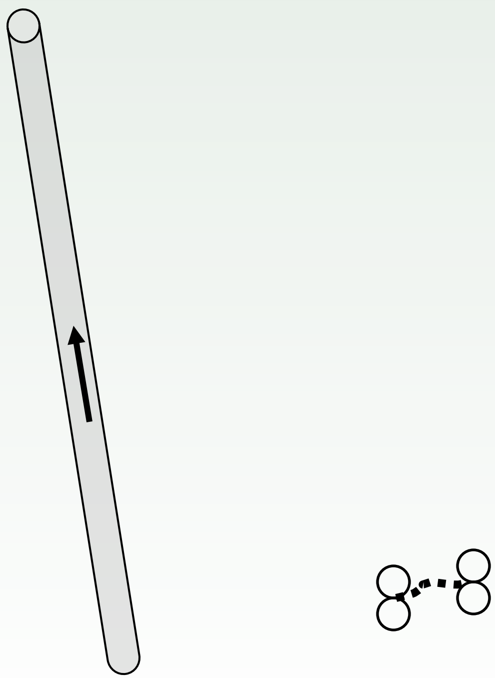
Active processes: alignment by the motors

- Molecular motors align microtubules (requires energy)
- Motors enforce fully aligned state



Mechanism of Self-Organization

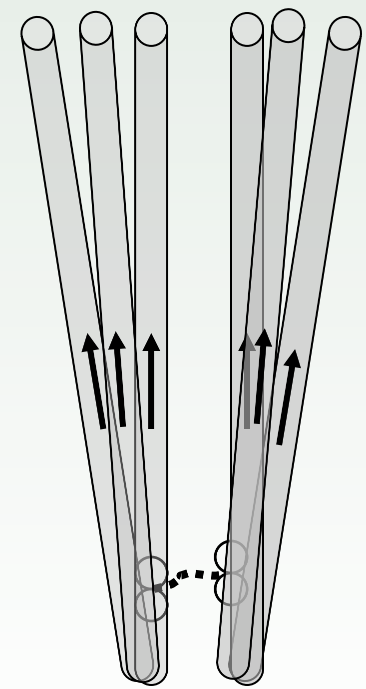
Motor binding to 1 MT – no effect



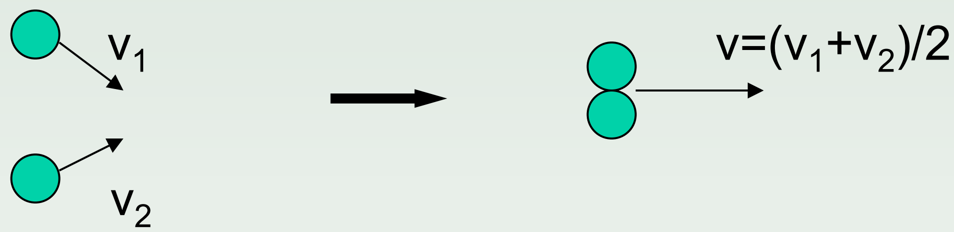
Motor binding to 2 MT –

mutual orientation after interaction

Zipper effect or inelastic collision



Collisions of Inelastic Grains



$$\begin{pmatrix} v_1^a \\ v_2^a \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ 1-\gamma & \gamma \end{pmatrix} \begin{pmatrix} v_1^b \\ v_2^b \end{pmatrix}$$

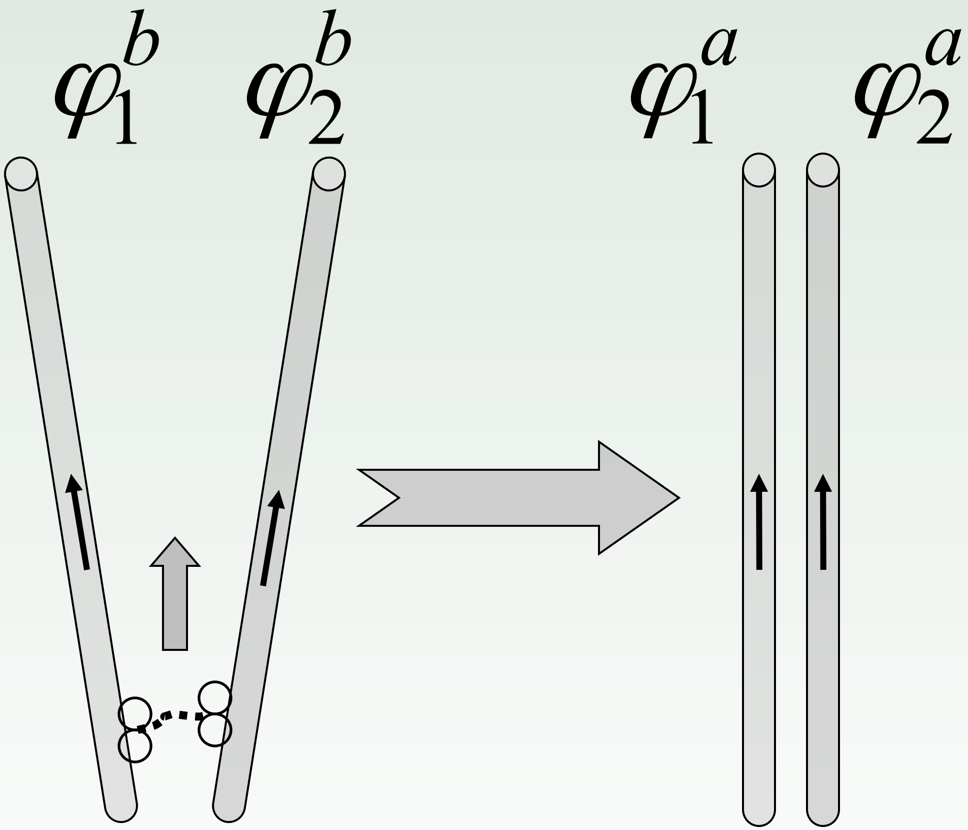
v^a & v^b velocities after/before collision

γ=0 – elastic collisions

γ=1/2 – fully inelastic collision

γ=*1* – no interaction

Inelastic Collision of Polar Rods



$$\varphi_1^a = \varphi_2^a = \frac{1}{2}\left(\varphi_1^b + \varphi_2^b\right)$$

$\varphi_{1,2}$ – orientation angles

Fully Inelastic Collision!!!

Molecular Dynamics Simulations of Stiff Inelastic Rods

- Simple rules

  -rigid rods of equal length

  -no explicit motors

  -fully inelastic collisions

  - rods diffuse anisotropically in 2 dim, $D_{parallel}$=2 $D_{perpendic}$

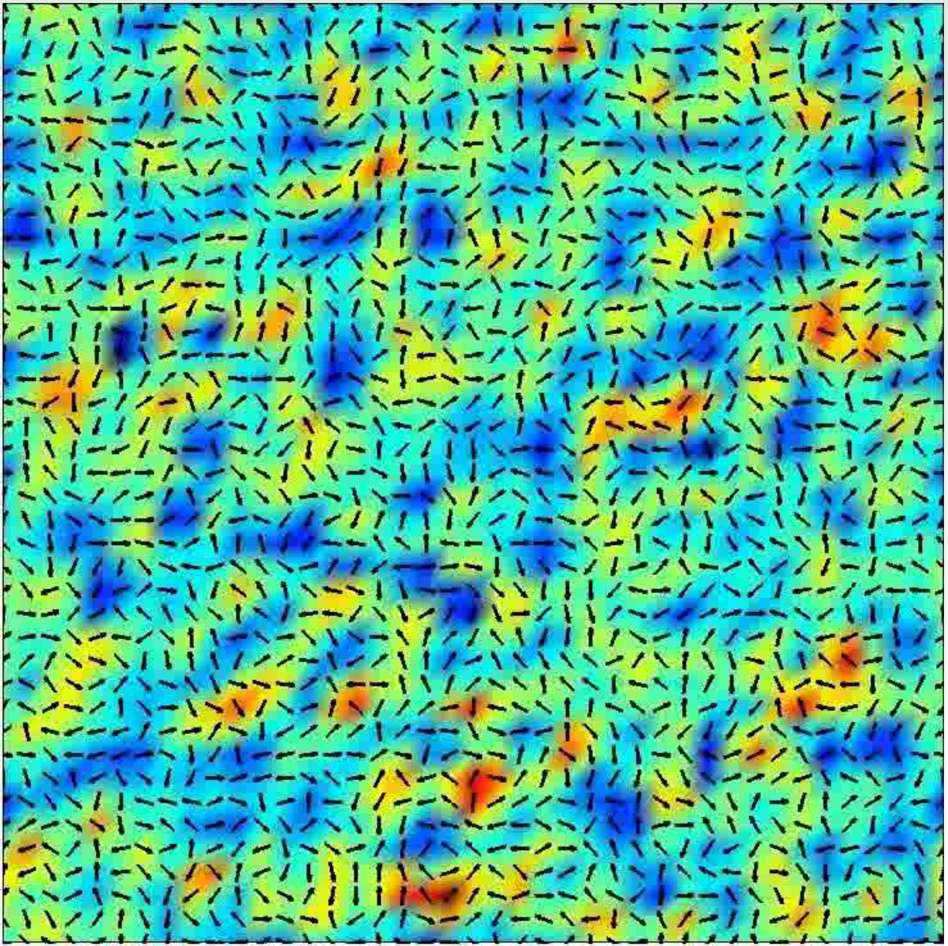
  -reorient upon collision with some probability P_{on}

  -probability of interaction depends on proximity to the end (dwelling)

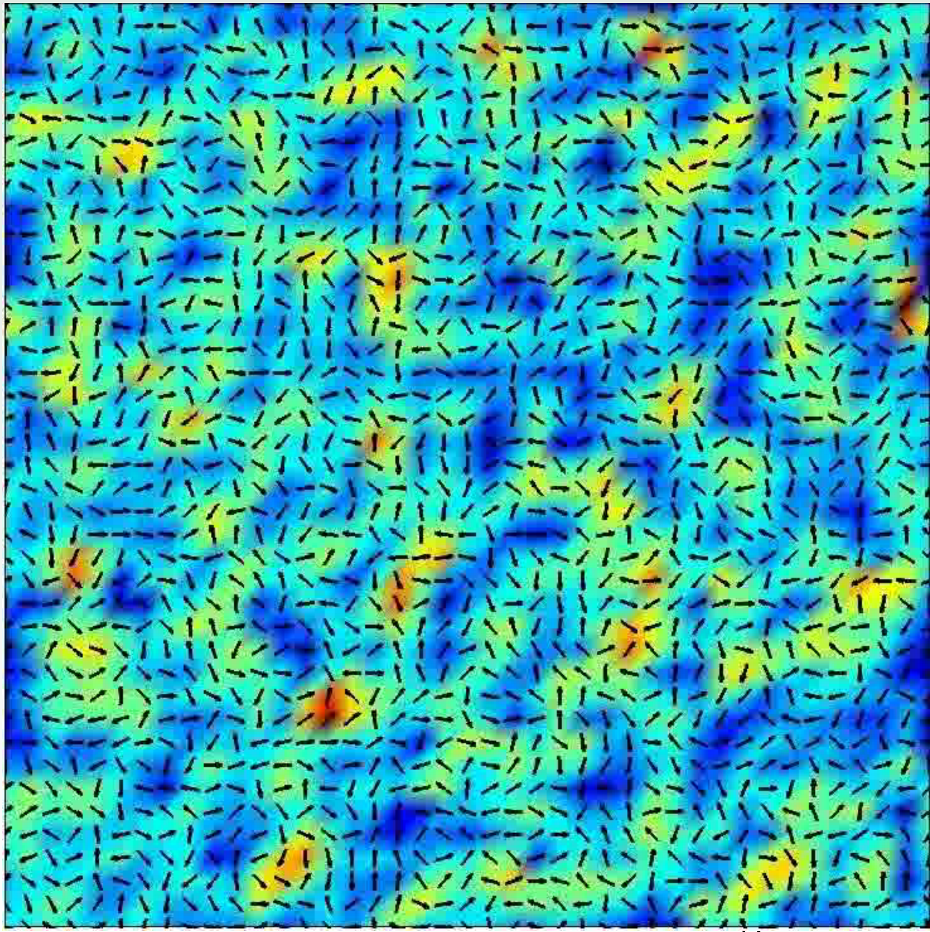
Jia, Bates, Karpeev, I.A. PRE 2008

Molecular Dynamics Simulations

Vortices



Asters



Random reorientation – diffusion

- Brownian motion: two types of description
- Stochastic equation – $$\frac{d\mathbf{r}}{dt} = \boldsymbol{\xi}(t)$$

  $\mathbf{r}$ – position of the particle

  $\boldsymbol{\xi}$ – random uncorrelated force

- Diffusion equation for the probability $P(\mathbf{r})$

$$\frac{\partial P(\mathbf{r})}{\partial t} = D\Delta P(\mathbf{r})$$

D – diffusion coefficient

Δ – Laplace operator

Langevin (stochastic) Equations

$$\frac{dx(t)}{dt} = f(x(t)) + \zeta(t)$$

$\zeta(t)$ – white Gaussian noise

$$\left\langle \zeta(t)\zeta(t') \right\rangle = D\delta(t - t')$$

$$\left\langle \zeta(t) \right\rangle = 0$$

D – noise intensity

$P(\zeta) = \frac{1}{\sqrt{2\pi D}} \exp\left(-\zeta^2 / 2D \right)$ – Gaussian (normal) distribution

Probability distributions for orientation angles $P(\varphi)$

- $P(\varphi)$ – probability to find a particle with orientation φ
- Consider small system – no spatial dependence
- Collision rate g does not depend on orientation (Maxwell molecules)
- Binary uncorrelated collisions
- Random reorientation of particles

Angular diffusion equation

$$\frac{\partial P(\varphi)}{\partial t} = D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2}$$

D_r – rotational diffusion coefficient

φ – orientation angle

- **However, φ is 2π – periodic function!!!**

$$P(\varphi,t) = \sum_{n=-\infty}^{\infty} C_n \exp\left(-D_\varphi n^2 t + in\varphi\right)$$

for $t \to \infty$ the distribution flattens: $P(\varphi,t) \to C_0 = const$

Binary collisions

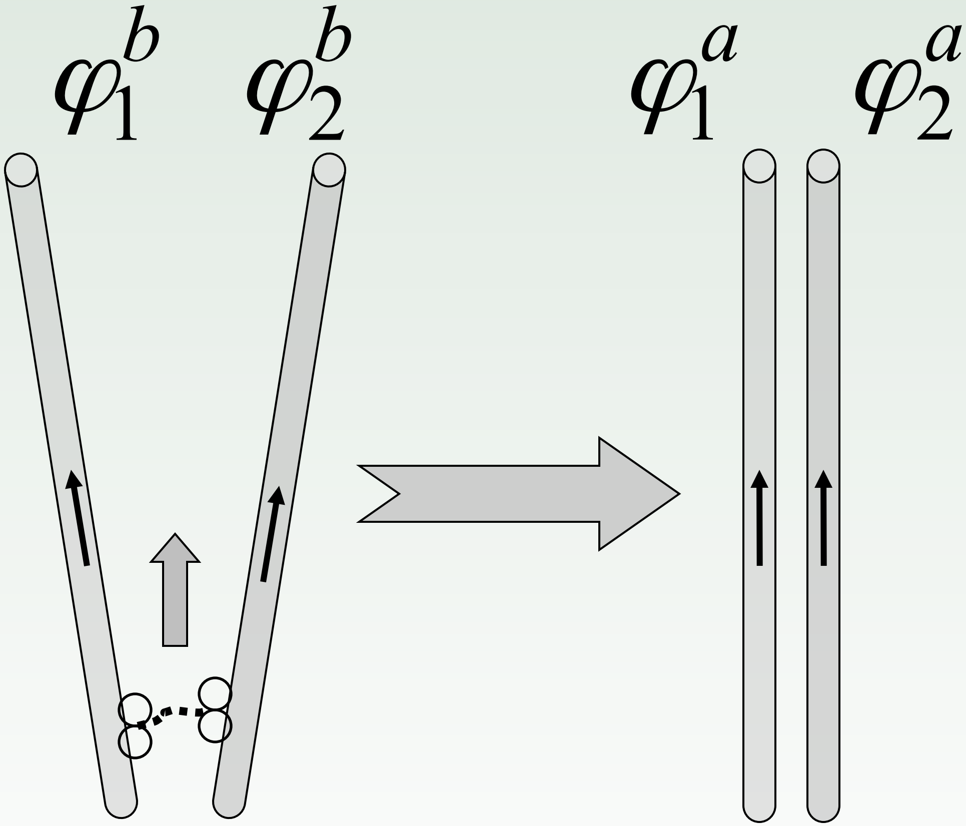
- **Collisions will favor aligned state**
- **If no noise, all rods will assume the same direction**

$$P(\varphi, t) \rightarrow \delta(\varphi - \varphi_0)$$

φ_0 some angle

- **noise (angular diffusion) will broaden the distribution**

Inelastic Collision of Polar Rods



$$\varphi_1^a = \varphi_2^a = \frac{1}{2}\left(\varphi_1^b + \varphi_2^b\right)$$

$\varphi_{1,2}$ – orientation angles

Fully Inelastic Collision!!!

Probability of binary collisions

- For two particles with orientations φ and ψ probability of binary interaction is $P(\varphi)\, P(\psi)$
- after the collision orientation are changed accordingly

$$\varphi_{new} \rightarrow \varphi - (\varphi - \psi)/2$$
$$\psi_{new} \rightarrow \psi + (\varphi - \psi)/2$$

- Therefore, $P(\varphi_{new})P(\psi_{new})$ added to the distribution and $P(\varphi)P(\psi)$ particles removed
- Total number of particles is conserved

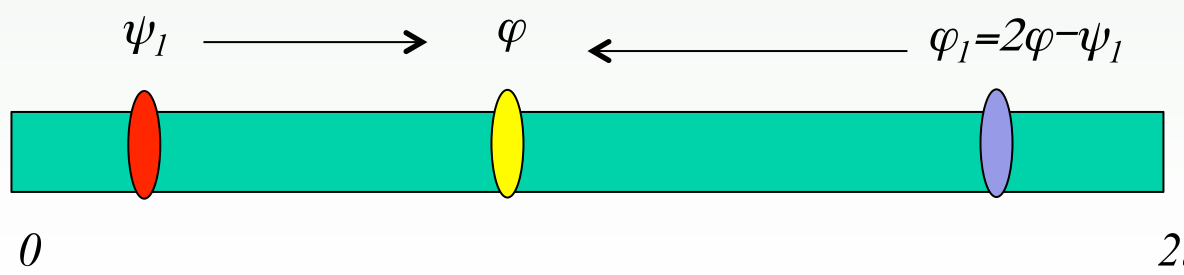
Collision Integral

- Now integrate over collision angle
- g – rate of collisions
- after the collision orientation are changed accordingly
- two regions contribute to $P(\varphi)$ (and, in fact, twice)

$$\frac{\partial P(\varphi)}{\partial t} = g\left[\int_{-\pi}^{\pi} d\psi_1 P(\varphi_1)P(\psi_1) - \int_{-\pi}^{\pi} d\psi P(\varphi)P(\psi)\right]$$

and $\dfrac{\varphi_1 + \psi_1}{2} = \varphi$

$$2P(2\varphi - \psi_1)P(\psi_1) \rightarrow P(\varphi)$$



Kinetic (balance) equation

- Now add rotational diffusion

$$\frac{\partial P(\varphi)}{\partial t} =$$

$$D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2} + g\left[\int d\psi 2P(2\varphi-\psi)P(\psi) - \int_{-\pi}^{\pi} d\psi P(\varphi)P(\psi)\right] =$$

substitute $\mathbf{u} \rightarrow 2(\varphi-\psi)$

$$D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2} + g\int_{-\pi}^{\pi} du P(\varphi - \frac{1}{2}u)P(\varphi + \frac{1}{2}u) - g\int_{-\pi}^{\pi} d\psi P(\varphi)P(\psi) =$$

$\mathbf{u} \rightarrow (\varphi-\psi)$

$$D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2} + g\int_{-\pi}^{\pi} du\left[P(\varphi - \frac{1}{2}u)P(\varphi + \frac{1}{2}u) - P(\varphi)P(\varphi - u)\right]$$

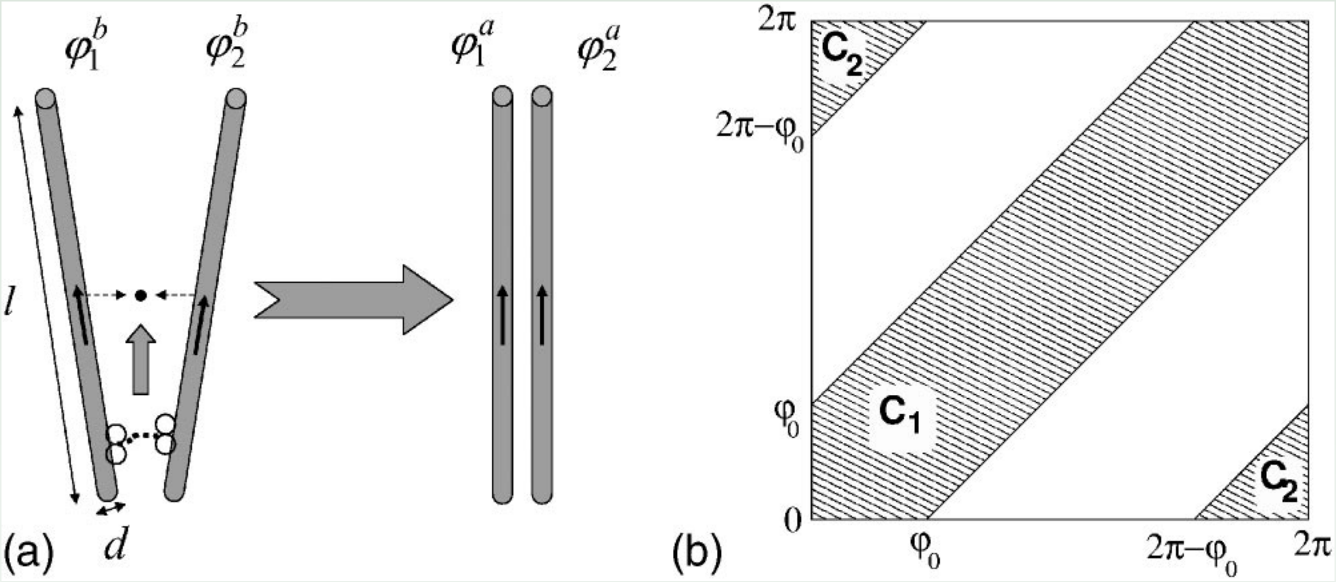
Kinetic equation

source *sink*

$$\frac{\partial P(\varphi)}{\partial t} = D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2} + g \int_{-\pi}^{\pi} du \left[P(\varphi - \frac{1}{2}u) P(\varphi + \frac{1}{2}u) - P(\varphi) P(\varphi - u) \right]$$

Main difference with the kinetic theory for non-ideal gases: finite limit of integration and periodic b.c.

Collision Integral



$$\varphi_1^a = \varphi_2^a = \frac{1}{2}\left(\varphi_1^b + \varphi_2^b\right) \text{ for } \left|\varphi_1^b + \varphi_2^b\right| < \varphi_0 < \pi$$

$$\varphi_1^{a,b} \rightarrow \varphi_1^{a,b} + \pi, \varphi_2^{a,b} \rightarrow \varphi_2^{a,b} - \pi \quad \text{for } 2\pi - \varphi_0 < \left|\varphi_1^b + \varphi_2^b\right| < 2\pi$$

Collision Integral: more systematic derivation

$$\frac{\partial P(\varphi)}{\partial t} = D_r \frac{\partial^2 P(\varphi)}{\partial \varphi^2}$$

$$+g\int_{C_1} d\varphi_1 d\varphi_2 P(\varphi_1)P(\varphi_2)\left[\delta\left(\varphi - \tfrac{1}{2}(\varphi_1+\varphi_2)\right) - \delta\left(\varphi - \varphi_2\right)\right]$$

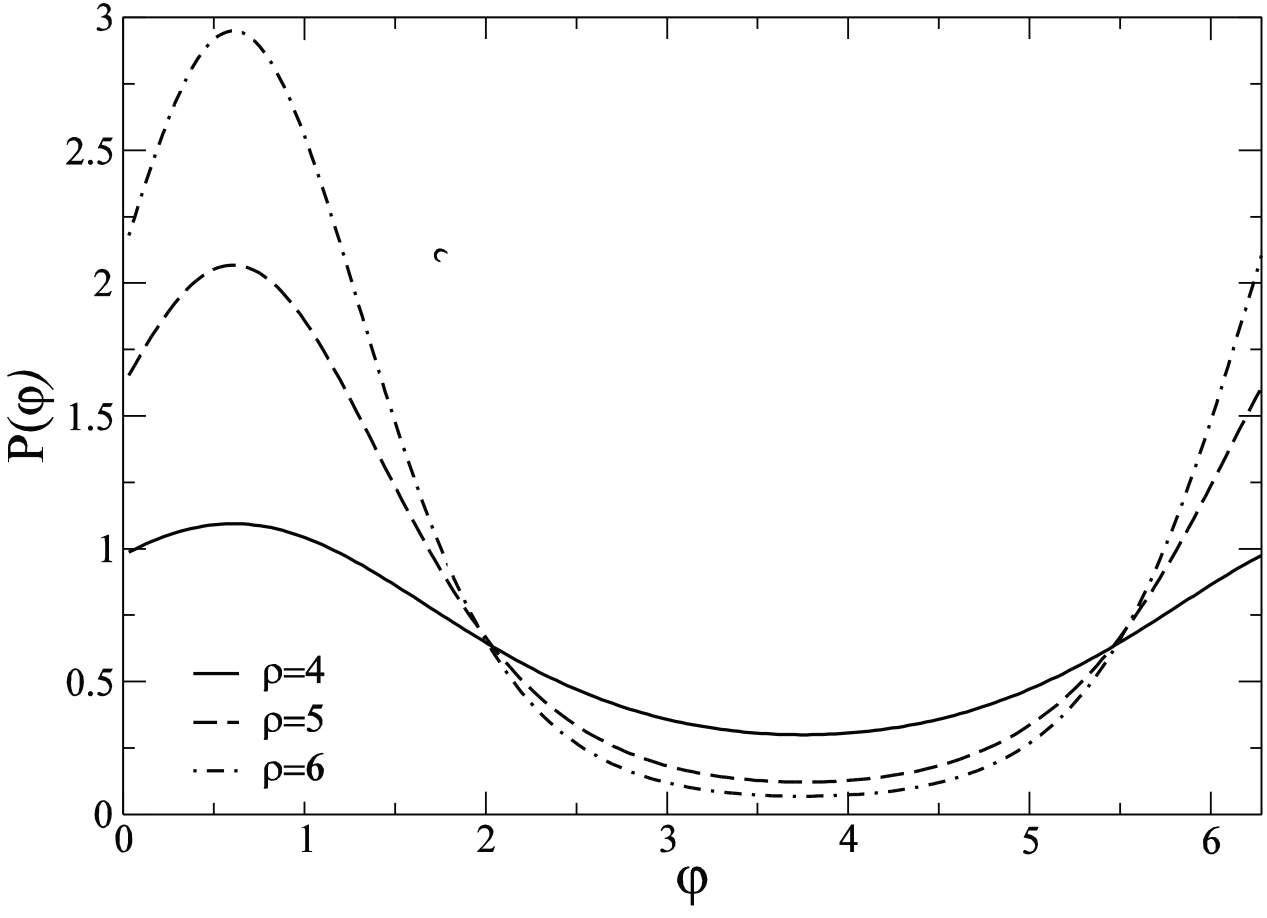
$$+g\int_{C_2} d\varphi_1 d\varphi_2 P(\varphi_1)P(\varphi_2)\left[\delta\left(\varphi - \tfrac{1}{2}(\varphi_1+\varphi_2) - \pi\right) - \delta\left(\varphi - \varphi_2\right)\right]$$

- *D_r - thermal rotational diffusion*
- *g – collision efficiency (~ concentration of motors)*

since diffusion of motors >> diffusion of microtubules
assume g=const

Stationary Orientation Distributions

Onset of a non-trivial distribution with the increase of the collision rate g



Stability of isotropic state

- Isotropic state: all orientations are equi-probable $P(\varphi)=1/2\pi$

- remember the norm condition $\int_0^{2\pi} d\varphi P(\varphi,t) = 1$

- Small perturbations of the isotropic state

$$P(\varphi,t) = \frac{1}{2\pi} + \xi = \frac{1}{2\pi} + \sum_{n=-\infty}^{\infty} \xi_n \exp[\lambda_n t + in\varphi]$$

Linearized system

$$\frac{\partial \xi(\varphi)}{\partial t} = D_r \frac{\partial^2 \xi(\varphi)}{\partial \varphi^2} + \frac{g}{2\pi} \int_{-\pi}^{\pi} du \left[\xi(\varphi + \tfrac{1}{2}u) + \xi(\varphi - \tfrac{1}{2}u) - \xi(\varphi) - \xi(\varphi - u) \right]$$

substituting for $n \neq 0$ $\quad \xi = \xi_n \exp[\lambda_n t + in\varphi]$

$$\lambda_n = -D_r n^2 + \frac{g}{2\pi} \int_{-\pi}^{\pi} du \left[\exp(\tfrac{in}{2}u) + \exp(-\tfrac{in}{2}u) \right] - g =$$

$$-D_r n^2 + \frac{4g}{n\pi} \sin(\pi n / 2) - g$$

for $n = 0$ $\quad \lambda_n = 0$ due to conservation of the # of particles

Linearized system

Eigenvalues $\lambda_n = \dfrac{4g}{\pi n}\sin(\pi n/2) - g - D_r n^2$

Most Unstable Mode (n=±1)

$$\lambda_0 = 0$$

$$\lambda_1 = g\left(4/\pi - 1\right) - D_r > 0$$

$$\lambda_2 = -g - 4D_r < 0$$

For $g > D_r/(4/\pi - 1) \approx 3.662\, D_r$ - isotropic state loses stability

Orientation phase transition above critical value of the collision rate g ~motor density !!!

Macroscopic Variables: Derivation of the Landau (Stuart) equation

- Concentration $$\rho = \int_{-\pi}^{\pi} P(\varphi)d\varphi$$

- Average orientation $\boldsymbol{\tau}=(\tau_x, \tau_y)$

$$\tau_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\varphi P(\varphi)d\varphi \qquad \tau_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\varphi P(\varphi)d\varphi$$

- "Complex orientation" $$\psi = \tau_x + i\tau_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\varphi} P(\varphi)d\varphi$$

Fourier Expansion

$$P(\varphi) = \sum_{n=-\infty}^{\infty} P_n e^{in\varphi}; \qquad P_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\varphi) e^{-in\varphi} d\varphi$$

Relation to observables

$$\rho = 2\pi P_0; \qquad \psi = P_{-1}; \qquad \psi^* = P_1$$

Asymptotic expansion for P_n

$$\dot{P}_k + (D_r k^2 + g)P_k = 2\pi g \sum_n \sum_m P_n P_m \frac{\sin[\pi(n-m)/2]}{\pi(n-m)/2} \delta_{n+m,k}$$

Scaling of variables $\quad t \to D_r t; \qquad P_n \to \dfrac{g}{D_r} P_n$

Introduce concentration
(or effective collision rate) $\quad \rho \to \dfrac{g}{D_r}$

$$\dot{P}_k + (k^2 + \rho)P_k = 2\pi \sum_n \sum_m P_n P_m \frac{\sin[\pi(n-m)/2]}{\pi(n-m)/2} \delta_{n+m,k}$$

Asymptotic expansion for P_n

$$\dot{P}_k + (k^2 + \rho)P_k = 2\pi \sum_n \sum_m P_n P_m \frac{\sin[\pi(n-m)/2]}{\pi(n-m)/2} \delta_{n+m,k}$$

- Diffusion $-k^2$ forces rapid decay higher harmonics
- Linear growth rates λ_n

$$\lambda_0 = 0 \qquad \rho = g/D_r$$

$$0 < \lambda_1 = \left(4/\pi - 1\right)\rho - 1 = \varepsilon \ll 1 - \text{near the threshold}$$

$\lambda_n < 0 \ \ for\ |n| \geq 2$ Neglect higher harmonics

Asymptotic expansion in the powers of ε

$$P(\varphi,t)=P_0(\varepsilon^2 t)+\varepsilon P_1(\varepsilon^2 t)e^{i\varphi}+\varepsilon^2 P_2(\varepsilon^2 t)e^{i2\varphi}+\text{compex conjugated}$$

$$t\to\varepsilon^2 t,\varepsilon-\text{small parameter},\;\; P(\varphi)=\sum_{n=-\infty}^{\infty}\varepsilon^{|n|}P_n(\varepsilon^2 t)e^{in\varphi},P_n=P_{-n}^*$$

$$\varepsilon^2\dot P_0=0$$

$$\varepsilon^3\dot P_1=\underbrace{\varepsilon(2P_0(4-\pi)-1)}_{\lambda_1\sim\varepsilon^2}P_1-\frac{8}{3}\varepsilon^3 P_2P_{-1}\;\text{ remember }\quad 2\pi\sum_n\sum_m P_nP_m\frac{\sin[\pi(n-m)/2]}{\pi(n-m)/2}\delta_{n+m,k}$$

$$\cancel{\varepsilon^4\dot P_2}=\underbrace{-\varepsilon^2(2\pi P_0+4)}_{\lambda_2\sim O(1)<0}P_2+2\pi\varepsilon^2P_1^2+O(P_3)$$

$$\Downarrow$$

$$P_2=-\frac{2\pi P_1^2}{2\pi P_0+4}\;\Longrightarrow\;\dot P_1=(2P_0(4-\pi)-1)P_1-\frac{8}{3}\frac{2\pi}{2\pi P_0+4}P_{-1}P_1^2$$

Asymptotic Landau Equation

- Truncation of series for $|n|>2$ and recall $\boldsymbol{\tau}=P_{-1}$

$$\frac{\partial \rho}{\partial t} = 0$$

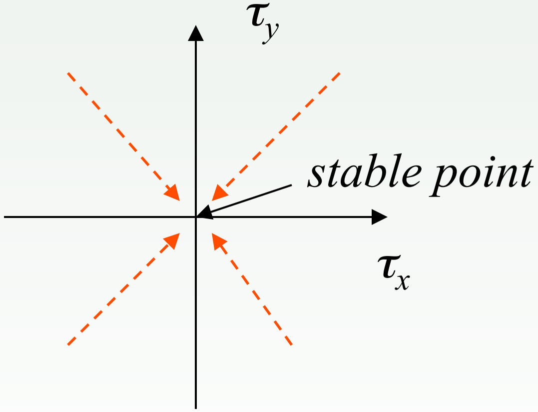
$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \left(\left(\tfrac{4}{\pi} - 1\right)\rho - 1\right)\boldsymbol{\tau} - \frac{16\pi}{3(4+\rho)} \,|\, \boldsymbol{\tau} \,|^2\, \boldsymbol{\tau} \approx \left(0.273\rho - 1\right)\boldsymbol{\tau} - 2.18 \,|\, \boldsymbol{\tau} \,|^2\, \boldsymbol{\tau}$$

- Second order phase transition for $\rho>\rho_c = 1/0.273 \approx 3.662$

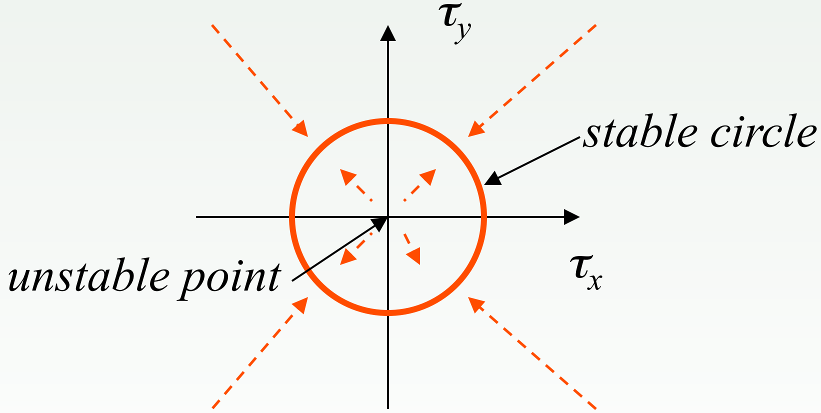
Second order phase transition for $\rho>\rho_c$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} \approx \left(\rho/\rho_c - 1\right)\boldsymbol{\tau} - 2.18\,|\boldsymbol{\tau}|^2\,\boldsymbol{\tau}$$

$\rho<\rho_c$ – no preferred orientation
$|\boldsymbol{\tau}| \to 0$, stable point $\boldsymbol{\tau}=0$

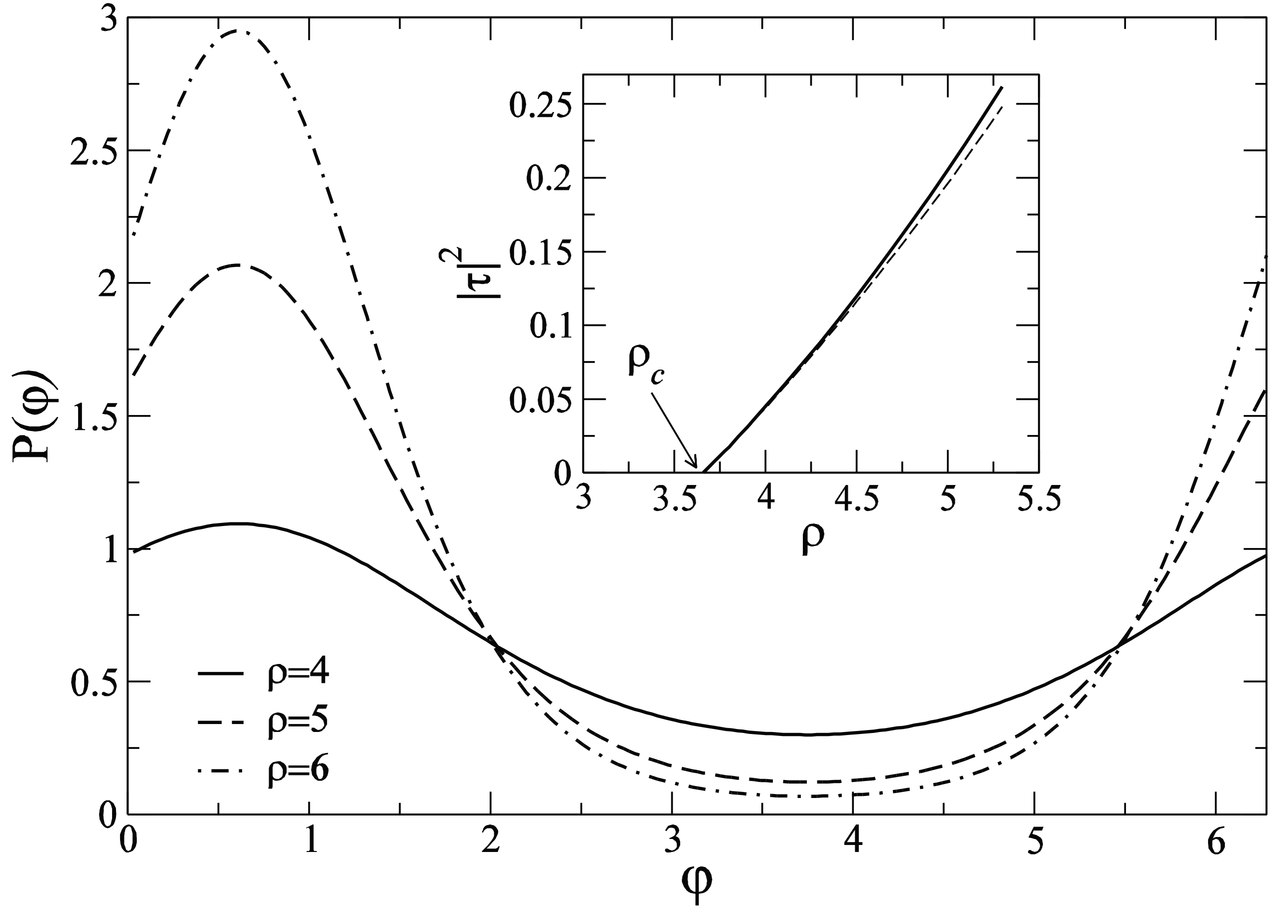


$\rho>\rho_c$ – onset of preferred orientation
$|\boldsymbol{\tau}| \to$ const, direction is determined by initial distribution, stable limit circle



Compare with the Hopf Bifurcation Scenario

Stationary Angular Distributions Comparison with Numerical Solution



Brownian motion of spherical particles

- Einstein-Stokes relation

$$D = \mu k_B T$$

$\mu -$ mobility

$k_B -$ Boltzmann constant

$D -$ diffusion coeffifient

$T -$ temperature (in Kelvins)

$$\frac{\partial P(\mathbf{r})}{\partial t} = D\Delta P(\mathbf{r})$$

Equivalent Langevin equation

$$\dot{\mathbf{r}} = \xi(t), \left\langle \xi(t)\xi(t') \right\rangle = 2D\delta(t - t')$$

Translational Diffusion for Spherical Particle

- In the limit of small Reynolds number mobility is the inverse drag coefficient

$$D = \mu k_B T = k_B T / \zeta$$

$$\mu = 1/\zeta$$

For translational motion

$$\zeta_{tr} = 6\pi\eta a$$

a – radius of the particle

η – dynamic viscosity of the liquid

$$D_{tr} = \frac{k_B T}{6\pi\eta a}$$

Rotational Diffusion for Spherical Particle

- In the limit of small Reynolds number mobility is inverse the drag coefficient

$$D_r = \mu_r k_B T = k_B T / \zeta_r$$

For rotational motion

$$\zeta_r = 8\pi\eta a^3$$

a – radius of the particle

η – dynamic viscosity of the liquid

$$D_r = \frac{k_B T}{8\pi\eta a^3}$$

Translation and Rotation of Rods

Anisotropic friction coefficients

$$\dot{x}_{\parallel} = -\Gamma_{\parallel} f_{\parallel}; \quad \dot{x}_{\perp} = -\Gamma_{\perp} f_{\perp}$$

Diffusion matrix depends on the angle

$$D_{ij}(\varphi) = \bar{D}\delta_{ij} + \Delta D \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

$$\bar{D} = (D_{\parallel} + D_{\perp}) / 2$$

$$\Delta D = (D_{\parallel} - D_{\perp}) / 2$$

Diffusion Matrix: different form

$$D_{ij} = D_{\|} n_i n_j + D_{\perp}(\delta_{ij} - n_i n_j) - \text{diffusion matrix}$$

$\mathbf{n}$=(cos(φ), sin(φ))- unit orientaional vector

$$D_{\|} = k_B T \frac{\log(l/d)}{2\pi\eta l} - \text{parallel diffusion}$$

$$D_{\perp} = D_{\|}/2 - \text{perpendicular diffusion}$$

$$D_r = k_B T \frac{12\log(l/d)}{\pi\eta l^3} - \text{rotational diffusion}$$

l – length of the rod, *d*- diameter,
η – dynamic viscosity of the fluid

Fokker-Plank eq for diffusing rods

$$\frac{\partial P(\varphi,\mathbf{r})}{\partial t} = D_r \frac{\partial^2 P(\varphi,\mathbf{r})}{\partial \varphi^2} + \partial_i D_{ij}(\varphi)\partial_j P(\varphi,\mathbf{r}) =$$

$$D_r \frac{\partial^2 P(\varphi,\mathbf{r})}{\partial \varphi^2} + \frac{D_\parallel + D_\perp}{2}\Delta P(\varphi,\mathbf{r}) + \frac{D_\parallel - D_\perp}{2}\partial_i Q_{ij}(\varphi)\partial_j P(\varphi,\mathbf{r})$$

$$Q_{ij}(\varphi) = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

Spatial Localization of Interaction

- Interaction between rods decay with the distance
- Translational and rotational diffusion of rods

$$\frac{\partial P(\varphi,\mathbf{r})}{\partial t} = D_r \frac{\partial^2 P(\varphi,\mathbf{r})}{\partial \varphi^2} + \partial_i D_{ij} \partial_j P(\varphi,\mathbf{r}) + gI(W:P)$$

$I(W:P)$ – collision integral

W - interaction kernel

$D_{ij} = D_{\|} n_i n_j + D_{\perp}(\delta_{ij} - n_i n_j)$ – translational diffusion matrix

Collision Integral

$$I(W:P) = \iint d\mathbf{r}_1 d\mathbf{r}_2 \iint d\phi_1 d\phi_2 P(\phi_1, \mathbf{r}_1) P(\phi_2, \mathbf{r}_2) W(\phi_1, \mathbf{r}_1, \phi_2, \mathbf{r}_2) \times$$

$$\times \left[\delta(\mathbf{r} - (\mathbf{r_1} + \mathbf{r_2})/2) \delta(\phi - (\phi_1 + \phi_2)/2) - \delta(\mathbf{r} - \mathbf{r_2}) \delta(\phi - \phi_2) \right]$$

Interaction Kernel

- Decays with distance between rods
- Depends on relative angle between rods
- Symmetric with respect permutation of rods

$$W(\mathbf{r}_1,\mathbf{r}_2,\varphi_1,\varphi_2)=\frac{1}{\pi b^2}\exp\left[-\frac{|\mathbf{r}_1-\mathbf{r}_2|^2}{b^2}\right]\left(1+\beta(\mathbf{r}_1-\mathbf{r}_2)(\mathbf{n}_1-\mathbf{n}_2)\right)$$

$b=O(l)$ interaction scale

β characterizes anisotropy of interaction between polar rods

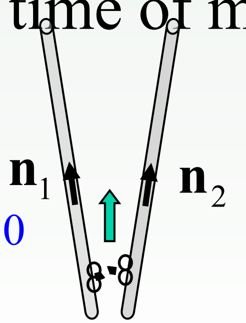
β ~dwelling time of motor at the end of MT

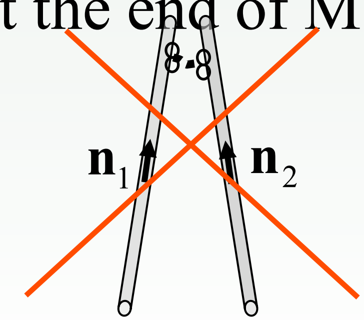
β small for kinesin

β large for NCD

$$\beta \approx V_0 - p_{end} l_0^2 M_{bound}$$

$(\mathbf{r}_1-\mathbf{r}_2)(\mathbf{n}_1-\mathbf{n}_2)>0$





$(\mathbf{r}_1-\mathbf{r}_2)(\mathbf{n}_1-\mathbf{n}_2)<0$

Generalized expansion in the powers of ε

$$P(\mathbf{r},\varphi,t) = P_0(\varepsilon\mathbf{r},\varepsilon^2 t) + \varepsilon P_1(\varepsilon\mathbf{r},\varepsilon^2 t)e^{i\varphi} + \varepsilon^2 P_2(\varepsilon\mathbf{r},\varepsilon^2 t)e^{i2\varphi} + \ldots + \text{compex conjugated}$$

$$t \to \varepsilon^2 t, \mathbf{r} \to \varepsilon\mathbf{r}, \varepsilon - \text{small parameter,}\ \ P(\mathbf{r},\varphi,t) = \sum_{n=-\infty}^{\infty} \varepsilon^{|n|} P_n(\varepsilon\mathbf{r},\varepsilon^2 t)e^{in\varphi}, P_n = P_{-n}^*$$

What does happen with the diffusion

$$\frac{\partial P(\varphi,\mathbf{r})}{\partial t} = D_r \frac{\partial^2 P(\varphi,\mathbf{r})}{\partial \varphi^2} + \frac{D_\parallel + D_\perp}{2}\Delta P(\varphi,\mathbf{r}) + \frac{D_\parallel - D_\perp}{2}\partial_i Q_{ij}(\varphi)\partial_j P(\varphi,\mathbf{r})$$

$$Q_{ij}(\varphi) = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

$$P(\mathbf{r},\varphi,t) = P_0(\varepsilon\mathbf{r},\varepsilon^2 t) + \varepsilon P_1(\varepsilon\mathbf{r},\varepsilon^2 t)e^{i\varphi} + \varepsilon^2 P_2(\varepsilon\mathbf{r},\varepsilon^2 t)e^{i2\varphi} + \ldots + \text{compex conjugated}$$

$$\frac{\partial P_0(\mathbf{r})}{\partial t} = \frac{D_\parallel + D_\perp}{2}\Delta P_0(\mathbf{r}) + \ldots$$

$$\frac{\partial P_1(\mathbf{r})}{\partial t} = \frac{D_\parallel + D_\perp}{2}\Delta P_1(\mathbf{r}) + \frac{D_\parallel - D_\perp}{4}\partial_i Q^0_{ij}\partial_j P_{-1}(\mathbf{r}) - D_r P_1(\mathbf{r}) + \ldots$$

$$Q^0_{ij} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

Some simplification

$$\frac{\partial P_1(\mathbf{r})}{\partial t} = \frac{D_\parallel + D_\perp}{2}\Delta P_1(\mathbf{r}) + \frac{D_\parallel - D_\perp}{4}\partial_i Q^0_{ij}\partial_j P_{-1}(\mathbf{r}) + \dots$$

$$Q^0_{ij} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

Average orientation $\boldsymbol{\tau}=(\tau_x,\tau_y)$

Complex orientation $\psi = \tau_x + i\tau_y = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\varphi} P(\varphi)d\varphi = P_{-1}$

$$\frac{\partial \psi^*}{\partial t} = \frac{D_\parallel + D_\perp}{2}\Delta\psi^* + \frac{D_\parallel - D_\perp}{4}\left(\partial_x^2\psi - 2i\partial_x\partial_y\psi - \partial_y^2\psi\right) - D_r\psi^*$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{D_\parallel + 3D_\perp}{4}\Delta\boldsymbol{\tau} + \frac{D_\parallel - D_\perp}{4}\nabla(\nabla\cdot\boldsymbol{\tau}) - D_r\boldsymbol{\tau}$$

Dealing with the Kernel in the Collision Integral

$$I(W:P)=\iint d\mathbf{r}_1 d\mathbf{r}_2 \iint d\phi_1 d\phi_2 P(\phi_1,\mathbf{r}_1)P(\phi_2,\mathbf{r}_2)W(\phi_1,\mathbf{r}_1,\phi_2,\mathbf{r}_2)\times$$

$$\times\left[\delta(\mathbf{r}-(\mathbf{r}_1+\mathbf{r}_2)/2)\delta(\phi-(\phi_1+\phi_2)/2)-\delta(\mathbf{r}-\mathbf{r}_2)\delta(\phi-\phi_2)\right]$$

$$W\left(\mathbf{r}_1,\mathbf{r}_2,\varphi_1,\varphi_2\right)=\frac{1}{\pi b^2}\exp\left[-\frac{\left|\mathbf{r}_1-\mathbf{r}_2\right|^2}{b^2}\right]\left(1+\beta\left(\mathbf{r}_1-\mathbf{r}_2\right)\left(\mathbf{n}_1-\mathbf{n}_2\right)\right)$$

Approximation of narrow kernel (small b): $\mathbf{r}_1=\mathbf{r}_2+\xi$

$$I(W:P)=\iint d\xi d\mathbf{r}_2 \iint d\phi_1 d\phi_2 P(\phi_1,\mathbf{r}_2+\xi)P(\phi_2,\mathbf{r}_2)W(\phi_1-\phi_2,\xi)\times$$

$$\times\left[\delta(\mathbf{r}-(\xi+2\mathbf{r}_2)/2)\delta(\phi-(\phi_1+\phi_2)/2)-\delta(\mathbf{r}-\mathbf{r}_2)\delta(\phi-\phi_2)\right]$$

Continuum Equations

$$\frac{\partial \rho}{\partial t} = \nabla^2 \left[\frac{\rho}{32} - \frac{B^2 \rho^2}{16} \right] - \frac{7B^4 \rho_0 \nabla^4 \rho}{256}$$

Friction anisotropy

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = (0.273\rho - 1)\boldsymbol{\tau} - 2.18\,|\,\boldsymbol{\tau}\,|^2\,\boldsymbol{\tau} + \frac{5\nabla^2 \boldsymbol{\tau}}{192} + \frac{\nabla\nabla \cdot \boldsymbol{\tau}}{96} + \frac{B^2 \rho\, \nabla^2 \boldsymbol{\tau}}{4\pi} +$$
$$+ H \left[\frac{\nabla \rho^2}{16\pi} - \left(\pi - \tfrac{8}{3}\right)\boldsymbol{\tau}(\nabla \cdot \boldsymbol{\tau}) - \tfrac{8}{3}(\boldsymbol{\tau}\nabla)\boldsymbol{\tau} \right]$$

Kernel anisotropy

$r \to \dfrac{r}{l}$ $\quad$ $B = \dfrac{b}{l}$ <1/2 normalized cuttoff length

$H = \dfrac{\beta b^2}{l}$ normalized kernel anisotropy (~ dwelling time at the end)

Asters and Vortices

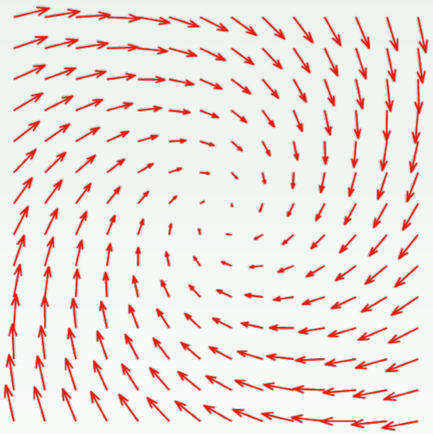
- For $HB^2 << 1$ equations split and become independent

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = (0.273\rho - 1)\boldsymbol{\tau} - |\boldsymbol{\tau}|^2 \boldsymbol{\tau} + \frac{5\nabla^2 \boldsymbol{\tau}}{192} + \frac{B^2 \rho \nabla^2 \boldsymbol{\tau}}{4\pi} + \frac{\nabla\nabla\cdot\boldsymbol{\tau}}{96} - H\left[0.321\boldsymbol{\tau}(\nabla\cdot\boldsymbol{\tau}) - 1.81(\boldsymbol{\tau}\nabla)\boldsymbol{\tau}\right]$$

- Without blue and red terms Eq possesses a "Vortex Solution"

(compare with Abrikosov vortices in type-II superconductors)

$$\psi = \tau_x + i\tau_y = F(r)\exp[i\theta + i\varphi]$$



r, θ-polar coordinates

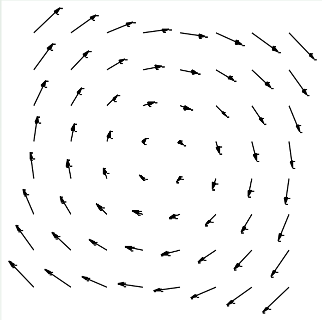
$\varphi = const$ arbitrary phase (tilt angle)

Vortices

- For H=0 (no red terms) the only stable solutions φ= ±π/2

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \left(0.273\rho - 1\right)\boldsymbol{\tau} - |\boldsymbol{\tau}|^2 \boldsymbol{\tau} + \frac{5\nabla^2\boldsymbol{\tau}}{192} + \frac{B^2\rho\nabla^2\boldsymbol{\tau}}{4\pi} + \frac{\nabla\nabla\cdot\boldsymbol{\tau}}{96}$$

-Vortex: MT circle around the center



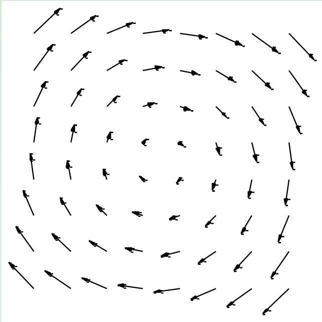
- Liquid crystals analogy: Frank Free Energy

$$F = -A|\boldsymbol{\tau}|^2 + \frac{1}{2}|\boldsymbol{\tau}|^4 + \underbrace{K_1}_{\textit{splay}}|\nabla\cdot\boldsymbol{\tau}|^2 + \underbrace{K_3}_{\textit{bend}}|\nabla\times\boldsymbol{\tau}|^2$$

Term $\nabla\nabla\cdot\boldsymbol{\tau}$ penalizes splay deformations → vortices

Aranson & Tsimring, PRE 2003

Analogy with the magnetic field



- Magnetic field is divergence-free

$$\nabla \cdot \mathbf{B} = 0$$

- Magnetic field lines are always closed loops!

- Similarly, the friction anisotropy $\nabla(\nabla \cdot \boldsymbol{\tau})$ favors closed loops of orientation field, i.e. vortices

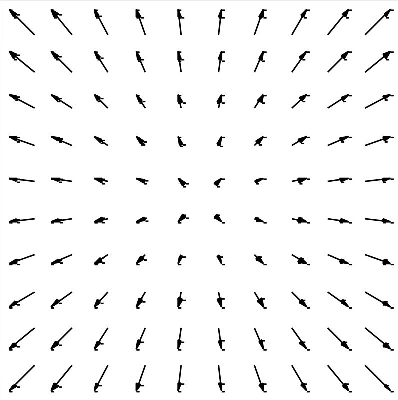
Aranson & Tsimring, PRE 2003

Asters

- For H≠0 (no blue terms) the only stable solution φ= 0

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \left(0.273\rho - 1\right)\boldsymbol{\tau} - |\boldsymbol{\tau}|^2 \boldsymbol{\tau} + \frac{5\nabla^2\boldsymbol{\tau}}{192} + \frac{B^2\rho\nabla^2\boldsymbol{\tau}}{4\pi} - H\left[0.321\boldsymbol{\tau}\left(\nabla\cdot\boldsymbol{\tau}\right) - 1.81\left(\boldsymbol{\tau}\nabla\right)\boldsymbol{\tau}\right]$$

No phase degeneracy: $\psi = \tau_x + i\tau_y = F(r)\exp[i\theta]$

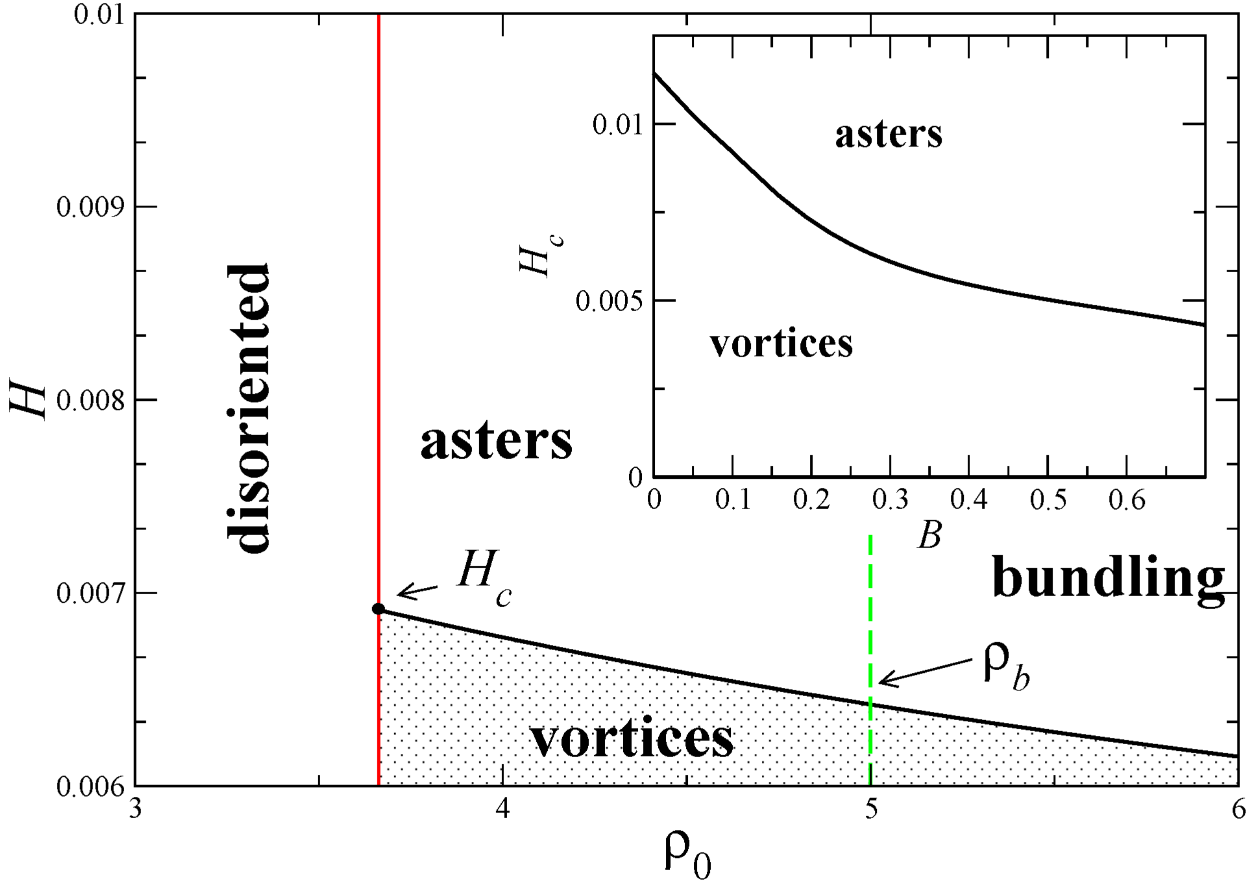


Aster: MT directed towards the center

$$\frac{\partial \psi}{\partial t} = \left(0.273\rho - 1\right)\psi - |\psi|^2 \psi + \frac{5\nabla^2\psi}{192} + \frac{B^2\rho\nabla^2\psi}{4\pi} + H\left[(\pi - 8/3)\psi \,\mathrm{Re}\,\bar{\nabla}\psi^* + 8/3 \,\mathrm{Re}(\psi^*\bar{\nabla})\psi\right]$$

$$\bar{\nabla} = \partial_x + i\partial_y = \exp[i\theta](\partial_r + i/r\partial_\theta)$$

Phase Diagram



Implications of the Analysis

- Asters stable for large MM density
- Vortices stable only for low MM density
- No stable vortices for $H>H_c$ for all MM density

(in experiments no vortices in Ncd for all densities)

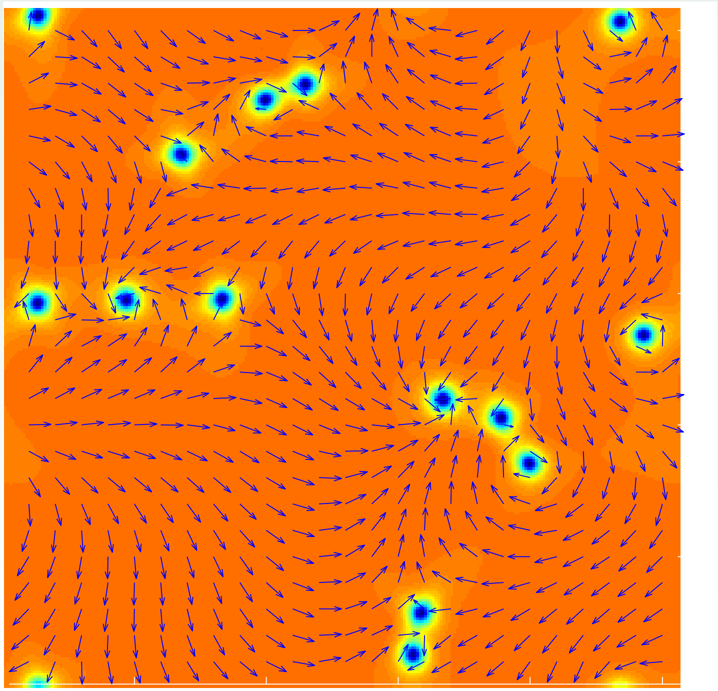
Experiment

- 2D mixture of MM & MT exhibits pattern formation
- In kinesin vortices are formed for low density of MM and asters are formed for higher density
- In Ncd only asters are observed for all MM densities
- For very high MM density asters disappear and bundles formed

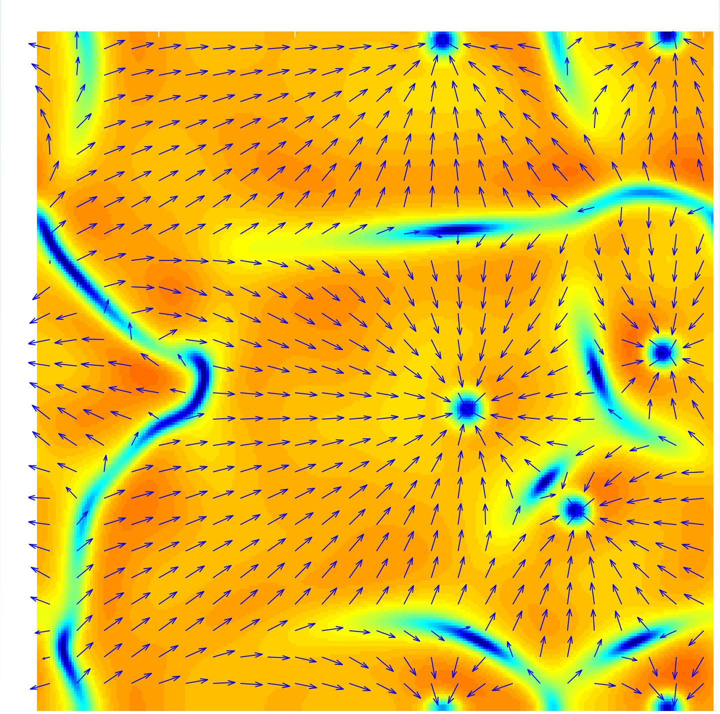
Numerical Solution

- Quasispectral Method ; 256x256 FFT harmonics
- Periodic boundary conditions
- Spontaneous creation of vortices and asters

H=0.004

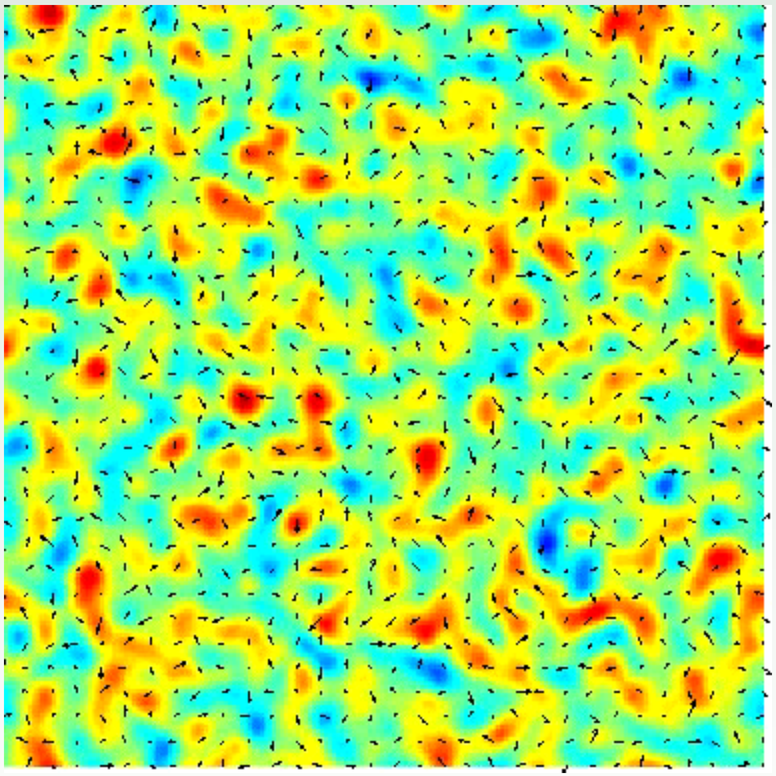


H=0.125



Evolution of Vortices and Asters

Large anisotropy H



Small anisotropy H