# Asymptotic-Preserving scheme for the Fokker-Planck-Maxwell system in the quasi-neutral regime.

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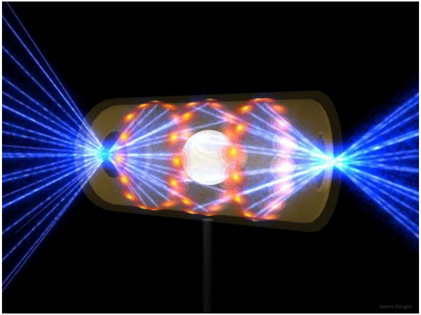


# Introduction

Two important effects in a plasma :

- Collisions between particles : Mean free path : $\lambda_{ei}$
- Collectif electromagnetic effects : Debye length : $\lambda_{De}$

$$\text{Quasi-neutrality regime} : \alpha = \frac{\lambda_{De}}{\lambda_{ei}} \to 0$$

Why Asymptotic-Preserving schemes ?

For Classical scheme :

$$\text{Stability condition :} \quad \Delta t \leq \frac{1}{\omega_{pe}} \approx \alpha^2$$

$$\text{Precision :} \quad \Delta x \leq \lambda_{De} \approx \alpha^2$$

$\hookrightarrow$ Total cost too expensive $\Rightarrow$ Asymptotic-Preserving (AP) schemes.

# State of art

Asymptotic-Preserving (AP) for the quasi-neutral regime :

Fluid :

- P. Crispel, P. Degond, M-H. Vignal, 2005.
- P. Degond, F. Deluzet, D. Savelief, 2011.
- ...

Kinetic :

- P. Degond, F. Deluzet, L. Navoret, A. Sun, M-H. Vignal, 2009.
- R. Belaouar, N. Crouseilles, P. Degond, E. Sonnendrücker, 2009.
- ...

Our contribution :

An AP scheme for the Fokker-Planck-Landau-Maxwell system in the quasi-neutral regime.

# Sommaire

# 2. The Model

# Model

## Kinetic description :

Electron distribution function : $f(t, \mathbf{x}, \mathbf{v})$, fixed ions

$\hookrightarrow$ Resolution of Fokker-Planck-Landau equation

$$\underbrace{\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f}_{advection \quad term} + \underbrace{\frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\nabla_{\mathbf{v}} f}_{force \quad term} = \underbrace{C_{e,e}(f,f) + C_{e,i}(f),}_{collisional \quad terms}$$

$$C_{ee}(f,f) = \alpha_{ee} div_{\mathbf{v}} \Big( \int_{\mathbf{v}' \in \mathbb{R}^3} S(\mathbf{v} - \mathbf{v}')[\nabla_{\mathbf{v}} f(\mathbf{v}) f(\mathbf{v}') - f(\mathbf{v}) \nabla_v f(\mathbf{v}')] d\mathbf{v}' \Big),$$

$$C_{ei}(f) = \alpha_{ei} div_{\mathbf{v}} \Big[ S(\mathbf{v}) \nabla_{\mathbf{v}} f(\mathbf{v}) \Big],$$

$$S(\mathbf{u}) = \frac{1}{|\mathbf{u}|^3}(|\mathbf{u}|^2 Id - \mathbf{u} \otimes \mathbf{u}).$$

# Model : collisions operators properties

$C_{ee}$ and $C_{ei}$ conservation properties

$$\int_{\mathbb{R}^3} C_{ee}(f,f) \begin{pmatrix} 1 \\ \mathbf{v} \\ \mathbf{v}^2 \end{pmatrix} d\mathbf{v} = 0, \qquad \int_{\mathbb{R}^3} C_{ei}(f) \begin{pmatrix} 1 \\ \mathbf{v}^2 \end{pmatrix} d\mathbf{v} = 0.$$

Entropy dissipation

$$\int_{\mathbb{R}^3} C_{ei}(f)\, log\, f\, d\mathbf{v} \leq 0, \qquad \int_{\mathbb{R}^3} C_{ee}(f,f)\, log\;\; f\, d\mathbf{v} \leq 0.$$

Consequence : the Boltzmann entropy

$$\mathcal{H}(f) = \int_{\mathbb{R}^3} (f \ln f - f)\, dv.$$

is a Lyapunov function for the Fokker-Planck-Landau equation.

# Collision operators equilibrium states

Equilibrium state of $C_{ei}$ $(C_{ei}(f) = 0)$ : set of isotropic functions $f = f(|\mathbf{v}|)$.

Equilibrium state of $C_{ee}$ $(C_{ee}(f, f) = 0)$ : Maxwellian distribution function

$$f = n \ \left(\frac{m_e}{2\pi k_B T}\right)^{3/2} \ exp\left( - \frac{m_e(\mathbf{v} - \mathbf{u_e})^2}{2k_B T}\right).$$

Isotropisation Maxwellisation

$$f(\mathbf{v}) \underset{\nu_{ei}}{\rightarrow} f(|\mathbf{v}|) \underset{\nu_{ee}}{\rightarrow} f(\mathbf{v}) = e^{-\frac{m\mathbf{v}^2}{2k_B t}}$$

# Maxwell equations

$$\begin{cases} \dfrac{\partial \mathbf{E}}{\partial t} - c^2 \nabla_{\mathbf{x}} \times \mathbf{B} = -\dfrac{\mathbf{j}}{\varepsilon_0}, & (Ampere) \\ \nabla_{\mathbf{x}}.\mathbf{E} = \dfrac{e}{\varepsilon_0}(n_i - n_e), & (Gauss) \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, & (Faraday) \\ \nabla_{\mathbf{x}}.\mathbf{B} = 0, & (Thomson) \end{cases}$$

$$n_{e,i}(\mathbf{x}, t) = \int_{\mathbb{R}^3} f_{e,i}(\mathbf{x}, \mathbf{v}, t) d\mathbf{v},$$

$$\mathbf{j} = q \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t) \mathbf{v} d\mathbf{v}.$$

Parameters for the collisional processes analysis :

- the mean free path : $\lambda_{ei}$,
- the thermal velocity : $v_{th} = \sqrt{\dfrac{k_B T}{m_e}}$,
- the electron-ion collision frequency : $\nu_{ei} = \dfrac{v_{th}}{\lambda_{e,i}}$.

# Dimensionless Fokker-Planck-Landeau-Maxwell system

Scaling used for collisional processes

$$\tilde{t} = \nu_{e,i} t, \quad \tilde{x} = x/\lambda_{e,i}, \quad \tilde{v} = v/v_{th}.$$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f - \nabla_{\mathbf{v}}.((\mathbf{E} + \mathbf{v} \times \mathbf{B})f) = C_{e,e}(f,f) + C_{e,i}(f), \\ \dfrac{\partial \mathbf{E}}{\partial t} - \dfrac{1}{\beta^2}\nabla_{\mathbf{x}} \times \mathbf{B} = -\dfrac{\mathbf{j}}{\alpha^2} \quad , \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla_{\mathbf{x}} \times \mathbf{E} = 0, \qquad \alpha = \dfrac{\lambda_{De}}{\lambda_{ei}}, \quad \beta = v_{th}/c. \\ \nabla_{\mathbf{x}}.\mathbf{E} = \dfrac{1}{\alpha^2}(1-n), \\ \nabla_{\mathbf{x}}.\mathbf{B} = 0, \end{cases}$$

# Simplified case

Electrostatic case ($B = 0$) with one dimension for space ($x \in \mathbb{R}$) and one for velocity ($v \in \mathbb{R}$)

$$(S_\alpha) \begin{cases} \dfrac{\partial f}{\partial t} + v\partial_x f - E\partial_v f = C_{e,e}(f,f) + C_{e,i}(f), \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\alpha^2}, \end{cases}$$

with Maxwell-Poisson satisfied at the initial time.

Limit system ( $\alpha \to 0$ ) (Theoretical results : [Han-Kwan, Hauray])

$$(S_0) \begin{cases} \dfrac{\partial f}{\partial t} + v\partial_x f - E\partial_v f = C_{e,e}(f,f) + C_{e,i}(f), \\ j = 0, \end{cases}$$

with $n = 1$ at the initial time.

# 3. Reformulation

# Reformulation

$\hookrightarrow$ Provide a reformulation of the Maxwell-Ampere equation.

$$\frac{\partial f}{\partial t} + v\partial_x f - E\partial_v f = C_{e,e}(f,f) + C_{e,i}(f),$$

Dimensionless electric current $\quad j = -\int_{\mathbb{R}} f\, v\, dv,$

$$\begin{cases} \dfrac{\partial j}{\partial t} - \partial_x(\displaystyle\int_{\mathbb{R}} v^2 f dv) + E \displaystyle\int_{\mathbb{R}} v\partial_v f dv = -\displaystyle\int_{\mathbb{R}} C_{e,i} v dv, \\ \dfrac{\partial E}{\partial t} = -\dfrac{j}{\alpha^2}. \end{cases}$$

$$-\alpha^2 \frac{\partial^2 E}{\partial t^2} + E\int_{\mathbb{R}} v\partial_v f dv = \partial_x(\int_{\mathbb{R}} v^2 f)dv - \int_{\mathbb{R}} C_{e,i} v dv.$$

If $\alpha \to 0$ we can obtain $E$.

# Reformulation

**Reformulated** system

$$\begin{cases} \dfrac{\partial f}{\partial t} + \partial_x(vf) - \partial_v(Ef) = C_{e,e}(f,f) + C_{e,i}(f), \\ -\alpha^2 \dfrac{\partial^2 E}{\partial t^2} + E\displaystyle\int_{\mathbb{R}} v\partial_v f dv = \partial_x(\int_{\mathbb{R}} v^2 f)dv - \int_{\mathbb{R}} C_{e,i} v dv, \end{cases}$$

with Maxwell-Poisson satisfied at the initial time.

Limit system when $\alpha \to 0$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \partial_x(vf) - \partial_v(Ef) = C_{e,e}(f,f) + C_{e,i}(f), \\ E = \dfrac{\partial_x(\int_{\mathbb{R}} v^2 f dv) - \int_{\mathbb{R}} C_{e,i} v dv}{-n}, \qquad \text{(Generalised Ohm's law)} \end{cases}$$

with $n = 1$ and $j = 0$ at the initial time.

# Reformulation for full Maxwell system

General Fokker-Planck-Maxwell **reformulated** system

$$\begin{cases} \frac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f - (\mathbf{E} + \mathbf{v} \times \mathbf{B}).\nabla_{\mathbf{v}} f = C_{e,e}(f,f) + C_{e,i}(f), \\ \alpha^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mathbf{E} \int_{\mathbb{R}^3} \mathbf{v}.\nabla_{\mathbf{v}} f d\mathbf{v} = -div_{\mathbf{x}}(\int_{\mathbb{R}^3} \mathbf{v} \otimes \mathbf{v} f d\mathbf{v}) + \frac{\alpha^2}{\beta^2}[\nabla_{\mathbf{x}} \times \frac{\partial \mathbf{B}}{\partial t}] + \int_{\mathbb{R}^3} C_{e,i}(f)\mathbf{v} d\mathbf{v}, \\ \frac{\partial \mathbf{B}}{\partial t} + div_{\mathbf{x}} \times \mathbf{E} = 0. \end{cases}$$

with Maxwell-Poisson and Maxwell-Thomson satisfied at the initial time.

# 4. Numerical schemes

# Discrete model / Problem of the classical scheme

Classical scheme for the Maxwell-Ampere electrostatic equation

$$\frac{E^{n+1} - E^n}{\Delta t} = -\frac{j^n}{\alpha^2}.$$

If $\lambda_{ei} >> \lambda_{De}$ , quasi-neutrality : $\alpha \to 0$

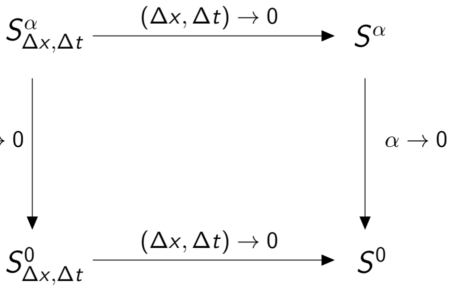
$$j^n \Longrightarrow 0$$

$$E^{n+1} = \ ?$$

Stability condition for the classical scheme : $\Delta t \approx \alpha^2$

# Asymptotic-Preserving methods

- Origins of the Asymptotic-Preserving methods [1].




No necessity to reduce $(\Delta t, \Delta x)$ when $\alpha \to 0$

1. S. Jin, SIAM J. Sci. Comp. (1999).

# Discrete model

Electrostatic case with one dimension for space ($x \in \mathbb{R}$) and one for velocity ($v \in \mathbb{R}$)

$$\frac{f^{n+1} - f^n}{\Delta t} + \nabla_x(vf^n) - \partial_v(E^{n+1} f^n) = C_{e,e}(f^n, f^n) + C_{e,i}(f^n).$$

Electric current : $\quad j^n = -\int_v f^n v \, dv,$

$$\begin{cases} \dfrac{j^{n+1} - j^n}{\Delta t} = \beta_1(f^n) E^{n+1} + \beta_2(f^n), \\ \dfrac{E^{n+1} - E^n}{\Delta t} = -\dfrac{j^{n+1}}{\alpha^2}. \end{cases}$$

$$E^{n+1} = \frac{-\dfrac{\alpha^2 E^n}{\Delta t^2} + \beta_2(f^n) + \dfrac{j^n}{\Delta t}}{-\dfrac{\alpha^2}{\Delta t^2} - \beta_1(f^n)}.$$

If $\alpha \to 0$ we can obtain $E^{n+1}$, $\Delta t$ is not constrained by $\alpha$.

# 5. Angular moments models

# Moments model

## Kinetic model

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla_{\mathbf{x}} f - (\mathbf{E} + \mathbf{v} \times \mathbf{B}).\nabla_{\mathbf{v}} f = C_{e,e}(f,f) + C_{e,i}(f).$$

Pb : Too costly

One solution : Fluid models : (Euler-Lorentz, ...).

Pb : Not precize

Good compromise : Angular moments models

Spherical Coordinates : $\Omega = v/|v|$ direction of propagation, $\zeta = |v|$.

- $f^0(\zeta) = \zeta^2 \int_{S_2} f(v)\, d\Omega$ : mass,
- $f^1(\zeta) = \zeta^2 \int_{S_2} \Omega f(v)\, d\Omega$ : impulsion,
- $f^2(\zeta) = \zeta^2 \int_{S_2} \Omega \otimes \Omega f(v)\, d\Omega$ : pressure.

# Notations

Spherical coordinates : $\mu = \cos(\theta)$, $\theta \in [0, \pi]$

$$\Omega = \begin{pmatrix} \mu \\ \sqrt{1-\mu^2}\cos\varphi \\ \sqrt{1-\mu^2}\sin\varphi \end{pmatrix}, \quad v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \zeta\Omega = \begin{pmatrix} \zeta\mu \\ \zeta\sqrt{1-\mu^2}\cos\varphi \\ \zeta\sqrt{1-\mu^2}\sin\varphi \end{pmatrix}.$$

Simplification : one angular dimension : $\mu \in [-1, 1]$.

**N first moments**

$$f^i = 2\pi\zeta^2 \int_{-1}^{1} f(\zeta,\mu)\mu^i d\mu = \zeta^2\langle f\mu^i\rangle,\ i \in \{0, N\},$$
$$F^i = \frac{f^i}{\zeta^2},\ \langle\Psi\rangle = 2\pi \int_{-1}^{1} \Psi(\mu)\, d\mu$$

$F^0$ : isotropic part of $f$

# Moments system

Moments extraction $\Rightarrow$ moments system

$$\partial_t \overline{f} + \zeta \partial_x \widetilde{f} = \overline{Q}$$

$$\overline{f} = \begin{pmatrix} f^0 \\ \vdots \\ f^N \end{pmatrix}, \widetilde{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^{N+1} \end{pmatrix} \quad \overline{Q} = \begin{pmatrix} Q^0 \\ \vdots \\ Q^N \end{pmatrix}.$$

Pb : Non closed system : $N+1$ equations for $N+2$ unknowns

$\Rightarrow$ Define a closure : $f \approx f^{closure}$

$f \Rightarrow (f^0(\zeta), f^1(\zeta), \ldots, f^N(\zeta))$ donnés

## Example : $P_1$ model

closure : $f^{P_1} = A_0(\zeta) + A_1(\zeta)\mu, \quad A_0(\zeta) = \dfrac{f_0(\zeta)}{4\pi}, \; A_1(\zeta) = \dfrac{3\, f_1(\zeta)}{4\pi}.$

$$f \geq 0 \;\Leftrightarrow\; |f_1/f_0| \leq 1/3$$

# Entropy minimisation principle/Closure

## Construction of angular closure

- Entropy minimisation principle on the distribution function [1,2]
- Constraints on the angular variable

Entropy minimisation problem

$$\min_{g \geq 0} \left\{ \mathcal{H}(g) / \ \forall \zeta \in \mathbb{R}_+, \ \zeta^2 \int_{-1}^{1} \mu^i g(\mu,\zeta) d\mu = f^i(\zeta), \ \ i \in \{0,\dots,N\} \right\}$$

(1) G.N. Minerbo, *Maximum entropy Eddington Factors*, J. Quant. Spectrosc. Radiat. Transfer (1978).
(2) B. Dubroca,J.L. Feugeas, *Entropic moment closure hierarchy for the radiative transfert equation*, C.R.A.S.(1999).

# The $M_N$ closure

## $M_N$ closure

$$f^{M_N}(t,x,\zeta,\mu) = \exp\left(\overline{\alpha}(\zeta,t,x)\,.\,\overline{\mu}\right) > 0, \quad \overline{\mu} = \begin{pmatrix} \mu^0 \\ \vdots \\ \mu^N \end{pmatrix}, \; \overline{\alpha} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_N \end{pmatrix}$$

Realizability problems in gas dynamics : more complcated [Junk, Schneider]. Contraintes sur $v \in \mathbb{R}^3$

Aim : Construct BGK models leading to correct transport coefficients ([Brull-Schneider]).

# The $M_1$ closure

Determination of $f_2$ function of $f_0$ and $f_1$ : $f \Rightarrow (f_0(\zeta), f_1(\zeta))$

## Entropy minimisation problem.

$$\min_{g \geq 0} \left\{ \mathcal{H}(g) \ / \ \forall \zeta \in \mathbb{R}^+, \int_{-1}^{1} g(\mu,\zeta) d\mu = f_0(\zeta), \ \int_{-1}^{1} g(\mu,\zeta)\mu d\mu = f_1(\zeta) \right\}$$

## $M_1$ closure :

$$f^{M_1} = \rho(\zeta) exp(-\mu a(\zeta)) \geq 0, \quad \rho(\zeta) \in \mathbb{R}^+ \quad a(\zeta) \in \mathbb{R}.$$

Expression of $f_2$ :

$$f_2 = \chi \, f_0 \qquad \text{with} \qquad \chi = \frac{|a|^2 - 2|a| coth(|a|) + 2}{|a|^2}.$$

# Realisability domain

## Realisablity domain :

$$\mathcal{A} = \left\{ \overline{f} = \begin{pmatrix} f^0 \\ \vdots \\ f^N \end{pmatrix} \in \mathbb{R}^{N+1} / \exists\ f \geq 0 \in L^1 \text{ and } \zeta^2 \langle \mu^i f \rangle = f^i,\ i \in \{0, N\} \right\}$$

Particular case : $N = 1$. $\mathcal{A} = \mathcal{B}$

$$\mathcal{B} = \left\{ \overline{f} = \begin{pmatrix} f^0 \\ f^1 \end{pmatrix} \in \mathbb{R}^2,\ f^0 > 0 \quad \text{and} \quad |f^1| < f^0 \right\} \cup \{(0,0)\} \ .$$

Generalisation to $M_N$ : Kershaw, 1976.

Tools : Hankel matrixes

# Classical approximation in plasmas physics

↪ $C_{ee}$ non-linear : complicated angular moment extraction

↪ Approximated collisional operators :[2], [3]

$$C_{ee}(f,f) \approx Q^0_{ee} = C_{ee}(F_0,F_0)$$

$$Q^0_{ee} = \partial_\zeta \left( \zeta \int_0^\infty \widetilde{J}(\zeta,\zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta (F^0(\zeta)) - F^0(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right)$$

$$\widetilde{J}(\zeta,\zeta') = \frac{2\alpha_{ee}}{3} \inf \left( \frac{1}{\zeta^3}, \frac{1}{\zeta'^3} \right) \zeta'^2 \zeta^2 \,.$$

Problem : No conservation of the realisibility domain [4].

2. Berezin, Khudick, Pekker, 1987
3. Buet, Cordier, 1998
4. J. Mallet, S. Brull, B. Dubroca. 2013.

# Moments system

Moments extraction w.r.t 1, $\mu$ on the kinetic equation

$$\partial_t f = Q_{ee}^0$$

$$\Rightarrow \begin{cases} \partial_t f^0 = Q_{ee}^0 \\ \partial_t f^1 = 0 \end{cases} \tag{1}$$

Initial conditions

$$f^0(t=0) = \frac{1}{3}\chi_{[0,3]}(\zeta) \text{ and } f^1(t=0) = \frac{1}{4}\chi_{[0,3]}(\zeta) .$$

Asymptotic state for $f^0$ and $f^1$

$$f^0 = \sqrt{\frac{2}{\pi}} \exp\left(\frac{-\zeta^2}{2}\right)\zeta^2 , \quad f^1(t=0) = \frac{1}{4}\chi_{[0,3]}(\zeta) .$$

# Asymptotic state

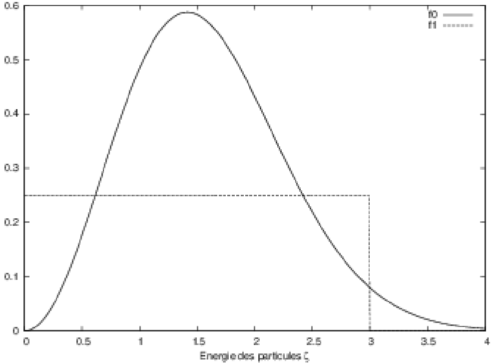



FIGURE: *$f^0$ et $f^1$ in function of $\zeta$ for the stationary state when $\zeta_{\max} = 4$.*

Conclusion : Non conservation of the realisability domain by $Q^0_{ee}$.
$\Rightarrow$ Derive a new model preserving the realisability domain

# New approximation of $C_{ee}$

$\hookrightarrow$ Linearisation of $C_{ee}$ around the isotropic state of $f$.

**New electrons/electrons collision operator to preserve $\mathcal{A}$**

$$Q_{e,e}(f) = \frac{1}{\zeta^2}\partial_\zeta\left(\zeta \int_0^{+\infty} \tilde{J}(\zeta,\zeta')\left[F^0(\zeta')\frac{1}{\zeta}\partial_\zeta f(\zeta) - f(\zeta)\frac{1}{\zeta'}\partial_{\zeta'}F^0(\zeta')\right]\zeta'^2 d\zeta'\right),$$

$f = F^0 \Rightarrow Q_{ee} = Q^0_{ee} \Rightarrow Q_{ee}$ extends $Q^0_{ee}$.

**Properties of the operateur $Q(f) = Q_{ee}(f) + Q_{ei}(f)$**

- Mass and energy conservation
- Entropy dissipation

# Moments extraction

$Q^i_{ee}$, $Q^i_{ei}$ : moments of order $i$ of $Q_{ee}$ and $Q_{ei}$

**Collision operator : $Q^i = Q^i_{ee} + Q^i_{ei}$**

$$Q^i_{ee} = \partial_\zeta \left( \zeta \int_0^\infty \tilde{J}(\zeta,\zeta') \left[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta F^i(\zeta) - F^i(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \right] \zeta'^2 d\zeta' \right)$$

$$Q^0_{ei} = 0, \;\; Q^1_{ei} = -\alpha_{ei} \frac{2f^1}{\zeta^3}$$

Conservation of the realisability domain :
$\overline{f}(t=0,\zeta) \in \mathcal{A} \Rightarrow \overline{f}(t,\zeta) \in \mathcal{A} \; \forall t \geq 0.$

# $M_1$ model

Angular extraction of the kinetic equation [5] :

$$\begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta\left(\frac{qE}{m} f_1\right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta\left(\frac{qE}{m} f_2\right) - \frac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1). \end{cases}$$

Collisions operators

$$Q_0(f_0) = \partial_\zeta\left(\zeta^2 A(\zeta)\partial_\zeta(\frac{f_0}{\zeta^2}) - \zeta B(\zeta) f_0\right),$$

$$Q_1(f_1) = -\frac{f_1}{\zeta^3},$$

$$A(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^2 f_0(\mu) d\mu, \quad B(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\mu^3})\mu^3 \partial_\mu(\frac{f_0(\mu)}{\mu^2}) d\mu.$$

5. J. Mallet, S. Brull, B. Dubroca, C.I.C.P. 2013.

# The M1-Maxwell model

Simplified electrostatic case ($B = 0$) with one space dimension ($x \in \mathbb{R}$)

$$(S_\alpha) \begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta \left( \frac{qE}{m} f_1 \right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta \left( \frac{qE}{m} f_2 \right) - \frac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1), \\ \frac{\partial E}{\partial t} = -\frac{j}{\alpha^2}, \end{cases}$$

with Maxwell-Poisson satisfied at the initial time.

Limit system ( $\alpha \to 0$ )

$$(S_0) \begin{cases} \partial_t f_0 + \nabla_x.(\zeta f_1) + \partial_\zeta \left( \frac{qE}{m} f_1 \right) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x.(\zeta f_2) + \partial_\zeta \left( \frac{qE}{m} f_2 \right) - \frac{qE}{m\zeta}(f_0 - f_2) = Q_1(f_1) + Q_0(f_1), \\ j = 0, \end{cases}$$

with $n = 1$ at the initial time.

# Reformulation of the M1-Maxwell model

Evolution equation on $f_1$ :

$$\frac{f_1^{n+1} - f_1^n}{\Delta t} + \nabla_x(\zeta f_2^n) - \partial_\zeta(E^{n+1} f_2^n) + \frac{E^{n+1}}{\zeta}(f_0^n - f_2^n) = Q_0(f_1^n) + Q_1(f_1^n).$$

Electric current : $j^n = -\int_\zeta \zeta f_1^n d\zeta,$

$$\begin{cases} \dfrac{j^{n+1} - j^n}{\Delta t} = \beta_1(f_0^n, f_1^n)E^{n+1} + \beta_2(f_0^n, f_1^n), \\ \dfrac{E^{n+1} - E^n}{\Delta t} = -\dfrac{j^{n+1}}{\alpha^2}. \end{cases}$$

$$E^{n+1} = \frac{-\dfrac{\alpha^2 E^n}{\Delta t^2} + \beta_2(f_0^n, f_1^n) + \dfrac{j^n}{\Delta t}}{-\dfrac{\alpha^2}{\Delta t^2} - \beta_1(f_0^n, f_1^n)}.$$

If $\alpha \to 0$ we can obtain $E^{n+1}$, $\Delta t$ is not constrained by $\alpha$.

# 6. Batishev test case

# Batishchev [6] test case

Relaxation of a localised temperature profile.

$\hookrightarrow$ Study of the non-local heat transport.

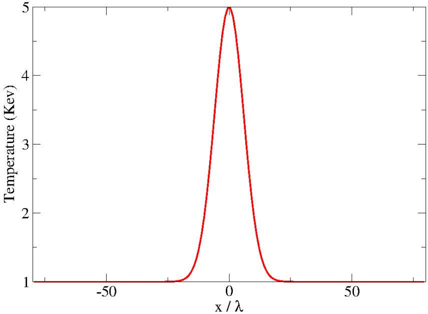
Initialy :

$$T(0,x) = T_0 + T_1 exp(-\frac{x^2}{(\delta L)^2})$$

$T_0 = 1$Kev, $T_1 = 4$Kev, $\frac{\lambda_{ei}}{\delta L} = 0.01$

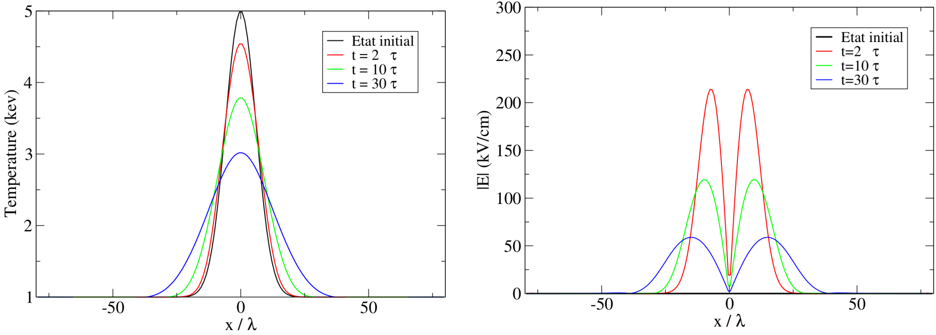
$$\begin{cases} f_0 = \sqrt{\frac{2}{\pi}}\Big(\frac{1}{T(x)}\Big)^{\frac{3}{2}} exp\Big(-\frac{\zeta^2}{2T(x)}\Big) \\ f_1 = 0 \end{cases}$$

$\alpha = 4.10^{-4}$



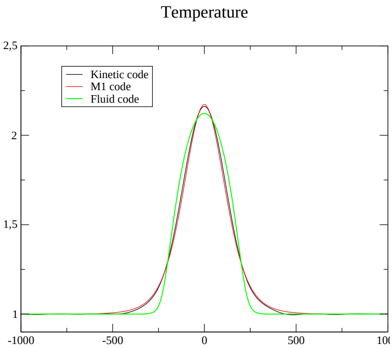

6. O.V Batishchev & al Physics of Plasmas (2002).

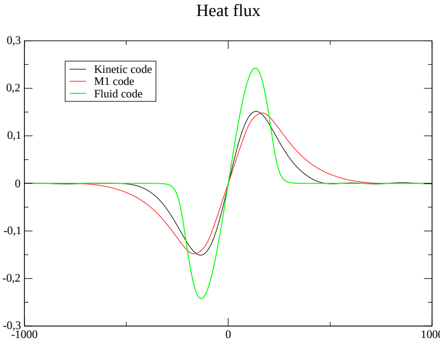
# Results




Temperature (left)-Electric field (right)

# Comparison




Kinetic : [A.V. Bobylev, I.F. Potapenko, JCP, 2013] .
M1-AP : 3 minutes with 1 process.

# Conclusions

- Numerical resolution of the Fokker-Planck-Maxwell system in the quasi-neutral regime for kinetic and moment models.
- Implementation and validation with Batishchev test case : $\alpha \to 0$.
- Accurate and fast scheme.

# Perspectives :

$M_1$-Maxwell system

- Diffusion limit
- Consider mobile ions (non conservative terms)

$M_2$ with medical applications (radiotherapy).

- Phd of T.Pichart
- Collaboration avec Martin Franck (RWTH-Aix-La-Chapelle)

# Thank you