On hybrid method for rarefied gas dynamics : Boltzmann vs. Navier-Stokes models

Francis FILBET, Thomas REY

University of Lyon

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Boltzmann-like Kinetic Equations

Study of a particle distribution function $f^\varepsilon(t, x, v)$, depending on time $t > 0$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^3$, solution to

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \dfrac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon), \\ f^\varepsilon(0, x, v) = f_0(x, v), \\ + \text{ boundary conditions}. \end{cases} \tag{1}$$

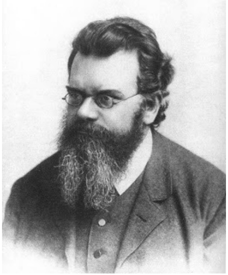
where ε is usually the *Knudsen number*, ratio of the mean free path before collision by the typical length scale of the problem.

The Boltzmann operator is

$$\mathcal{Q}(f)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \left[f'_* f' - f_* f \right] B(|v - v_*|, \cos\theta) \, d\sigma \, dv_*,$$

where B is the collision kernel, $\cos\theta := (v - v_*) \cdot \sigma$ and

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \qquad v'_* = \frac{v + v^*}{2} - \frac{|v - v_*|}{2} \sigma.$$

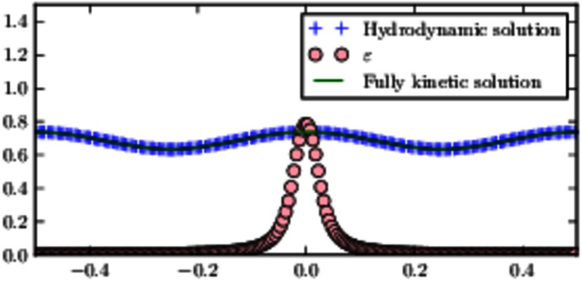


Fluid limit of the Boltzmann-like Kinetic Equations

First order *fluid dynamic limit* $\varepsilon \to 0$ given by the Euler equations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho\, \mathbf{u}) = \mathbf{0}, \\ \partial_t(\rho\, \mathbf{u}) + \mathrm{div}_{\mathbf{x}}\left(\rho\, \mathbf{u} \otimes \mathbf{u} + \rho \mathbf{T}\, \mathbf{I}\right) = \mathbf{0}, \\ \partial_t E + \mathrm{div}_x\left(\mathbf{u}\left(\mathbf{E} + \rho \mathbf{T}\right)\right) = 0, \quad \textit{with}\, \rho T = \frac{1}{3}\left(2E - \rho|\mathbf{u}|^2\right). \end{cases}$$

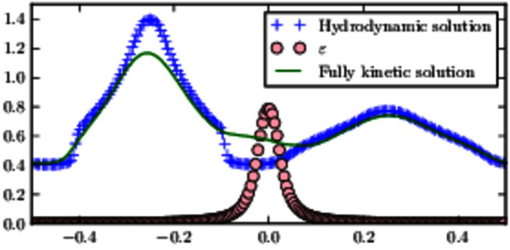
Kinetic ($1d_x \times 3d_v$) vs. Euler $1d_x$



Outline of the Talk

1. Chapman-Enskog expansion for the Boltzmann equation
   - A hierarchy of models
   - The Moment Realizability Criterion
   - From Fluid to Kinetic
   - From Kinetic to Fluid (an additional condition) based on the relative entropy
2. Numerical Scheme
   - Hybrid method coupling kinetic and fluid solvers
   - Implementation
3. Treatment of the Boundary Conditions
4. Numerical Simulations
   - Test 1 : the Riemann problem
   - Test 2 : Blast wave problem
   - Test 3 : Smooth, Far from Equilibrium, Variable Knudsen Number
   - Test 4 : Flow generated by gradients of temperature
5. Conclusion and work in progress

A Hybrid Scheme?



- How to identify efficiently these zones?
- To pass from hydrodynamical model to the kinetic one, is the knowledge of the hydrodynamic fields *enough* to do so?
- Can we design a scheme able to *connect* different types of spatial mesh cells (hydrodynamic and kinetic)?
- Finally, can we do so *dynamically*?

State of the Art

Asymptotic-Preserving schemes give uniformly accurate and stable (with respect to the Knudsen number ε) approximate solutions but the kinetic equation is solved everywhere

- huge computational cost (S. Jin and FF & S. Jin for Boltzmann).

Hybrid methods : Two "different" hydrodynamic break-up criteria:

- Based on the value of the local Knudsen number: Boyd, Chen and Chandler, *Phys. Fluid* (1994); Kolobov, Arslanbekov *et al.*, *JCP* (2007); Degond and Dimarco, *JCP* (2012). *Problem dependent criterion*
- Based on the heat flux: Tiwari, *JCP* (1998); Tiwari, Klar and Hardt, *JCP* (2009); Degond, Dimarco and Mieussens, *JCP* (2010); Alaia and Puppo, *JCP* (2012). *Can miss the variations of the local velocity*
- Decomposition of the particle distribution function: Dimarco and Pareschi, *MMS* (2008). *Need to use a Monte-Carlo approach for the tail*

A major problem: In all these works, the criteria cannot "see" if the kinetic distribution is far from equilibrium

Hydrodynamic Description of a Rarefied Gas

Let us derive a systematic criteria to choose between fluid and kinetic models.

- write a hierarchy of models using a Chapman Enskog expansion
- derive criteria based on this hierarchy.

By performing the expansion

$$f^\varepsilon = \mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}\left[1 + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \ldots\right],$$

we find that, without closure,

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho\,\mathbf{u}) = \mathbf{0}, \\ \partial_t(\rho\,\mathbf{u}) + \mathrm{div}_\mathbf{x}\left(\rho\mathbf{u}\otimes\mathbf{u} + \rho\mathbf{T}\,(\mathbf{I} + \mathbf{A})\right) = \mathbf{0}, \\ \partial_t E + \mathrm{div}_x\left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u} + \frac{\mathbf{3}}{\mathbf{2}}\rho\mathbf{T}\,(\mathbf{I}+\mathbf{A}) + \rho\mathbf{T}^{3/2}\mathbf{B}\right) = 0, \end{cases}$$

where **A** is the traceless stress tensor and **B** the dimensionless heat flux:

$$\mathbf{A} := \frac{1}{T}\int_{\mathbb{R}^3}\left[(v-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) - \frac{|\mathbf{v}-\mathbf{u}|^2}{\mathbf{3}}\mathbf{I}\right](f^\varepsilon - \mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(v))\,dv,$$

$$\mathbf{B} := \int_{\mathbb{R}^3}\left[\frac{|v-\mathbf{u}|^2}{2T} - \frac{5}{2}\right]\frac{(v-\mathbf{u})}{T^{1/2}}(f^\varepsilon - \mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(v))\,dv.$$

Examples

We set $\mathbf{V} = (\mathbf{v} - \mathbf{u})/\sqrt{\mathbf{T}}$, hence

- *The zeroth order: Compressible Euler.* Cutting the expansion at ε^0 yields

$$\mathbf{A}_{\mathbf{Euler}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{A}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\,\mathbf{dv} = \mathbf{0}_{\mathbf{M_3}},$$
$$\mathbf{B}_{\mathbf{Euler}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\,\mathbf{dv} = \mathbf{0}_{\mathbf{R^3}}.$$

- *The first order: Compressible Navier-Stokes.* Cutting at ε^1 yields

$$\mathbf{A}^{\varepsilon}_{\mathbf{NS}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{A}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\left[\mathbf{1} + \varepsilon\,\mathbf{g}^{(1)}(\mathbf{v})\right]\mathbf{dv} = -\varepsilon\frac{\mu}{\rho\,\mathbf{T}}\mathbf{D}(\mathbf{u}),$$
$$\mathbf{B}^{\varepsilon}_{\mathbf{NS}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\left[\mathbf{1} + \varepsilon\,\mathbf{g}^{(1)}(\mathbf{v})\right]\mathbf{dv} = -\varepsilon\frac{\kappa}{\rho\,\mathbf{T}^{3/2}}\nabla_{\mathbf{x}}\mathbf{T}.$$

The *viscosity* μ and the *thermal conductivity* κ depend on the collision operator. The *deformation tensor* is given by

$$\mathbf{D}(\mathbf{u}) = \nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^{\top} - \frac{\mathbf{2}}{\mathbf{3}}(\mathrm{div}_{\mathbf{x}}\,\mathbf{u})\,\mathbf{I}.$$

Examples Continued

The second order: Burnett equations. At order ε^2, we have (in the BGK case...)

$$\mathbf{A}^{\varepsilon}_{\mathrm{Burnett}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{A}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\left[\mathbf{1}+\varepsilon\,\mathbf{g}^{(1)}(\mathbf{v})+\varepsilon^{2}\mathbf{g}^{(2)}(\mathbf{v})\right]\mathbf{dv}$$

$$= -\varepsilon\frac{\mu}{\rho\,T}\mathbf{D}(\mathbf{u}) - \mathbf{2}\varepsilon^{2}\frac{\mu^{2}}{\rho^{2}\mathbf{T}^{2}}\Big\{-\frac{\mathbf{T}}{\rho}\mathrm{Hess}_{\mathbf{x}}(\rho)+\frac{\mathbf{T}}{\rho^{2}}\nabla_{\mathbf{x}}\rho\otimes\nabla_{\mathbf{x}}\rho-\frac{\mathbf{1}}{\rho}\nabla_{\mathbf{x}}\mathbf{T}\otimes\nabla_{\mathbf{x}}\rho$$

$$+(\nabla_{x}\mathbf{u})(\nabla_{x}\mathbf{u})^{\top}-\frac{1}{3}\mathbf{D}(\mathbf{u})\,\mathrm{div}_{\mathbf{x}}(\mathbf{u})+\frac{\mathbf{1}}{\mathbf{T}}\nabla_{\mathbf{x}}\mathbf{T}\otimes\nabla_{\mathbf{x}}\mathbf{T}\Big\};$$

$$\mathbf{B}^{\varepsilon}_{\mathrm{Burnett}} := \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{V})\mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}(\mathbf{v})\left[\mathbf{1}+\varepsilon\,\mathbf{g}^{(1)}(\mathbf{v})+\varepsilon^{2}\mathbf{g}^{(2)}(\mathbf{v})\right]\mathbf{dv}$$

$$= -\varepsilon\frac{\kappa}{\rho\,T^{3/2}}\nabla_{x}T-\varepsilon^{2}\frac{\mu^{2}}{\rho^{2}\,T^{5/2}}\Big\{+\frac{25}{6}(\mathrm{div}_{x}\,\mathbf{u})\nabla_{x}T$$

$$-\frac{5}{3}\left[T\,\mathrm{div}_{x}(\nabla_{x}\mathbf{u})+(\mathrm{div}_{x}\,\mathbf{u})\nabla_{x}T+6(\nabla_{x}\mathbf{u})\nabla_{x}T\right]$$

$$+\frac{2}{\rho}\mathbf{D}(\mathbf{u})\nabla_{\mathbf{x}}(\rho\mathbf{T})+\mathbf{2}\,\mathbf{T}\,\mathrm{div}_{\mathbf{x}}(\mathbf{D}(\mathbf{u}))+\mathbf{16}\mathbf{D}(\mathbf{u})\nabla_{\mathbf{x}}\mathbf{T}\Big\}.$$

Moment Realizability

- By construction, the following matrix is *positive definite*:

$$\mathbf{I} + \mathbf{A}^\varepsilon = \frac{1}{\rho}\int_{\mathbb{R}^3} \mathbf{V}\otimes\mathbf{V}\,\mathbf{f}^\varepsilon(\mathbf{v})\,\mathbf{dv}.$$

In particular,

If its eigenvalues are *nonpositive*, the truncation of the expansion of f^ε is *wrong* $\Rightarrow$ the regime considered is not correct but this criterion does not account for $\nabla_x T$.

- Following the work of Levermore et al.[1] we define the **moment realizability** matrix $\mathbf{M}$ for $\mathbf{m} := \left(1, \mathbf{V}, \left(\frac{2}{3}\right)^{1/2}\left(\frac{|\mathbf{V}|^2}{2} - \frac{5}{2}\right)\right)$ by

$$\mathbf{M} := \int_{\mathbb{R}^3} \mathbf{m}\otimes\mathbf{m}\,\mathbf{f}^\varepsilon(\mathbf{v})\,\mathbf{dv} = \begin{pmatrix} 1 & \mathbf{0}_{\mathbb{R}^3}^\top & 0 \\ \mathbf{0}_{\mathbb{R}^3} & \mathbf{I}+\mathbf{A}^\varepsilon & \left(\frac{2}{3}\right)^{1/2}\mathbf{B}^\varepsilon \\ 0 & \left(\frac{2}{3}\right)^{1/2}(\mathbf{B}^\varepsilon)^\top & C^\varepsilon \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \mathbf{0}_{\mathbb{R}^3}^\top & 0 \\ \mathbf{0}_{\mathbb{R}^3} & \mathbf{I}+\mathbf{A}^\varepsilon - \dfrac{2}{3\,C^\varepsilon}\mathbf{B}^\varepsilon\otimes\mathbf{B}^\varepsilon & 0 \\ 0 & \mathbf{0}_{\mathbb{R}^3}^\top & C^\varepsilon \end{pmatrix}, \quad C^\varepsilon > 0.$$

[1] Levermore, Morokoff, Nadiga, *Phys. Fluid* (1998)

A Hierarchy of Criteria

Since by construction, **M** is positive definite, the following 3×3 matrix is positive definite too:

$$\mathcal{V} := \mathbf{I} + \mathbf{A}^{\varepsilon} - \frac{2}{3\,\mathbf{C}^{\varepsilon}} \mathbf{B}^{\varepsilon} \otimes \mathbf{B}^{\varepsilon}.$$

Example

$$\begin{cases} \mathcal{V}_1 & = & \mathcal{V}_{Euler} = \mathbf{I}; \\ \mathcal{V}_{\varepsilon} & = & \mathcal{V}_{NS} = \mathcal{V}_{Euler} - \varepsilon \frac{\mu}{\rho T} \mathbf{D}(\mathbf{u}) - \varepsilon^2 \frac{2}{3} \frac{\kappa^2}{\rho^2 T^3} \nabla_x T \otimes \nabla_x T; \\ \mathcal{V}_{\varepsilon^2} & = & \mathcal{V}_{Burnett} = \mathcal{V}_{NS} - \ldots \end{cases}$$

A fluid breakdown criterion (k^{th} order cloture)

If the deviation of f^{ε} from the thermodynamic equilibrium is *too large*, then the hydrodynamic description has broken down,

- the eigenvalues of $\mathcal{V}_{\varepsilon^k}$ are *nonpositive*
- if

$$|\lambda_{\varepsilon^k} - \lambda_{\varepsilon^{k+1}}| > \varepsilon^{k+1}, \qquad \forall \lambda_{\varepsilon^k} \in \mathrm{Sp}(\mathcal{V}_{\varepsilon^k}),$$

the *fluid regime considered is wrong*

From Kinetic to Fluid (an additional condition) based on the relative entropy

Let us consider f^ε the solution to the Boltzmann equation and the truncation

$$f_k^\varepsilon = \mathcal{M}_{\rho,\mathbf{u},\mathbf{T}}\left[1 + \varepsilon\, g^{(1)} + \varepsilon^2\, g^{(2)} + \ldots + \varepsilon^k\, g^{(k)}\right].$$

Then, we have:

A fluid breakdown criterion (k^{th} order closure)

The kinetic description corresponds to an hydrodynamic closure of order k if

$$\| f^\varepsilon - f_k^\varepsilon \|_{L^1} \leq \delta_0.$$

For *numerical purposes*, it can be interesting to take an additional criteria on

$$\frac{\Delta t}{\varepsilon} \gg 1,$$

where Δt is the time step. Indeed, the relaxation time towards the Maxwellian distribution is of order $\varepsilon/\Delta t$. Hence for small ε or large time step Δt the solution is at thermodynamical equilibrium.

The Hybrid Scheme

At time t^n, the space domain $\Omega = \Omega_f \sqcup \Omega_h$ is decomposed as

- *Fluid cells* $x_i \in \Omega_f$, described by the hydrodynamic fields

$$U_i^n := \left(\rho_i^n, \mathbf{u_i^n}, \mathbf{T_i^n}\right) \simeq \left(\rho(t^n, x_i), \mathbf{u}(\mathbf{t^n}, \mathbf{x_i}), (\mathbf{t^n}, \mathbf{x_i})\right);$$

- *Kinetic cells* $x_j \in \Omega_h$, described by the particle distribution function

$$f_j^n(v) \simeq f(t^n, x_j, v), \quad \forall v \in \mathbb{R}.$$

Before evolving the equation:

- In a fluid cell $x_i \in \Omega_f$, compute the eigenvalues of $\mathbf{M}$ corresponding to the model of order k and $k+1$:
  - If they are positive and close enough, solve the fluid equation to obtain U_i^{n+1};
  - In the other case, set $f_i^n(v) := f_{k,i}^n$.
- In a kinetic cell $x_j \in \Omega_h$, compute $\mathcal{H}\left[f_j^n | \mathcal{M}_{\rho(f_j^n), \mathbf{u}(\mathbf{f_j^n}), \mathbf{T}(\mathbf{f_j^n})}\right]$ and the eigenvalues of $\mathbf{M}$ at order k and $k+1$:
  - If it is "big", solve the kinetic equation to obtain f_j^{n+1};
  - In the other case, set for $\varphi(v) = (1, v, |v - \mathbf{u}|^2)$,

$$U_j^n := \int_{\mathbb{R}^d} f_j^n \varphi(v)\, dv.$$

Numerical schemes

- Kinetic part:

$$\frac{\partial f}{\partial t} + \mathrm{div}_x(v\, f) = \frac{1}{\varepsilon}\mathcal{Q}(f)$$

→ Collision term $\mathcal{Q}_{\mathcal{B}}$: spectral scheme[2] or simple relaxation for ES-BGK;
→ Free transport term $\mathrm{div}_x(v\, f)$: finite volume Lax-Friedrichs method with Van Leer's flux limiter[3];

- Macroscopic part:

$$\frac{\partial U}{\partial t} + \nabla_x \cdot F(U) = G(U), \;\; U \in \mathbb{R}^5$$

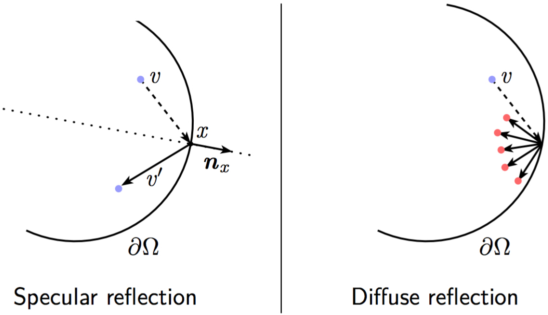
→ Reconstruction of the fluxes either with a WENO-5 procedure or with Lax-Friedrichs method;

- Projection of kinetic to fluid and lifting of fluid to kinetic: discrete velocity Maxwellian distribution[4];
- *IMEX* time stepping.

[2] Pareschi - Perthame, *Transport Theory Statist. Phys.* (1996)
[3] Van Leer, *JCP* (1977)
[4] Berthelin, Tzavaras, Vasseur, *JSP* (2009)

Maxwell Boundary Conditions

Let $x \in \partial\Omega$, $\mathbf{n_x}$ the outer normal to Ω and $\alpha \in [0,1]$



For an *outgoing velocity* $v \in \mathbb{R}^3$ $(v \cdot \mathbf{n}_x \geq 0)$, we set

$$f(t,x,v) = \alpha \mathcal{R} f(t,x,v) + (1-\alpha)\mathcal{M} f(t,x,v), \ \forall x \in \partial\Omega, \ v \cdot \mathbf{n}_x \geq 0,$$

where $\mathcal{R}$ stands for *specular* reflections and $\mathcal{M}$ for *diffusive* reflections.

Maxwell Boundary Conditions

Let $x \in \partial\Omega$ and $\mathbf{n_x}$ the outer normal to Ω.

→ The specular boundary operator is given by

$$\mathcal{R}f(t,x,v) = f(t,x,v - 2(\mathbf{n_x}\cdot\mathbf{v})\,\mathbf{n_x}), \;\; \mathbf{v}\cdot\mathbf{n_x} \geq \mathbf{0}.$$

Its macroscopic counterpart corresponds to the so called *no-slip condition*

$$\mathbf{u}\cdot\mathbf{n_x} = \mathbf{0};$$

→ The diffusive boundary operator is given by

$$\mathcal{M}f(t,x,v) = \mu(t,x)\,\mathcal{M}_{1,\mathbf{u_w},\mathbf{T_w}}, \;\; v\cdot\mathbf{n_x} \geq \mathbf{0}$$

where $\mathcal{M}_{1,\mathbf{u_w},\mathbf{T_w}}$ is the wall Maxwellian and μ insures global mass conservation.
Its macroscopic counterpart corresponds to

$$\mathbf{u}\cdot\mathbf{n_x} = \mathbf{0}, \quad \mathbf{E}(\mathbf{x}) = \frac{\mathbf{d}}{\mathbf{2}}\,\rho(\mathbf{x})\,\mathbf{T_w} + \rho\,|\mathbf{u_w}|^{\mathbf{2}}.$$

Test 1 : the Riemann problem

- Initial condition:

$$f^{in}(x,v) = \mathcal{M}_{\rho(x),\mathbf{u}(\mathbf{x}),\mathbf{T}(\mathbf{x})}(v), \quad \forall x \in [-0.5, 0.5], v \in [-8, 8]^2,$$

with

$$(\rho(x), \mathbf{u}(\mathbf{x}), \mathbf{T}(\mathbf{x})) = \begin{cases} (1, 0, 0, 1) & \text{if } x < 0, \\ (0.125, 0, 0, 0.25) & \text{if } x \geq 0. \end{cases};$$

- $\mathcal{Q} = \mathcal{Q}_{\mathcal{BGK}}$;
- Zero flux at the boundary in space;
- Various ε;
- Kinetic mesh: $N_x = 200$, $N_v = 32^2$;
- Fluid (Euler) mesh: $N_x = 200$.

Test 1 : Riemann problem (macroscopic quantities), $\varepsilon = 10^{-2}$

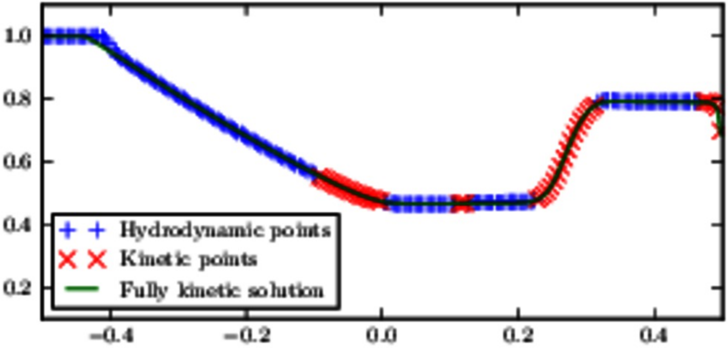
Figure : **Test 1 - Riemann problem with** $\varepsilon = 10^{-2}$ **:** *Order 0 (Euler)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 1 : Riemann problem (macroscopic quantities), $\varepsilon = 10^{-2}$

Figure : **Test 1 - Riemann problem with** $\varepsilon = 10^{-2}$ **:** *Order 1 (CNS)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 1 : Riemann problem (Temperature), $\varepsilon = 10^{-3}$

Hybrid 3 times faster than full kinetic



Test 1 : Riemann problem (macroscopic quantities), $\varepsilon = 10^{-3}$

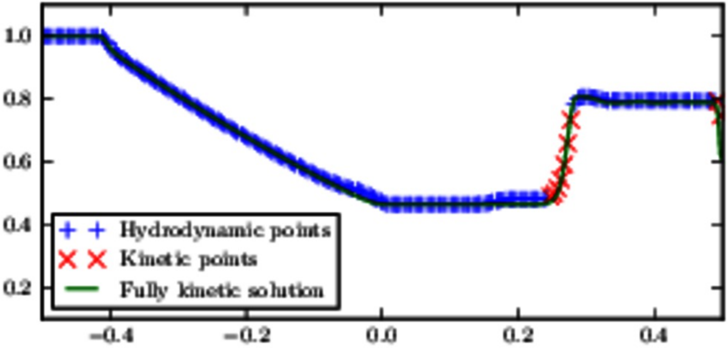
Figure : **Test 1 - Riemann problem with $\varepsilon = 10^{-3}$:** *Order 0 (Euler)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 1 : Riemann problem (macroscopic quantities), $\varepsilon = 10^{-3}$

Figure : **Test 1 - Riemann problem with** $\varepsilon = 10^{-3}$ **:** *Order 1 (CNS)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 1 : Riemann problem (Temperature), $\varepsilon = 10^{-4}$

Hybrid 9 times faster than full kinetic



Test 2 : Blast wave problem

- Initial condition:

$$f^{in}(x,v) = \mathcal{M}_{\rho(x),\mathbf{u}(\mathbf{x}),\mathbf{T}(\mathbf{x})}(v), \quad \forall x \in [-0.5, 0.5], v \in [-7.5, 7.5]^2,$$

with

$$(\rho(x), \mathbf{u}(\mathbf{x}), \mathbf{T}(\mathbf{x})) = \begin{cases} (1,1,0,2) & \text{if } x < -0.3, \\ (1,0,0,0.25) & \text{if } -0.3 \leq x \leq 0.3, ; \\ (1,-1,0,2) & \text{if } x \geq 0.3. \end{cases}$$

- $\mathcal{Q} = \mathcal{Q}_{\mathcal{BGK}}$;
- Specular boundary conditions ($\alpha = 1$ in the Maxwell boundary conditions);
- $\varepsilon = 10^{-2}$ and 0.005;
- Kinetic mesh: $N_x = 200$, $N_v = 32^2$;
- Fluid (Euler) mesh: $N_x = 200$.

Test 2 : Blast wave problem (Density)

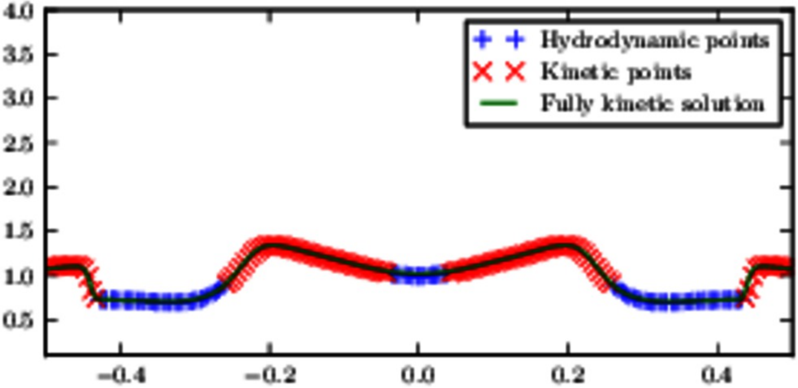
Figure : **Test 2 - Blast wave with** $\varepsilon = 10^{-2}$ **:** *Order 0 (Euler)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 2 : Blast wave problem (Density)

Figure : **Test 2 - Blast wave with** $\varepsilon = 10^{-2}$ **:** *Order 1 (CNS)*; Density, mean velocity, temperature and heat flux at time $t = 0.20$.

Test 2 : Blast wave problem (Density)

Hybrid 2 times faster than full kinetic



Test 2 - Blast wave with $\varepsilon = 5.10^{-3}$ **:** *Order 0 (Euler)*; Density.

Test 3 : Smooth, Far from Equilibrium, Variable Knudsen Number

- Initial condition:

$$f^{in}(x,v) = \frac{1}{2}\left(\mathcal{M}_{\rho(x),\mathbf{u}(\mathbf{x}),\mathbf{T}(\mathbf{x})}(v) + \mathcal{M}_{\rho(x),-\mathbf{u}(\mathbf{x}),\mathbf{T}(\mathbf{x})}(v)\right),$$

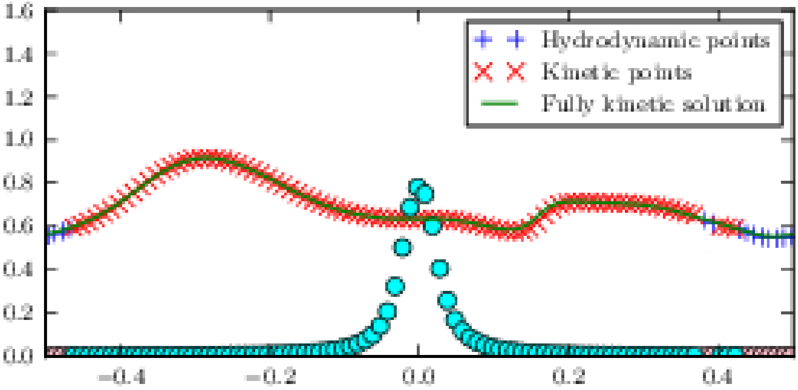
  for $x \in [-0.5, 0.5]$, $v \in [-7.5, 7.5]^2$, with

$$(\rho(x), \mathbf{u}(\mathbf{x}), \mathbf{T}(\mathbf{x})) = \left(1 + \frac{1}{2}\sin(\pi x), 0.75, 0, (5 + 2\cos(2\pi x))/20\right);$$

- $\mathcal{Q} = \mathcal{Q}_{\mathcal{B}}$;
- Periodic boundary conditions;
- $\varepsilon(x) = 10^{-4} + \frac{1}{2}\left(\arctan(1 + 30x) + \arctan(1 - 30x)\right)$;
- Kinetic mesh: $N_x = 100$, $N_v = 32^2$;
- Fluid (Euler) mesh: $N_x = 100$.

Test 3 : Smooth, Far from Equilibrium, Variable Knudsen Number

Hybrid 1.7 times faster than full kinetic



Test 4 : gradient of temperature

We consider the Boltzmann equation

$$\begin{cases} \dfrac{\partial f}{\partial t} + v_x \dfrac{\partial f}{\partial x} = \dfrac{1}{\varepsilon} Q(f,f), x \in (-1/2, 1/2),\ v \in \mathbb{R}^2, \\ f(t=0,x,v) = \dfrac{1}{2\pi\, k_B\, T_0(x)} \exp\left(-\dfrac{|v|^2}{2 k_B\, T_0(x)} \right), \end{cases}$$

with $k_B = 1$, $T_0(x) = 1 + 0.44\,(x - 1/2)$ and we assume purely diffusive boundary conditions [3,4].

[3] K. Aoki and N. Masukawa, Gas flows caused by evaporation and condensation on two parallel condensed phases and the negative temperature gradient: Numerical analysis by using a nonlinear kinetic equation. *Phys. Fluids,* **6** 1379-1395, (1994).

[4] D.J. Rader, M.A. Gallis, J.R. Torczynski and W. Wagner, Direct simulation Monte Carlo convergence behavior of the hard-sphere-gas thermal conductivity for Fourier heat flow. *Phys. Fluids* 18, 077102 (2006)

Test 4 : gradient of temperature

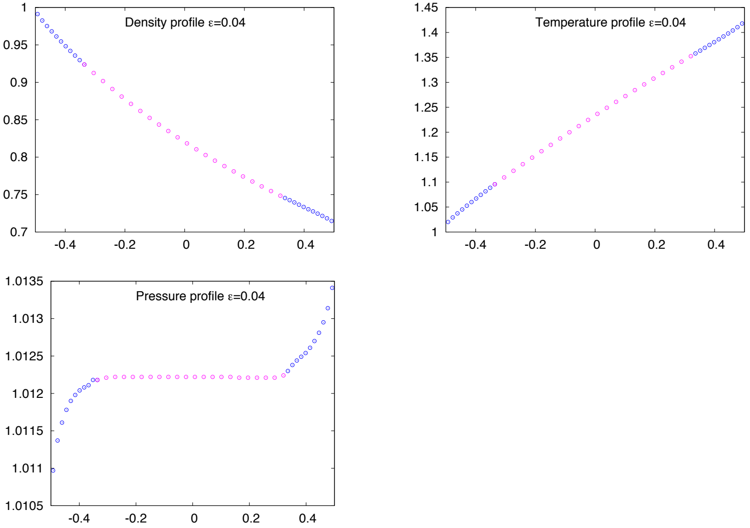


Figure : steady state of Density, Temperature and Pressure.

Work in Progress

Solve the time evolution Boltzmann equation $(x, v) \in \Omega \times \mathbb{R}^3_v$, with $\Omega \subset \mathbb{R}^2$.

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{Kn} \mathcal{Q}(f).$$

We consider a Mach number $Ma = 0.3$ and a Reynolds number $Re = 3000$.
The Mach, Reynolds and Knudsen numbers relation is given by:

$$Kn = \frac{Ma}{Re}\sqrt{\frac{\gamma\pi}{2}}, \quad \gamma = 1.4$$

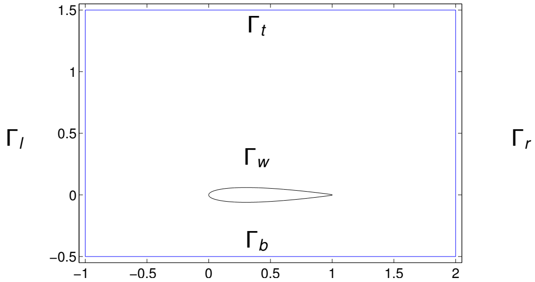


Figure : Flow around an object. *Domain including an airfoil.*

Flow around an airfoil in 2D