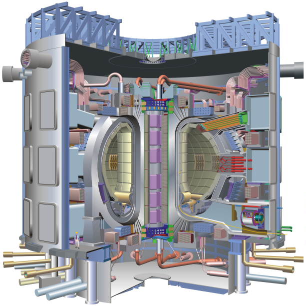
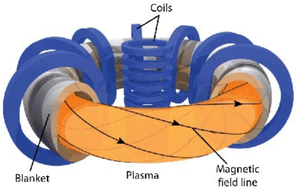
Mathematical properties of hierarchies of reduced MHD models





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with Rémy Sart (PLDV)

Section 1

Introduction

Reduced MHD models

Numerical modeling of the MHD stability of Tokamaks

$$\begin{cases} \partial_t \rho = -\nabla\cdot(\rho\mathbf{v}) + \nabla\cdot(D_\perp\nabla_\perp\rho) + S_\rho, \\ \rho\partial_t T = -\rho\mathbf{v}\cdot\nabla T - p\nabla\cdot\mathbf{v} + \nabla\cdot(\kappa_\perp\nabla_\perp T + \kappa_\parallel\nabla_\parallel T) + S_T, \\ \frac{1}{R^2}\partial_t\psi = \eta(T)\nabla\cdot\left(\frac{1}{R^2}\nabla_\perp\psi\right) - \mathbf{B}\cdot\nabla u, \\ \mathbf{e}_\theta\cdot\nabla\wedge\left(\rho\partial_t\mathbf{v} = -\rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p + \mathbf{J}\wedge\mathbf{B} + \mu\Delta\mathbf{v}\right), \\ \mathbf{B}\cdot\left(\rho\partial_t\mathbf{v} = -\rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p + \mathbf{J}\wedge\mathbf{B} + \mu\Delta\mathbf{v}\right), \end{cases}$$

with

$$\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\theta + \frac{1}{R}\nabla\psi\wedge\mathbf{e}_\theta \text{ and } \mathbf{v} = v_\parallel\mathbf{B} - R\nabla\varphi\wedge\mathbf{e}_\theta.$$

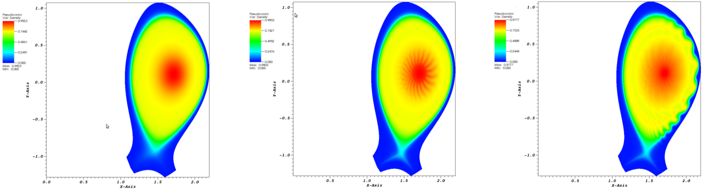
Note that $\mathbf{e}_\theta\cdot\nabla\wedge\mathbf{v} = \omega$ is the poloidal vorticity and φ is the poloidal velocity potential.

Pressure law provided by : $p = (\gamma - 1)\rho T$.

- Czarny-Huysmans : Bézier surfaces and finite elements for MHD simulations, JCP 2008.
- Hözl and al, Reduced-MHD Simulations of Toroidally and Poloidally Localized ELMs, 2012.
- Callen, notion of Extended-MHD, Cemracs 2014.

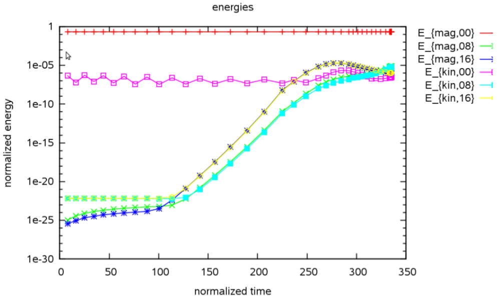
ELM's calculations (courtesy of Franck-Sonnendrucker IPP)

Control of Edge Localized Modes (ELMs) fundamental for ITER



Reduced MHD models used to compute
the growth rate of unstable modes (and much more things of course)

Growth rate computed mode per mode



Mathematical questions addressed in this talk :

- structure of reduced MHD models and
- comparison principle for the linear growth rate of reduced models.
- new models

Section 2

Reduction

Reduction in the language of MHD

Toy model (before reduction) : a resistive MHD non linear system modeling the interaction of a ionized fluid with a strong magnetic field

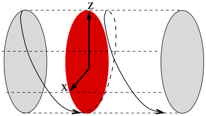
$$\begin{cases} \partial_t \rho & +\nabla \cdot (\rho \mathbf{u}) & = 0, \\ \partial_t \mathbf{B} & -\nabla \wedge (\mathbf{u} \wedge \mathbf{B}) & = -\eta \nabla \wedge (\nabla \wedge \mathbf{B}) \quad +\eta \nabla \wedge \mathbf{J}_b, \\ \partial_t(\rho \mathbf{u}) & +\nabla \cdot \left(\rho \mathbf{u}^2 + \frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B}^2 + p\mathbf{I} \right) & = \nu \Delta \mathbf{u}. \end{cases}$$

- Viscosity is $\nu > 0$, resistivity is $\eta > 0$. The free divergence constraint must be added : $\nabla \cdot \mathbf{B} = 0$. An additional equation should be added for the temperature/entropy/pressure/total energy.
- Source is mandatory to study stationary solutions defined by

$$\nabla \wedge \mathbf{B} = \mathbf{J}_b, \qquad \nabla \cdot \left(\frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B}^2 + p\mathbf{I} \right) = 0.$$

Mathematical structure :
hyperbolic (non linear)-parabolic (linear) with source.

Slab geometry



- Usual potential assumptions : F_0 is given and ψ is unknown such that

$$\mathbf{B} = F_0 \mathbf{e}_y + \nabla_\perp \psi \wedge \mathbf{e}_y = (\partial_z \psi(t,x,z), F_0, -\partial_x \psi(t,x,z))$$

- "Incompressiblity" $\nabla \cdot \mathbf{u} = 0$ yields :
$\mathbf{u} = \nabla \phi(t,x,z) \wedge \mathbf{e}_y = (-\partial_z \phi(t,x,z), 0, \partial_x \phi(t,x,z))$.
- That is we seek the unknowns in a linear space

$$(\mathbf{B}, \mathbf{u}) \in \mathcal{K}_0 = U_0 + \mathcal{K}$$

where $\mathbf{x} \mapsto U_0 = (F_0 \mathbf{e}_y, 0)$ is a given function (can be a constant) and

$$\mathcal{K} = \mathrm{Span}\,\{\nabla_\perp \psi \wedge \mathbf{e}_y, \nabla_\perp \varphi \wedge \mathbf{e}_y\}$$

is a closed vectorial subspace of infinite dimension.

Weak formulation (hyp. part)

Assume for simplicity $\rho \equiv 1$: just two equations remain.

Let $\mathcal{C} = \Omega \times \mathbb{R}$ be the infinite cylinder in the y direction.

$$\begin{cases} \int_{\mathcal{C}} \left(\partial_t \mathbf{u} + \nabla \cdot \mathbf{u} \otimes \mathbf{u} + \frac{1}{\mu_0} (\nabla \wedge \mathbf{B}) \wedge \mathbf{B} \right) \cdot \widehat{\mathbf{u}} \; dv = 0, & \forall \widehat{\mathbf{u}}, \\ \int_{\mathcal{C}} \left(\partial_t \mathbf{B} - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) \right) \cdot \widehat{\mathbf{B}} \; dv = 0, & \forall \widehat{\mathbf{B}}. \end{cases}$$

The test functions are $\widehat{\mathbf{u}} = \nabla_\perp \widehat{\phi}(x,z) \wedge \mathbf{e}_y$ and $\widehat{\mathbf{B}} = \nabla_\perp \widehat{\psi}(x,z) \wedge \mathbf{e}_y$, that is $\left(\widehat{\mathbf{u}}, \widehat{\mathbf{B}}\right) \in \mathcal{K}$.

The unknowns are

$$\mathbf{u} = \nabla_\perp \phi(t,x,z) \wedge \mathbf{e}_y \text{ and } \mathbf{B} = F_0 \mathbf{e}_y + \nabla_\perp \psi(t,x,z) \wedge \mathbf{e}_y$$

that is $(\mathbf{u}, \mathbf{B}) \in \mathcal{K}_0$ where $\mathcal{K}_0 = (0, F_0 \mathbf{e}_y) + \mathcal{K}$.

After integration by parts (with vanishing Dirichlet boundary data) and some amount of differential calculus such as $\nabla_\perp \widetilde{\phi} \wedge \mathbf{e}_y = \mathrm{curl}(\mathbf{e}_y \widetilde{\phi})$, the end result is ...

Usual model

. . .the incompressible model in the 2D domain

$$\begin{cases} \partial_t \psi = [\psi, \varphi], & \mathbf{B} = \vec{\text{curl}}\ \psi, \\ \partial_t \omega = [\omega, \varphi] + [\psi, \Delta_\perp \psi], & \\ \Delta_\perp \varphi = \omega, & \mathbf{u} = \vec{\text{curl}}\ \varphi. \end{cases}$$

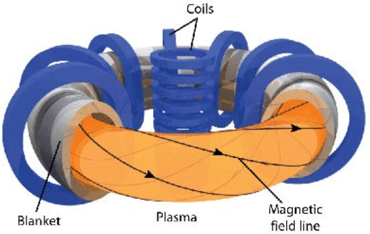
The Poisson bracket is $[a, b] = \partial_x a \partial_y b - \partial_y a \partial_x b : (x, y) \in \Omega$.

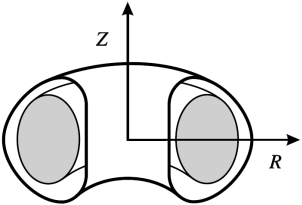
$\psi(t, x, y)$ is the magnetic potential
$\omega(t, x, y)$ is the fluid vorticity
$\varphi(t, x, y)$ is the fluid potential

- Strauss 76', 82', . . .

Next question : is it possible to generalize to more general sets $\mathcal{K}$ needed for Tokamaks ?

Answer = YES





Change of coordinates : $(R, Z) \in \Omega$ and $\theta \in [0, 2\pi]$, with
$x = R\cos\theta,\ x = R\sin\theta,\ z = Z$.

Local frame is $(\mathbf{e}_R, \mathbf{e}_Z, \mathbf{e}_\theta) = (\nabla R, \nabla Z, R\nabla\theta)$ and

$$\begin{aligned} \mathbf{B} - F_0\nabla\theta &= \nabla\psi \wedge \nabla\theta &= \text{curl}\ (\psi\nabla\theta) &= \tfrac{1}{R}\nabla\psi \wedge \nabla\mathbf{e}_\theta \\ \mathbf{u} &= \nabla\varphi \wedge \nabla\theta &= \text{curl}\ (\varphi\nabla\theta) &= \tfrac{1}{R}\nabla\varphi \wedge \nabla\mathbf{e}_\theta. \end{aligned}$$

By construction $\nabla \cdot \mathbf{B} = 0,\ \nabla \cdot \mathbf{u} = 0,\ \mathbf{B} \cdot \mathbf{e}_\theta = \frac{F_0}{R},\ \mathbf{u} \cdot \mathbf{e}_\theta = 0$.

It defines a linear space $\mathcal{K}_0$ such that $(\mathbf{B}, \mathbf{u}) \in \mathcal{K}_0$.

Reduction in the language of hyper. cons. laws

• Let $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ is a given open domain, typically a cylinder $\Omega = \mathcal{C}$ or a torus $\Omega = \mathcal{T}$.

• Start with a system of non linear conservation laws in dimension $d \geq 1$

$$\partial_t U + \nabla \cdot f(U) = 0.$$

• Assume an **additional compatible conservation law** $\partial_t S(U) + \nabla \cdot F(U) = 0$, where the function $U \mapsto S(U) \in \mathbb{R}$ is strictly convex : define the adjoint variable

$$V = \nabla S(U).$$

- Godunov 60'.
- In practice S is the energy or the entropy.
- $S^*(V) = (V, U) - S(U(V))$, $F^*(V) = (V, f(U)) - F(U(V))$
- $U = \nabla S^*(V)$

Let $\mathcal{K} \subset \mathbb{R}^n$ be a given vectorial subspace

$$\mathcal{K} = \text{Span}\left\{Z_1, \ldots, Z_p\right\}, \qquad p < n.$$

A reduced model (in dimension p) writes

$$\begin{cases} \partial_t(U, Z_i) + \nabla \cdot (f(U)Z_i) = 0, & 1 \leq i \leq p, \\ V \in \mathcal{K}. \end{cases}$$

More precisely $V(t,x) = \lambda_1(t,x)Z_i + \cdots + \lambda_p(t,x)Z_p$.

Theorem (Boillat-Ruggieri, Chen-Levermore-Liu '94') :
the reduced model is conservative and hyperbolic.

- If $S = \frac{1}{2}|U|^2$ and $V \equiv U$, it is the standard Galerkin projection.
- Algebra is the same in infinite dimension.

Reduction in both languages

• Consider any **linear** subspace of $X^n = \mathcal{C}^1(\Omega)^n$ defined by

$$\mathcal{K}_0 = V_0(\mathbf{x}) + \mathcal{K} \subset X^n \quad (1)$$

where $\mathbf{x} \mapsto V_0(\mathbf{x}) \in X^n$ is a given function and $\mathcal{K} \subset X^n$ is a closed vectorial subspace of **infinite dimension**.

• Neglect boundary conditions and consider the reduced system in **weak formulation**

$$\begin{cases} \displaystyle\int_\Omega [\partial_t U + \nabla \cdot f(U), Z] \, dv = 0, \quad \forall \textbf{ test function } Z \in \mathcal{K}, \\ V \in \mathcal{K}_0. \end{cases}$$

We say the model is **hyperbolic-compatible**.

Notice that

$$V - V_0 \in \mathcal{K}$$

can be thought of as being an "infinite sum" (an integral) of "basis" (individual) functions in $\mathcal{K}$.

First property : entropy/energy preservation

Set the **relative entropy** $\widehat{S}_0(U, \mathbf{x}) = S(U) - (V_0(\mathbf{x}), U)$.

Prop. A model with hyperbolic compatibility satisfies

$$\frac{d}{dt} \int_\Omega \widehat{S}_0(U, \mathbf{x}) dv = \int_\Omega \left(\nabla V_0(\mathbf{x}) : f(U) \right) dv + \text{b.c.} \qquad (2)$$

where b.c. represents integrals on the boundary $\partial\Omega$
and : is the contraction of tensors.

Proof : by definition $\mathcal{K}_0$ is affine and $V - V_0 \in \mathcal{K}$. So

$$\int_\Omega \left[(\partial_t U, V - V_0) + (\nabla \cdot f(U), V - V_0) \right] dv = 0.$$

It yields

$$\int_\Omega \partial_t \widehat{S}_0(U, \mathbf{x}) dv = - \int_\Omega \nabla \cdot F(U) dv + \int_\Omega (\nabla \cdot f(U), V_0) \, dv.$$

Integration by parts yields the result.

Second property : comparison principle

Assume U_0 is a special rest state $f(U_0) = 0$
(and that V_0 corresponds to U_0).
Add source, dissipation (and simplify)

$$\begin{cases} \int_\Omega [\partial_t U + \nabla \cdot f(U) - \nu \Delta (U - U_0), Z] \, dv = 0, \quad \forall Z \in d\mathcal{K}, \\ V \in \mathcal{K}. \end{cases}$$

Notice that U_0 is incorporated in the dissipation term to respect the rest state. For Tokamaks, $\mathbf{J}_c = \nu \Delta U_0$ corresponds to the bootstrap current. Other dissipative terms can be accounted for.

Fundamental question : determine the growth rate of perturbations around rest states.

Theorem : one can prove

$$\mathcal{K}_1 \subset \mathcal{K}_2 \Longrightarrow \lambda(\mathcal{K}_1) \leq \lambda(\mathcal{K}_2).$$

Proof

One linearizes : $V = V_0 + \varepsilon V_1 \dots$.
One founds out

$$\int_\Omega \left(\partial_t \left(A_0(x) V_1\right) + \nabla \cdot \left(B_0(x) V_1\right) - \nu \nabla \cdot \left(A_0(x) \nabla V_1\right) \quad , Z\right) dv = 0$$

where $A_0(x) = \nabla S^*_{V_0} = A_0^t(x) > 0$ and $B_0(x) = \nabla F^*_{V_0} = B_0^t(x)$.
The symmetry of the tensors is fundamental.

Here S^* and F^* are the Legendre and polar transforms of S and F.

Next step : take $Z = V_1$ and integrate by parts.

Introduction
Reduction
Models
Conclusion

One has

$$\frac{d}{dt}\int_\Omega (V_1, A_0 V_1)\, dv = \int_\Omega (\nabla \cdot B_0 V_1, V_1)\, dv - 2\nu \int_\Omega (A_0 \nabla V_1 : \nabla V_1)\, dv.$$

Define the space : $Y(\mathcal{K}) =$ closure of $\mathcal{K} \subset H_0^1(\Omega)^n$.
Define the real number $\lambda(\mathcal{K}) \in \mathbb{R}$

$$\lambda(\mathcal{K}) = \max_{V_1 \in Y(\mathcal{K})} \frac{\int_\Omega ((\nabla \cdot B_0(x))\, V_1, V_1)\, dv - 2\nu \int_\Omega (A_0(x) \nabla V_1 : \nabla V_1)\, dv}{\int_\Omega (V_1, A_0(x) V_1)\, dv}.$$

- Rest state constant in space $\Longrightarrow \lambda(\mathcal{K}) \leq 0$. This is the usual hyperbolic criterion.
- Concavity $\nabla \cdot B_0(x) \leq 0 \Longrightarrow \lambda(\mathcal{K}) \leq 0$
- Gronwall lemma states that $\left(\int_\Omega (V_1, A_0 V_1)\, dv\right)(t) \leq Ce^{\lambda(\mathcal{K})t}$
- $\lambda(\mathcal{K})$ is an upper bound of the rate of growth of linear perturbations.
- By definition $\mathcal{K}_1 \subset \mathcal{K}_2 \Longrightarrow \lambda(\mathcal{K}_1) \leq \lambda(\mathcal{K}_2)$.

- Simpler Rayleigh-Ritz quotients exist in ideal MHD, see Schnack.
- Comparison between Schnack approach and new approach to be done.

Section 3

Models

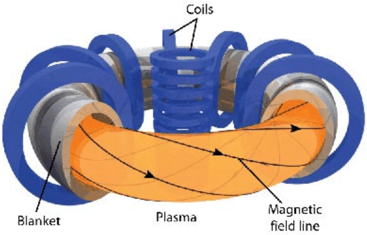
Goals

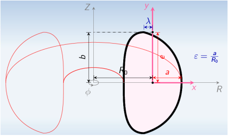
$$\begin{cases} \partial_t \rho & +\nabla \cdot (\rho \mathbf{u}) & = 0, \\ \partial_t \mathbf{B} & -\nabla \wedge (\mathbf{u} \wedge \mathbf{B}) & = -\eta \nabla \wedge (\nabla \wedge \mathbf{B}) \quad +\eta \nabla \wedge \mathbf{J}_b, \\ \partial_t (\rho \mathbf{u}) & +\nabla \cdot \left(\rho \mathbf{u}^2 + \frac{|\mathbf{B}|^2}{2} \mathbf{I} - \mathbf{B}^2 + p \mathbf{I} \right) & = \nu \Delta \mathbf{u}. \end{cases}$$

Closure is with the energy $\rho e = \rho \epsilon + \rho \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2$, that is $S = \rho e$.

- Consider the toroidal case,
- Define set $\mathcal{K}_0$ with increasing geometrical structures, and so with increasing complexity : choosing $\mathcal{K}_0$ means filtering the dynamics chosing physical glasses
- Make all calculations for the weak formulation (not shown),
- Write down the strong formulations of the corresponding entropy-Petrov-Galerkin. They will be of Navier-Stokes type,
- Provide additional comments (well posedness, . . .).

Models in the torus : $\mathcal{T} = \Omega \times [0, 2\pi]$





The curvature is $\varepsilon = \frac{R_+ - R_-}{R_+ + R_-} < 1$ and $R = 1 + \varepsilon x$.
Remark : small curvature is used in the usual derivation of the model : but $\varepsilon = 0.3$ for ITER.

Additional small physical parameter is $\beta = \frac{p}{|\mathbf{B}|}$.

Representation formulas in the torus

New coordinates $X = R\cos\Theta$, $Y = R\sin\Theta$ Z with $(R, Z) \in \Omega$. The **toroidal variable** is $\Theta \in [0, 2\pi]$.
The local directions are $(\mathbf{e}_R, \mathbf{e}_Z, \mathbf{e}_\Theta)$

$$\begin{cases} \mathbf{e}_R = \nabla R & = & (\cos\Theta, \sin\Theta, & 0), \\ \mathbf{e}_\Theta = R\nabla\Theta & = & (-\sin\Theta, \cos\Theta, & 0), \\ \mathbf{e}_Z = \nabla Z & = & (0, 0, & 1). \end{cases}$$

Consider the Ansatz

$$\begin{array}{llll} \mathbf{B} - F_0\nabla\Theta & = \nabla\psi \wedge \nabla\Theta & = \text{curl}\,(\psi\nabla\Theta) & = \frac{1}{R}\nabla\psi \wedge \nabla\mathbf{e}_\Theta \\ \mathbf{u} & = \nabla\varphi \wedge \nabla\Theta & = \text{curl}\,(\varphi\nabla\Theta) & = \frac{1}{R}\nabla\varphi \wedge \nabla\mathbf{e}_\Theta. \end{array}$$

- By construction $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot \mathbf{u} = 0$, $\mathbf{B} \cdot \mathbf{e}_\Theta = \frac{F_0}{R}$, $\mathbf{u} \cdot \mathbf{e}_\Theta = 0$.

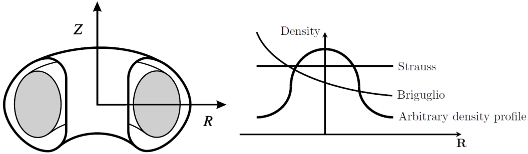
- Note that representation is more $\mathbf{u} = -R\nabla\varphi \wedge \mathbf{e}_\theta$ in Jorek models.

2D model with general density

Let $\rho = \rho_0 > 0$ be a given density (i.e. a given positive function).

$$\begin{aligned} \mathbf{B} - F_0 \nabla\Theta &= \nabla\psi \wedge \nabla\Theta &= \text{curl}\,(\psi\nabla\Theta) &= \tfrac{1}{R}\nabla\psi \wedge \nabla\mathbf{e}_\Theta \\ \rho_0 \mathbf{u} &= \tfrac{1}{R}\nabla\varphi \wedge \nabla\Theta &= \tfrac{1}{R}\,\text{curl}\,(\varphi\nabla\Theta) &= \tfrac{1}{R}\nabla\varphi \wedge \nabla\mathbf{e}_\Theta. \end{aligned}$$

$$\begin{cases} \partial_t \psi = \frac{1}{\rho R}[\psi, \varphi] + \eta \Delta^\star \psi - J_b, \quad J_b \text{ is a source term,} \\ \partial_t \omega = \frac{1}{\rho R}[\omega, \varphi] - 2\frac{1}{(\rho R)^2}[\rho R, \varphi]\,\omega + \rho R \left[\psi, \frac{1}{\rho R^2}\Delta^\star \psi\right] - \nu \Delta_\perp^\rho \omega, \\ \Delta_\rho \varphi = \omega. \end{cases}$$



The domain is $(R, Z) \in \Omega$. The system is supplemented with natural boundary conditions $\psi = \varphi = \omega = 0$ on $\partial\Omega$ or $\psi = \varphi = \partial_n \varphi = 0$ on $\partial\Omega$.

The fundamental energy estimate

Property $J_c = 0$, $\eta > 0$ and $\nu > 0$.
For regular solutions, one has

$$\frac{d}{dt}\int_\Omega \frac{|\nabla\psi|^2}{2R} + \frac{|\nabla\varphi|^2}{2\rho R}$$

$$= -\eta\int_\Omega \frac{(\Delta^\star_\perp\psi)^2}{R} - \nu\int_\Omega \frac{\omega^2}{\rho R} \leq 0, \quad (\text{measure} : d\Omega = dRdZ)\,.$$

Proof : Multiply the first eq. by $-\frac{\Delta^\star\psi}{R}$, the second eq. by $-\frac{\varphi}{\rho R}$. Then integrate by parts and use basic identities.

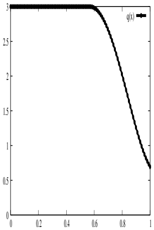
- D.-Sart : Reduced resistive MHD in Tokamaks with general density, M2AN 2012. *Assume* $\eta, \nu > 0$ *: Existence of weak solutions based on energy estimates plus compactness in convenient spaces* $H^2 \cap \{b.c.\}$.
- Same structure as potential formulations of Navier-Stokes equation : Chorin, Temam, . . .

Numerical example (data from Fujita).

Computations performed in a **2D simple FreeFem++ code** initiated at Cemracs ...

Dirichlet conditions for all variables $\eta = 10^{-4}$, $\nu = 10^{-5}$, $\rho =$ given, $\varepsilon = 0.3$, $J_{\mathrm{boot}} = -\partial_r \rho$.

ρ



Fujita

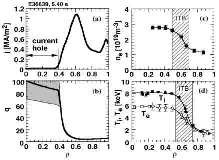
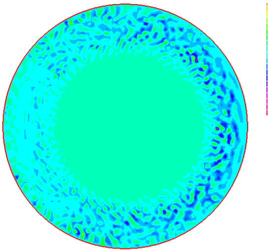


FIG. 2. Radial profiles of (a) current density j, (b) safety factor q, (c) electron density n_e, and (d) ion and electron temperatures (T_i and T_e). Here, n_e was measured by YAG (yttrium-aluminum-garnet) laser Thomson scattering, T_i by charge exchange recombination spectroscopy (CXRS) on carbon ions, and T_e by YAG laser Thomson scattering (open circles) and electron cyclotron emission (ECE) (open rectangles).

J_h



- Unconditionally stable numerical simulations of a new generalized reduced resistive magnetohydrodynamics model, Malapaka-D.-Sart, IJNMF, 2014.

2D with non constant F and $\rho = 1$

Start from $\mathbf{B} = F(R,Z)\nabla\theta + \nabla\psi(R,Z) \wedge \nabla\theta$ and $\mathbf{u} = \frac{1}{R}\nabla\varphi(R,Z) \wedge \nabla\Theta$.
One gets

$$\begin{cases} \partial_t \psi = \dfrac{1}{\rho R}[\psi, \phi] + \eta \Delta_\perp^\star \psi, \\ \partial_t \omega = \rho r \left[\dfrac{1}{(\rho R)^2}\omega, \phi \right] - \rho R \left[\dfrac{F}{\rho R^2}, F \right] - \rho R \left[\dfrac{1}{\rho R^2}\Delta_\perp^\star \psi, \psi \right] + \nu \Delta_\perp^\rho \omega, \\ \omega = \Delta_\perp^\rho \phi. \end{cases}$$

Prop : Consider a solution of the Grad-Shafranov equation

$$\Delta_\perp^\star \overline{\psi} = -R^2 \frac{d\overline{p}}{d\overline{\psi}} - \frac{1}{2}\frac{d\overline{F}^2}{d\overline{\psi}}.$$

Setting $\rho = \rho(\overline{p})$ and $(\psi, \omega) = (\overline{\psi}, 0)$, it yields a stationary solution of the dynamical model.

3D with constant F_0 but $\partial_\theta \neq 0$

$\Omega = \{(x,z) \in \mathcal{D} \text{ and } \theta \in [0, 2\pi]\}$.
Start from $\mathbf{B} = F_0 \nabla\theta + \nabla\psi(R,Z,\theta) \wedge \nabla\theta$ and $\mathbf{u} = \frac{1}{R}\nabla\varphi(R,Z,\theta) \wedge \nabla\Theta$.

One gets

$$\begin{cases} \partial_t \psi = \dfrac{1}{R}[\psi, \phi] + \eta_\perp \Delta^\star \psi + \eta_\parallel \partial_\theta^2 \psi + \dfrac{1}{R^2} F_0 \partial_\theta \phi + Q, \\ \partial_t \omega = r\left[\dfrac{1}{R^2}\omega, \phi\right] + R\left[\psi, \dfrac{1}{R^2}\Delta^\star\psi\right] + \nu_\perp \Delta^\star \omega + \nu_\parallel \partial_\theta^2 \omega + \dfrac{1}{R^2} F_0 \Delta^{\star\star\star} \partial_\theta \psi, \\ \omega = \Delta^\star \phi. \end{cases}$$

- The coupling of different toroidal modes is with ∂_θ derivative.
- The source term Q is defined by the weak form

$$\int_{\mathcal{D}} \frac{1}{R}\left(\partial_R Q \partial_R \widetilde{\psi} + \partial_Z Q \partial_R \widetilde{\psi}\right) = 2F_0 \int_{\mathcal{D}} \frac{1}{R^4} \partial_\theta \phi \partial_R \widetilde{\psi}, \quad \forall \widetilde{\psi} \in H_0^1(\mathcal{D}).$$

B.C. are : $\psi = \phi = \frac{\partial \phi}{\partial n} = 0$. Energy identity is

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{1}{R}\left(|\nabla_{R,Z}\psi|^2 + |\nabla_{R,Z}\phi|^2\right)$$

$$+\eta_\perp \int_\Omega \frac{|\Delta^\star \psi|^2}{R} + \nu_\perp \int_\Omega \frac{|\Delta^\star \phi|^2}{R}$$

$$+\eta_\parallel \int_\Omega \frac{|\partial_\theta \nabla_{R,Z}\psi|^2}{R} + \nu_\parallel \int_\Omega \frac{|\partial_\theta \nabla_{R,Z}\phi|^2}{R} = 0$$

Theor. : Assume $\eta_\perp, \nu_\perp, \eta_\parallel, \nu_\parallel > 0$.
There exists a weak solution $(\psi, \phi) \in L^2([0, T] : H^2(\Omega)^2)$ with initial data $(\psi_0, \phi_0) \in H^2(\Omega)^2 \cap \{b.c.\}$. Regularity can be precized.

Proof : use the energy identity as an a priori identity, and follow Temam, Navier-Stokes Equations : Theory and Numerical Analysis, AMS(2001)

Bounded number of Fourier modes : $= 2D\frac{1}{2}$

- One can plug

$$\psi_N = \sum_{-N \leq n \leq N} \psi_n(R, Z) e^{in\theta}$$

in the weak formulation.

- All terms can be computed explicitly in a code, as in Jorek code.

- Denote λ_N the maximal growth rate of eigenmodes.
One has the inequality

$$\lambda_N \leq \lambda_{N+1} \leq \cdots \leq \lambda_\infty .$$

Section 4

Conclusion

Conclusion

- Physical basis : reduced MHD models obtained by filtering out the non essential part helps to get understanding of the physics
- Hierarchy of Navier-Stokes models for the modeling of reduced MHD in Tokamaks. Mathematical basis is Weak formulation.
- The geometry of the torus is taken into account by construction.
- Two mathematical results are
  - Comparison principle for the growth rate of instability (inherited from the Hyperbolic theory)
  - Well-posedness (existence) of viscous formulations in Sobolev spaces (inherited from the Navier-Stokes theory)
- Use of these structure and results for precontionning of "real" calculations still to be done.

  Weak formulation also appealing for numerical purposes (Nkonga).

- Navier-Stokes hierarchies of reduced MHD models in Tokamak geometry, HAL preprint server, D.-Sart.