High magnetic field averaged models for plasma physics

Mihaï BOSTAN and Céline CALDINI
University of Aix-Marseille, FRANCE
mihai.bostan@univ-amu.fr

Cemracs Numerical modeling of plasmas
21/07 - 29/08 2014 Cirm Marseille

Motivations

Transport of charged particles under strong magnetic fields

Magnetic confinement fusion (MCF)

Kinetic description with collisional mechanism

Landau equation

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}\left(E(t,x) + v \wedge \frac{B(t,x)}{\varepsilon}\right) \cdot \nabla_v f = Q(f^\varepsilon)$$

f^ε : particle distribution function in (x, v)

$f^\varepsilon(t, x, v)\mathrm{d}x\mathrm{d}v$: particle number inside the volume $\mathrm{d}x\mathrm{d}v$

$Q(f^\varepsilon, f^\varepsilon)$: Fokker-Planck-Landau bilinear collision kernel

Main purposes

Efficient resolution of problems involving disparate scales

$\frac{m_e}{m_i} << 1, \quad \frac{v_\parallel}{v_\perp} << 1, \quad \frac{T_c}{T_{\mathrm{obs}}} << 1$

Oscillation treatment : numerical instabilities, averaging, smoothing out fluctuations

Averaged Fokker-Planck-Landau equation for strongly magnetized plasmas

Fluid models for strongly magnetized plasmas

Strongly anisotropic diffusion

Main difficulties

- Complete explicit averaged kernels
- Preserve all the balances (mass, momentum, kinetic energy, entropy)
- Xu & Rosenbluth (linearized around equilibria, implementation seems hard), Garbet (variational principles), Brizard & Hahm
- Explain the perpendicular diffusion in space by averaging
- H theorem, equilibrium, long time behavior, etc
- Transport averaging : Frénod & Sonnendrücker, Frénod & Mouton, Golse & Saint-Raymond, Ghendrih & Hauray & Nouri, Han-Kwan.

Finite Larmor radius

$$B^\varepsilon = \left(0, 0, \frac{B}{\varepsilon}\right), \quad B > 0$$

$$\omega_c^\varepsilon = \frac{qB^\varepsilon}{m}, \quad T_{\text{obs}}\, \omega_c^\varepsilon \approx \frac{1}{\varepsilon} >> 1$$

$$\rho_{Larmor} = \frac{|v_\perp|}{\omega_c^\varepsilon} \approx \varepsilon, \quad \frac{L_\perp}{\rho_{Larmor}} \approx 1, \quad \frac{L_\parallel}{\rho_{Larmor}} \approx \frac{1}{\varepsilon} >> 1$$

Landau equation

Notations : $\overline{x} = (x_1, x_2),\ \overline{v} = (v_1, v_2),\ {}^\perp\overline{v} = (v_2, -v_1),\ \omega_c = qB/m$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\overline{v}\cdot\nabla_{\overline{x}} f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E\cdot\nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon}(v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = Q(f^\varepsilon, f^\varepsilon)$$

Fast cyclotronic motion

$$\mathcal{T} = b \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c \, {}^{\perp}\overline{v} \cdot \nabla_{\overline{v}} f$$

$$a \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$$

$$\partial_t f^{\varepsilon} + a \cdot \nabla_{x,v} f^{\varepsilon} + \frac{1}{\varepsilon} b \cdot \nabla_{x,v} f^{\varepsilon} = Q(f^{\varepsilon}, f^{\varepsilon})$$

Slow and fast time variables

t : slow time variable, $s = t/\varepsilon$: fast time variable

Particle trajectories

$$(X^{\varepsilon}(t), V^{\varepsilon}(t)) = Y^{\varepsilon}(t) = Y(t, t/\varepsilon) + \varepsilon Y^1(t, t/\varepsilon) + ...$$

$$\frac{dY^{\varepsilon}}{\mathrm{d}t} = a(Y^{\varepsilon}) + \frac{1}{\varepsilon} b(Y^{\varepsilon}) \implies \partial_s Y = b(Y)$$

Ansatz

$$f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 ...$$

$$\mathcal{T} f := b \cdot \nabla_{x,v} f = \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c \, {}^\perp \overline{v} \cdot \nabla_{\overline{v}} f = 0$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f,f)$$

Goal : close the evolution equation for f; eliminate the multiplier f^1 thanks to the divergence constraint

Expected limit model

$$\partial_t f + A \cdot \nabla_{x,v} f = \tilde{Q}(f,f), \;\; \mathcal{T} f = 0$$

The constraint

$$b \cdot \nabla_{x,v} f = \mathrm{div}_{x,v}\{fb\} = 0 \leftrightarrow \frac{\mathrm{d}}{\mathrm{d}s}\{f(X(s), V(s))\} = 0$$

Flow of b

$$\frac{\mathrm{d}\overline{X}}{\mathrm{d}s} = \overline{V}(s), \quad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c\,{}^\perp\overline{V}(s), \quad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0$$

Invariants

$$x_1 + \frac{v_2}{\omega_c},\ x_2 - \frac{v_1}{\omega_c},\ x_3,\ r = |\overline{v}|,\ v_3$$

$$b \cdot \nabla_{x,v} f = 0 \leftrightarrow \exists\, g : f(t,x,v) = g\left(t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right)$$

Closure

$$\mathrm{Range}(b \cdot \nabla_{x,v}) \perp \ker(b \cdot \nabla_{x,v})$$

$$P = \mathrm{Proj}_{\ker(b\cdot\nabla_{x,v})} \implies P(\mathrm{Range}(b \cdot \nabla_{x,v})) = 0$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 = Q(f,f)$$

$$\partial_t f + P(v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f) = P(Q(f,f))$$

How to compute P on transport and collision operators ?

Average along a flow

$$\overline{V}(s) = R(-\omega_c s)\overline{v}, \;\; \overline{X}(s) = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} - \frac{{}^\perp\overline{V}(s)}{\omega_c}, \;\; X_3(s) = x_3, \;\; V_3(s) = v_3$$

Definition (average operator)

$$\langle u \rangle (x, v) = \frac{1}{T_c} \int_0^{T_c} u(X(s; x, v), V(s; x, v)) \, \mathrm{d}s \in \ker b \cdot \nabla_{x,v}$$

Proposition The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$ *i.e.,*

$$\langle u \rangle \in \ker \mathcal{T} \;\; : \;\; \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \varphi \, \mathrm{d}v\mathrm{d}x = 0, \;\; \forall \, \varphi \in \ker \mathcal{T}.$$

Average and first order differential operators

$$\langle a \cdot \nabla_{x,v} f \rangle = \langle \mathrm{div}_{x,v}\{fa\} \rangle = ... = \mathrm{div}_{x,v}\{\langle f \rangle A\} = A \cdot \nabla_{x,v} f$$

Change of coordinates

$$\psi_1 = x_1 + \frac{v_2}{\omega_c}, \ \ \psi_2 = x_2 - \frac{v_1}{\omega_c}, \ \ \psi_3 = x_3, \ \ \psi_4 = \sqrt{(v_1)^2 + (v_2)^2}, \ \ \psi_5 = v_3$$

$$\psi_0 = -\frac{\alpha}{\omega_c}, \ \ \overline{v} = |\overline{v}| e^{i\alpha} = |\overline{v}|(\cos\alpha, \sin\alpha), \ \ \mathcal{T}\psi_0 = 1$$

$$u(x,v) = U(\psi_0(x,v), \psi_1(x,v), ..., \psi_5(x,v))$$

Derivations along the invariants

$$b^i \cdot \nabla_{x,v} u = \frac{\partial U}{\partial \psi_i}(\psi(x,v)), \ \ 0 \le i \le 5$$

Expressions for b^i

$$b^0 \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c \,^\perp\overline{v} \cdot \nabla_{\overline{v}}, ..., b^4 \cdot \nabla_{x,v} = -\frac{^\perp\overline{v}}{\omega_c|\overline{v}|} \cdot \nabla_{\overline{x}} + \frac{\overline{v}}{|\overline{v}|} \cdot \nabla_{\overline{v}}$$

Remark

$$[b^i, b^j] = 0, \quad 0 \leq i, j \leq 5.$$

Proposition Assume that $[c, b] = 0$. Then the operator $\mathrm{div}_{x,v}(\cdot\, c)$ is commuting with the average operator associated to the flow of $b \cdot \nabla_{x,v}$ (derivation w.r.t. a parameter under the integral sign)

$$\mathrm{div}_{x,v}(\langle u \rangle\, c) = \langle \mathrm{div}_{x,v}(uc) \rangle\,, \quad c \cdot \nabla_{x,v} \langle u \rangle = \langle c \cdot \nabla_{x,v} u \rangle\,.$$

Proof

$$[c, b] = 0 \leftrightarrow Z(h; Y(s; y)) = Y(s; Z(h; y))$$

How average and divergence commute

$$\xi = \sum_i (\xi \cdot \nabla_{x,v} \psi_i) b^i$$

$$\langle \mathrm{div}_{x,v} \xi \rangle = \left\langle \sum_{i=0}^{5} \mathrm{div}_{x,v} \{(\xi \cdot \nabla_{x,v} \psi_i) b^i\} \right\rangle = \mathrm{div}_{x,v} \left\{ \sum_{i=0}^{5} \langle \xi \cdot \nabla_{x,v} \psi_i \rangle \, b^i \right\}$$

$$\langle a \cdot \nabla_{x,v} f \rangle = ?, \quad a \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$$

$$\left\langle v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \frac{\langle {}^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \, \partial_{v_3} f$$

How to average the Landau kernel?

Average and collisions

Fokker-Planck-Landau kernel : integral differential operator (second order derivatives and convolution)

Relaxation operator

$$Q_B(f)(x,v) = \frac{1}{\tau}\int_{\mathbb{R}^3} s(v,v')\{M(v)f(x,v') - M(v')f(x,v)\}\,\mathrm{d}v'$$

$$\int_{\mathbb{R}^3} Q_B(f)(v)f(v)\,\frac{\mathrm{d}v}{M} = -\frac{1}{2\tau}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} sMM'\left[\frac{f(v)}{M(v)} - \frac{f(v')}{M(v')}\right]^2 \mathrm{d}v'\,\mathrm{d}v \leq 0$$

Proposition

For any function $f \in \ker \mathcal{T}$ we have

$$\left\langle \int_{\mathbb{R}^3} C(v,v') f(x,v')\,\mathrm{d}v' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f(\overline{x'}, x_3, v') \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

where $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c \overline{x'} + {}^\perp\overline{v'})$.

Corollary Assume that $s(v,v') = \sigma(|v-v'|),\ \ v,v' \in \mathbb{R}^3$. For any $f \in \ker \mathcal{T}$ we have

$$\langle Q_B f \rangle = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v) f(\overline{x'}, x_3, v') - M(v') f(x,v)\}$$

with $z = \omega_c \overline{x} + {}^\perp\overline{v} - (\omega_c \overline{x'} + {}^\perp\overline{v'})$ and

$$\mathcal{S}(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,)\chi(r, r', z)$$

$$\chi(r,r',z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2 \sqrt{|z|^2 - (r-r')^2}\sqrt{(r+r')^2 - |z|^2}}$$

Averaged relaxation operator

1. non local in space
2. similar properties (mass balance, negativity) but globally in $(\overline{x}, v)$
3. averaging leads to convolution with respect to the invariants

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle Q_B \rangle (f) \frac{f}{M} \,\mathrm{d}v\mathrm{d}x = -\frac{\omega_c^2}{2\tau} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M M'$$
$$\times \left[\frac{f(x,v)}{M(v)} - \frac{f(\overline{x'}, x_3, v')}{M(v')} \right]^2 \,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x \leq 0.$$

The Fokker-Planck kernel

$$\partial_t f + \frac{\langle {}^\perp\overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_{FP} \rangle (f)$$

$$f(0,x,v) = \langle f^{\mathrm{in}} \rangle (x,v)$$

$$Q_{FP}(f) = \frac{\theta}{m\tau} \mathrm{div}_v \left(\nabla_v f + \frac{m}{\theta} v f \right) = \frac{\theta}{m\tau} \mathrm{div}_v \left\{ M \nabla_v \left(\frac{f}{M} \right) \right\}$$

$$\langle Q_{FP} \rangle f(x,v) = \frac{\theta}{m\tau} \mathrm{div}_{\omega_c x, v} \left\{ M \mathcal{L} \nabla_{\omega_c x, v} \left(\frac{f}{M} \right) \right\}$$

$$\mathcal{L} = \begin{pmatrix} 2(I_3 - e_3 \otimes e_3) & -E \\ E & I_3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Fokker-Planck-Landau kernel

$$Q_{FPL}(f,f)(v) = \mathrm{div}_v \int_{\mathbb{R}^3} \sigma S(v-v') \left(f(v')(\nabla_v f)(v) - f(v)(\nabla_{v'} f)(v') \right) \mathrm{d}v'$$

Mass, momentum, kinetic energy balances

$$\int_{\mathbb{R}^3} Q_{FPL}(f,f)\,\mathrm{d}v = 0, \int_{\mathbb{R}^3} v Q_{FPL}(f,f)\,\mathrm{d}v = 0, \int_{\mathbb{R}^3} \frac{|v|^2}{2} Q_{FPL}(f,f)\,\mathrm{d}v = 0$$

Entropy production

$$D := -\int_{\mathbb{R}^3} \ln f \; Q_{FPL}(f,f)\,\mathrm{d}v \geq 0$$

The gain kernel Q^+_{FPL}

For any function $f = f(x,v)$ satisfying the constraint $\mathcal{T} f = 0$ we have

$$\langle Q^+_{FPL}(f,f)\rangle = \mathrm{div}_{\omega_c x, v}\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(x_1', x_2', x_3, v') \times \chi(|\overline{v}|, |\overline{v'}|, z)\, A^+ \nabla_{\omega_c x, v} f(x,v)\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\}$$

The loss kernel Q^-_{FPL}

For any function $f = f(x,v)$ satisfying the constraint $\mathcal{T} f = 0$ we have

$$\langle Q^-_{FPL}(f,f)\rangle = \mathrm{div}_{\omega_c x, v}\{\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}\,) f(x,v) \times \chi(|\overline{v}|, |\overline{v'}|, z)\, A^- \nabla_{\omega_c x', v'} f(x_1', x_2', x_3, v')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\}$$

$$
\begin{aligned}
&\langle Q_{FPL}(f,f)\rangle (x,v) \\
&= \mathrm{div}_{\omega_c x, v} \left\{ \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(\overline{x'}, x_3, v') A^+ \nabla_{\omega_c x, v} f(x,v)\, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\} \\
&- \mathrm{div}_{\omega_c x, v} \left\{ \omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \sigma\chi f(x,v) A^- \nabla_{\omega_c x', v'} f(\overline{x'}, x_3, v')\, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\}
\end{aligned}
$$

and

$$
\sigma\chi A^+(r, v_3, r', v_3', z) = \sum_{i=1}^{4} \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x}, v, \overline{x'}, v')
$$

$$
\sigma\chi A^-(r, v_3, r', v_3', z) = \sum_{i=1}^{4} \varepsilon_i \xi^i(\overline{x}, v, \overline{x'}, v') \otimes \xi^i(\overline{x'}, v', \overline{x}, v)
$$

for some vector fields $(\xi^i)_{1\le i\le 4}$ and $\varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \varepsilon_4 = 1$

$$\xi^1 = \{\sigma\chi\}^{1/2} \frac{r' \sin\varphi \, (v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|} \right)$$

$$\xi^2 = \{\sigma\chi\}^{1/2} \left[\frac{r - r' \cos\varphi}{|z|} \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|} \right) + \left(\frac{({}^\perp z, 0)}{|z|}, 0 \right) \right]$$

$$\xi^3 = \{\sigma\chi\}^{1/2} \frac{r' \sin\varphi}{|z|} \left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|} \right)$$

$$\frac{\xi^4}{\{\sigma\chi\}^{1/2}} = \frac{(r' \cos\varphi - r)(v_3 - v_3')}{|z|\sqrt{|z|^2 + (v_3 - v_3')^2}} \left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|} \right) + \frac{\left((v_3 - v_3') \frac{(z,0)}{|z|}, -|z| e_3 \right)}{\sqrt{|z|^2 + (v_3 - v_3')^2}}$$

Averaged Fokker-Planck-Landau kernel

1. non local in space
2. averaging leads to diffusion both in perpendicular space directions and velocity and convolution with respect to the invariants
3. similar properties (mass/momentum/kinetic energy balances, entropy decreasing) but globally in $(\overline{x}, v)$

Theorem H Consider two functions $f = f(x,v), \varphi = \varphi(x,v)$. We have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\langle Q_{FPL}\rangle(f,f)\varphi \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} ff'$$

$$(\xi^i \cdot \nabla \ln f - \varepsilon_i(\xi^i)'\nabla' \ln f')(\xi^i\cdot\nabla\varphi - \varepsilon_i(\xi^i)'\nabla'\varphi')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2$$

where

$$f = f(x,v),\ f' = f'(x_1', x_2', x_3, v')$$

$$\nabla\varphi = \nabla_{\omega_c x, v}\varphi(x,v),\ \nabla'\varphi' = \nabla_{\omega_c x', v'}\varphi(x_1', x_2', x_3, v')$$

$$\xi^i = \xi^i(x_1, x_2, v, x_1', x_2', v'),\ \ (\xi^i)' = \xi^i(x_1', x_2', v', x_1, x_2, v).$$

In particular the entropy $f \ln f$ (globally in (x_1, x_2, v)) decreases

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\ln f\,\langle Q_{FPL}\rangle(f,f)\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0.$$

Average collision invariants

$$\xi^i \cdot \nabla \varphi - \varepsilon_i (\xi^i)' \cdot \nabla' \varphi' = 0,\ \forall i \Leftrightarrow \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \langle Q_{FPL} \rangle (f, f) \varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = 0 \ \forall f$$

$$1,\ \omega_c \overline{x} + {}^\perp \overline{v},\ v_3,\ \frac{|v|^2}{2},\ \frac{|\omega_c \overline{x} + {}^\perp \overline{v}|^2 - |\overline{v}|^2}{2}$$

Gyro-kinetic equilibria

$$\xi^i \cdot \nabla \ln f - \varepsilon_i (\xi^i)' \cdot \nabla' \ln f' = 0,\ \forall i \Leftrightarrow \langle Q_{FPL} \rangle (f, f) = 0$$

$$f = \frac{\mathcal{R}}{(2\pi\theta)^{3/2}} \exp \left(- \frac{\left| \overline{v} - \frac{\theta}{\mu} {}^\perp (\omega_c \overline{x} - \overline{u}) \right|^2 + (v_3 - u_3)^2}{2\theta} \right)$$

$$\mathcal{R} = \frac{\rho(x_3) \omega_c^2}{2\pi \frac{\mu^2}{\mu - \theta}} \exp \left(- \frac{|\omega_c \overline{x} - \overline{u}|^2}{2 \frac{\mu^2}{\mu - \theta}} \right)$$

mean velocity $(\frac{\theta}{\mu} {}^\perp (\omega_c \overline{x} - \overline{u}), u_3)$ and temperature θ.

Linearization around equilibria

$$\langle Q_{FPL}\rangle(f,f) = \langle Q_{FPL}\rangle(f,f) - \langle Q_{FPL}\rangle(\mathcal{E}_f,\mathcal{E}_f)$$
$$\approx \langle Q_{FPL}\rangle(\mathcal{E}_f, f-\mathcal{E}_f) + \langle Q_{FPL}\rangle(f-\mathcal{E}_f,\mathcal{E}_f) := \mathcal{L}(f)$$

Theorem H Consider two functions $f = f(x,v), \varphi = \varphi(x,v)$. We have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{L}(f)\varphi \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\frac{\omega_c^2}{2}\sum_{i=1}^{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{E}_f\mathcal{E}_f'$$
$$(\xi^i\cdot\nabla\frac{f}{\mathcal{E}_f} - \varepsilon_i(\xi^i)'\nabla'\frac{f'}{\mathcal{E}_f'})(\xi^i\cdot\nabla\varphi - \varepsilon_i(\xi^i)'\nabla'\varphi')\,\mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2$$

Collisional invariants

$$\xi^i\cdot\nabla\varphi - \varepsilon_i(\xi^i)'\cdot\nabla'\varphi' = 0,\ \forall i \Leftrightarrow \int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\mathcal{L}(f)\varphi\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = 0\ \forall f$$

Negativity

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3}\frac{f}{\mathcal{E}_f}\mathcal{L}(f)\,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \le 0$$

Equilibria parametrization

For any $(\rho, u_1, u_2, u_3, K, G) \in \mathbb{R}^6$, $\rho > 0, K > 0, K + G > 0$ there is a unique local (in x_3) equilibrium $f = f(\overline{x}, v)$ for $\langle Q_{FPL} \rangle$ satisfying

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho, \quad \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} (\omega_c \overline{x} + {}^\perp\overline{v}, v_3) f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho u$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho \frac{(u_3)^2}{2} + \rho K$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{|\omega_c \overline{x} + {}^\perp\overline{v}|^2 - |\overline{v}|^2}{2} f \,\mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = \rho \frac{|\overline{u}|^2}{2} + \rho G$$

$$\frac{\mu\theta}{\mu - \theta} + \frac{\theta}{2} = K, \quad \mu - \frac{\mu\theta}{\mu - \theta} = G$$

Fluid models around equilibria

$$\partial_t f^\tau + v_3 \partial_{x_3} f^\tau + \frac{q}{m} E_3(t, x_3) \partial_{v_3} f^\tau = \frac{1}{\tau} \langle Q_{FPL} \rangle (f^\tau, f^\tau)$$

$$f^\tau = f + \tau f^1 + \tau^2 f^2 + \ldots$$

$$\langle Q_{FPL} \rangle (f, f) = 0 \Leftrightarrow f = \mathcal{E}_{\rho, u, \theta, \mu}$$

Collision invariants

$$\varphi \in \left\{ 1, \;\; \omega_c \overline{x} + {}^\perp \overline{v}, \;\; v_3, \;\; \frac{|v|^2}{2}, \;\; \frac{|\omega_c \overline{x} + {}^\perp \overline{v}|^2 - |\overline{v}|^2}{2} \right\}$$

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \left\{ \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E_3(t, x_3) \partial_{v_3} f \right\} \varphi \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2 = 0$$

Gyrokinetic Euler equations

$$\partial_t \rho + \partial_{x_3}(\rho u_3) = 0$$

$$\partial_t(\rho u) + \partial_{x_3}(\rho(u_3 u + (0,0,\theta))) - \rho \frac{q}{m}(0,0,E_3) = 0$$

$$\partial_t \left[\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{\theta + u_3^2}{2}\right)\right] + \partial_{x_3}\left[u_3\rho\left(\frac{\mu\theta}{\mu-\theta} + \frac{3\theta + u_3^2}{2}\right)\right] - \frac{q}{m}E_3\rho u_3 = 0$$

$$\partial_t \left[\rho\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] + \partial_{x_3}\left[\rho u_3\left(\mu - \frac{\mu\theta}{\mu-\theta}\right)\right] = 0$$

Entropy

$$\partial_t \left(\rho \ln \frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) + \partial_{x_3}\left(\rho u_3 \ln \frac{\rho(\mu-\theta)}{\mu^2\theta^{3/2}}\right) = 0$$

Strongly anisotropic diffusion

image processing, edge detection

heat diffusion in a tokamak

porous media

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$${}^t D = D, \ \ \exists\, d > 0 \ \text{ t.q. } \ D(y)\xi\cdot\xi + (b(y)\cdot\xi)^2 \geq d\,|\xi|^2, \ \ \xi \in \mathbb{R}^m, \ \ y \in \mathbb{R}^m.$$

Stability condition $\displaystyle \frac{d}{\varepsilon}\frac{\Delta t}{|\Delta y|^2} \leq \frac{1}{2}$

$$\lim_{\varepsilon \searrow 0} u^\varepsilon = u, \ \ b \cdot \nabla_y u = 0$$

$$\partial_t u - \mathrm{div}_y(\tilde{D}(y)\nabla_y u) = 0, \ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$$\frac{d}{\varepsilon}\frac{\Delta t}{|\Delta y|^2} \leq \frac{1}{2}$$

Main ideas

- Averaging along the flow of $b \cdot \nabla_y$

$$\mathrm{div}_y(b \otimes b\, \nabla_y u) = b \cdot \nabla_y (b \cdot \nabla_y u)$$

$[\mathrm{div}_y(D\nabla_y), b \cdot \nabla_y] = 0 \implies [\mathrm{div}_y(D\nabla_y), \langle \cdot \rangle] = 0$ and then

$$\partial_t \langle u^\varepsilon \rangle - \mathrm{div}_y(D(y)\nabla_y \langle u^\varepsilon \rangle) = 0, \ \ (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

- In the above case the effective matrix is D.
- More generaly, compute the commutator between $\mathrm{div}_y(D\nabla_y)$ and $b \cdot \nabla_y$.

Averaged diffusion matrix

$$[b(y)\cdot\nabla_y, \mathrm{div}_y(A(y)\nabla_y)] = \mathrm{div}_y([b,A]\nabla_y) \text{ in } \mathcal{D}'(\mathbb{R}^m)$$

$$[b,A] := (b\cdot\nabla_y)A - \partial_y b A(y) - A(y)\,{}^t\partial_y b, \quad y\in\mathbb{R}^m$$

Proposition Let $Y(s;y)$ be the flow of $b(y)$. The second order operator $\mathrm{div}_y(A(y)\nabla_y)$ commutes with $b(y)\cdot\nabla_y$ iff $[b,A]=0$, that is iff

$$A(Y(s;y)) = \partial_y Y(s;y)A(y)\,{}^t\partial_y Y(s;y), \quad s\in\mathbb{R}, \quad y\in\mathbb{R}^m$$

or

$$(\partial_y Y)^{-1}(s;\cdot)A(Y(s;\cdot))\,{}^t(\partial_y Y)^{-1}(s;\cdot) = A(y), \quad s\in\mathbb{R}, \quad y\in\mathbb{R}^m.$$

Unitary C^0-group

$$G(s)A = (\partial_y Y)^{-1}(s;\cdot)A(Y(s;\cdot))\,{}^t(\partial_y Y)^{-1}(s;\cdot)$$

Functional framework and hypotheses

$$^tP = P, \;\; P(y) > 0 \;\; y \in \mathbb{R}^m, \;\; P^{-1}, P \in L^2_{\mathrm{loc}}(\mathbb{R}^m), \;\; [b, P] = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m).$$

Hilbert space

$$H_Q = \{A = A(y) \;:\; \int_{\mathbb{R}^m} Q(y)A(y) : A(y)Q(y)\,\mathrm{d}y < +\infty\}$$

where $Q = P^{-1}$.

Scalar product

$$(A, B)_Q = \int_{\mathbb{R}^m} QA : BQ\,\mathrm{d}y, \;\; A, B \in H_Q.$$

Infinitesimal generator

$$L : \mathrm{dom}(L) \subset H_Q \to H_Q, \;\; \mathrm{dom} L = \{A \in H_Q \; : \; \exists \lim_{s \to 0} \frac{G(s)A - A}{s}\}$$

$$L(A) = \lim_{s \to 0} \frac{G(s)A - A}{s}, \;\; A \in \mathrm{dom}(L)$$

$$C_c^1(\mathbb{R}^m) \subset \mathrm{dom}(L), \;\; L(A) = b \cdot \nabla_y A - \partial_y b A - A \,{}^t\partial_y b, \;\; A \in C_c^1(\mathbb{R}^m)$$

Key points

L becomes skew-adjoint on the weighted L^2 space H_Q

Its kernel coincides with $\{A \in H_Q \subset L^1_{\mathrm{loc}}(\mathbb{R}^m) : [b, A] = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m)\}$

The averaged matrix field denoted $\langle D \rangle_Q$, associated to any $D \in H_Q$ appears as the long time limit of the solution of

$$\partial_t A - L(L(A)) = 0, \;\; t \in \mathbb{R}_+.$$

Theorem For all $D \in H_Q \cap L^\infty(\mathbb{R}^m)$ the solution of

$$\partial_t A - L(L(A)) = 0, \quad t \in \mathbb{R}_+, \quad A(0) = D$$

converges towards the field of averaged diffusion matrix

$$\lim_{t \to +\infty} A(t) = \langle D \rangle_Q$$

$$\nabla_y u \cdot \langle D \rangle_Q \nabla_y v = \langle \nabla_y u \cdot D \nabla_y v \rangle, \quad u, v \in H^1(\mathbb{R}^m) \cap \ker b \cdot \nabla_y.$$

Theorem(First order approximation)

Let $D \in H_Q$ be a field of symmetric positive matrices and $(u^\varepsilon_{\mathrm{in}})_\varepsilon \subset L^2(\mathbb{R}^m)$ a family of initial conditions such that $(\langle u^\varepsilon_{\mathrm{in}} \rangle)_\varepsilon$ converges weakly in $L^2(\mathbb{R}^m)$, as $\varepsilon \searrow 0$, towards some function u^{in}. Let u^ε, u satisfying

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \quad u^\varepsilon(0,\cdot) = u^\varepsilon_{\mathrm{in}}$$

$$\partial_t u - \mathrm{div}_y(\langle D \rangle_Q \nabla_y u) = 0, \quad u(0,\cdot) = u^{\mathrm{in}}.$$

Then we have the convergences

$$\lim_{\varepsilon \searrow 0} u^\varepsilon = u \ \text{ weakly} \star \ \text{ in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

$$\lim_{\varepsilon \searrow 0} \nabla_y u^\varepsilon = \nabla_y u \ \text{ weakly in } L^2(\mathbb{R}_+; L^2(\mathbb{R}^m)).$$

Averaged Boltzmann kernel

Paraxial approximation

Beams with optical axis (the particles remain close to the optical axis, having about the same kinetic energy)

Strongly magnetized beams

Collisional mechanism (Boltzmann kernel)

$$\partial_t \mathcal{F} + \mathcal{V} \cdot \nabla_{\mathcal{X}} \mathcal{F} + \frac{q}{m} (\mathcal{V} \wedge \mathcal{B}^\varepsilon) \cdot \nabla_{\mathcal{V}} \mathcal{F} = Q(\mathcal{F}, \mathcal{F}), \ \ (t, \mathcal{X}, \mathcal{V}) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$$

$$\mathcal{F}^\varepsilon(t, \mathcal{X}, \mathcal{V}) = \frac{1}{\varepsilon^3} f^\varepsilon \left(t, \frac{\mathcal{X}_1}{\varepsilon^2}, \frac{\mathcal{X}_2}{\varepsilon^2}, \mathcal{X}_3, \frac{\mathcal{V}_1}{\varepsilon}, \frac{\mathcal{V}_2}{\varepsilon}, \frac{\mathcal{V}_3 - u_3}{\varepsilon} \right)$$

The Larmor radius scales like ε^2, since $\omega_c \sim \varepsilon^{-1}$ and $v_\perp \sim \varepsilon$.

Therefore we take $x_\perp \sim \varepsilon^2$.

Boltzmann operator

$$Q = \int_{S^2}\int_{\mathbb{R}^3} \sigma(\mathcal{V}-\mathcal{V}',\omega)\{\mathcal{F}(\mathcal{V}-\omega\otimes\omega(\mathcal{V}-\mathcal{V}'))\mathcal{F}(\mathcal{V}'+\omega\otimes\omega(\mathcal{V}-\mathcal{V}'))$$
$$-\mathcal{F}(\mathcal{V})\mathcal{F}(\mathcal{V}')\}\,\mathrm{d}\mathcal{V}'\mathrm{d}\omega$$

with $\sigma(z,\omega) = |z|^\gamma b(z/|z|\cdot\omega)$.

$$Q^\varepsilon = \frac{\varepsilon^\gamma}{\varepsilon^3}\int_{S^2}\int_{\mathbb{R}^3}\sigma\{f^\varepsilon(v-\omega\otimes\omega(v-v'))f^\varepsilon(v'+\omega\otimes\omega(v-v'))$$
$$-f^\varepsilon(v)f^\varepsilon(v')\}\,\mathrm{d}v'\mathrm{d}\omega$$
$$=\varepsilon^{\gamma-3}Q(f^\varepsilon,f^\varepsilon)$$

Maxwell molecules $\gamma = 0$

$$\partial_t f^\varepsilon + (u_3 + \varepsilon v_3)\partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon + \frac{(\overline{v} \cdot \nabla_{\overline{x}} + \omega_c \, {}^\perp \overline{v} \cdot \nabla_{\overline{v}}) f^\varepsilon}{\varepsilon} = Q(f^\varepsilon, f^\varepsilon)$$

The limit model comes by averaging with respect to the fast cyclotronic motion

Transport averaging $\langle \partial_t f^\varepsilon + (u_3 + \varepsilon v_3)\partial_{x_3} f^\varepsilon - v_3 \partial_{x_3} u_3 \partial_{v_3} f^\varepsilon \rangle$

Collision averaging

$$\langle Q \rangle (f^\varepsilon, f^\varepsilon) := \langle Q(f^\varepsilon, f^\varepsilon) \rangle$$

$$Q(f,f) = Q_+(f,f) - Q_-(f,f)$$

$$Q_+(f,f) = \int_{S^2}\int_{\mathbb{R}^3} \sigma(v-v',\omega) f(V) f(V') \,\mathrm{d}v'\mathrm{d}\omega$$

$$Q_-(f,f) = \int_{S^2}\int_{\mathbb{R}^3} \sigma(v-v',\omega) f(v) f(v') \,\mathrm{d}v'\mathrm{d}\omega.$$

Probability density $\chi(r, r', \cdot)$

$$\chi(r,r',\overline{z}) = \frac{\mathbf{1}_{\{|r-r'|<|\overline{z}|<r+r'\}}}{\pi^2\sqrt{|\overline{z}|^2-(r-r')^2}\sqrt{(r+r')^2-|\overline{z}|^2}},\quad r,r'\in\mathbb{R}_+,\quad \overline{z}\in\mathbb{R}^2.$$

Proposition (Average of the loss part)

Let $y = (\omega_c \overline{x} + {}^\perp\overline{v}, v_3)$, $r = |\overline{v}|$.

$$\left\langle \int_{S^2}\int_{\mathbb{R}^3} \sigma(v-v',\omega) f(\overline{x},v) f'(\overline{x},v') \,\mathrm{d}v'\mathrm{d}\omega \right\rangle (\overline{x},v)$$

$$= 2\pi \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} \sigma(y-y',e) g(y,r) g'(y',r')\, \chi(r,r',\overline{y}-\overline{y'})\, r'\mathrm{d}r'\mathrm{d}y'\mathrm{d}e.$$

The averaged loss part has similar strucure : integral operator w.r.t. the pre-collisional quantities (y', r') and a collision parameter $e \in S^2$. The averaged gain part will express in terms of post-collisional quantities

$$Y = y - (y - y', e)e, \quad Y' = y' + (y - y', e)e, \quad R = ?, R' = ?$$

Collisions between Larmor circles

1. For any pair of Larmor circles intersecting at I fix a direction $d \in S^2$ and take a pre-collision pair $(\overline{x}, x_3, v), (\overline{x'}, x_3, v')$ (with same x_3).
2. Consider the characteristics $(\overline{X}(s), V(s))$ and $(\overline{X'}(s'), V'(s'))$ starting from $(\overline{x}, v), (\overline{x'}, v')$, meeting at I after s, s'.
3. Perform a Boltzmann collision at I.
4. Move backwards during s, s' on the new Larmor circles associated to I and the post-collisional Boltzmann velocities.

Post-collisional centers and radii

$$R = \left| r\mathcal{R}(-\psi)\frac{\overline{y'}-\overline{y}}{|\overline{y'}-\overline{y}|} - (y-y', e)\overline{e} \right|$$

$$R' = \left| r'\mathcal{R}(-(\psi-\varphi))\frac{\overline{y'}-\overline{y}}{|\overline{y'}-\overline{y}|} + (y-y', e)\overline{e} \right|$$

$$|\overline{y}-\overline{y'}|^2 = r^2+(r')^2-2rr'\cos\varphi, \quad (r')^2 = r^2+|\overline{y}-\overline{y'}|^2+2r|\overline{y}-\overline{y'}|\cos\psi.$$

Conservations

$$Y + Y' = y + y' \quad \text{(Larmor center and parallel velocity conservation)}$$

$$\frac{R^2+(Y_3)^2}{2}+\frac{(R')^2+(Y_3')^2}{2} = \frac{r^2+(y_3)^2}{2}+\frac{(r')^2+(y_3')^2}{2} \quad \text{(kinetic energy)}$$

$$\frac{|\overline{Y}|^2-R^2}{2}+\frac{|\overline{Y'}|^2-(R')^2}{2} = \frac{|\overline{y}|^2-r^2}{2}+\frac{|\overline{y'}|^2-(r')^2}{2} \quad \text{(circle power).}$$

Proposition (Average of the gain part)

$$\left\langle \int_{S^2}\int_{\mathbb{R}^3} \sigma(v - v', \omega) f(\overline{x}, V(v, v', \omega)) f'(\overline{x}, V'(v, v', \omega))\, \mathrm{d}v'\mathrm{d}\omega \right\rangle (\overline{x}, v)$$
$$= 2\pi \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} \sigma(y - y', e) g(Y, R) g'(Y', R')\, \chi(r, r', \overline{y} - \overline{y'})\, r'\mathrm{d}r'\mathrm{d}y'\mathrm{d}e.$$

H theorem for the averaged Boltzmann kernel

1. Weak formulation

$$\omega_c^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} m(\overline{x}, v)\, \langle Q \rangle (f, f)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 = -\pi^2 \int_{S^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+} \sigma$$

$$\times \{n(Y, R) + n(Y', R') - n(y, r) - n(y', r')\}$$

$$\times \{g(Y, R)g(Y', R') - g(y, r)g(y', r')\}\, \chi(r, r', \overline{y} - \overline{y'})\, r\mathrm{d}r\mathrm{d}y r'\mathrm{d}r'\mathrm{d}y'\mathrm{d}e$$

2. We have the inequality

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \ln f(\overline{x}, v)\, \langle Q \rangle (f, f)\, \mathrm{d}v\mathrm{d}x_1\mathrm{d}x_2 \leq 0$$

with equality iff

$$\ln g(Y, R) + \ln g(Y', R') = \ln g(y, r) + \ln g(y', r'), |r - r'| < |\overline{y} - \overline{y'}| < r + r'$$

Conclusions

- exact computations of the averaged collison kernels

- mass, momentum, kinetic energy balances

- H theorem

- complete description of the gyrokinetic equilibria and collision invariants

- Euler equations

Perspectives

- numerical simulation by well adaptated schemes

- macro-micro decomposition (implicite scheme for the zero average part, explicite scheme for the average part)

- general magnetic shape

- coupling with Maxwell equations