Time implicit-explicit high-order discontinuous Galerkin method with reduced evaluation costs

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Motivation and goal

- DGM:
  - compact stencil: well-suited for unstructured meshes, algorithm parallelization, BC application.
  - time explicit discretization: strongest CFL restriction associated to parabolic term
  - time implicit discretization: high computational cost induced by the large number of DOFs in practical app. (e.g. 3D Navier-Stokes eq.)
- Aim:
  - efficient implicit-explicit procedure for the DGM:
    - ⇝ partial uncoupling of DOFs in neighbouring elements
  - application to solution of nonlinear 2nd order PDE
  - numerical experiments with BR2 scheme

[Cockburn & Shu '01, Kroll *et al.* '10, Bassi *et al.* '97]

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Outline

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Outline

Advection-reaction equation

Consider the linear scalar advection-reaction equation for $u(\mathbf{x}) \in \mathbb{R}$:

$$\nabla \cdot (\mathbf{c}u) + \mu u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d \qquad (1a)$$

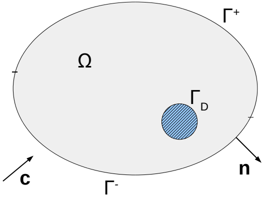
$$u = u_D \quad \text{on} \quad \Gamma_D^- = \Gamma_D \cup \Gamma_- \qquad (1b)$$

with $\mu \in L^2(\Omega)$ s.t. $\mu(\mathbf{x}) + \nabla \cdot \mathbf{c}(\mathbf{x}) \geq \mu_0 > 0$, $\mathbf{c} \in L^\infty(\Omega)$, $u_D \in L^2(\Gamma_D^-)$ and inflow boundary:

$$\Gamma_- = \{\mathbf{x} \in \partial\Omega : \ \mathbf{c}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\}$$

The exact solution u is sought in the function space

$$\mathcal{V} = \{v \in L^2(\Omega) : \ \nabla \cdot (\mathbf{c}v) + \mu v \in L^2(\Omega)\}$$



Space discretization

- shape-regular subdivision of Ω into N cells (simplices): $\Omega_h = \{\kappa\}$ s.t. $\overline{\Omega} = \bigcup_{\kappa\in\Omega_h} \overline{\kappa}$
- sets of internal and boundary faces:

  $\mathcal{E}_i = \{e \in \mathcal{E}_h :\ e\cap\partial\Omega_h = \emptyset\}, \quad \mathcal{E}_b = \{e \in \mathcal{E}_h :\ e \in \partial\Omega_h\}, \quad \mathcal{E}_D^- = \mathcal{E}_D \cup \mathcal{E}_-$
- function space of discontinuous polynomials:

$$\mathcal{V}_h^p = \{v \in L^2(\Omega_h) :\ v|_\kappa \in \mathcal{P}_p(\kappa),\ \forall\kappa \in \Omega_h\},$$

- let $(\phi^1, \ldots, \phi^{N_p})$ be a basis of $\mathcal{P}_p(\kappa)$;
- look for an approximate solution $u_h \in \mathcal{V}_h^p$ of (1):

$$u_h(\mathbf{x}) = \sum_{l=1}^{N_p} \phi^l(\mathbf{x}) V_\kappa^l \quad \forall \mathbf{x} \in \kappa$$

Remark: function space dimension: $\dim \mathcal{V}_h^p = N \times N_p = N \times \prod_{i=1}^{d} \frac{p+i}{i}$

Remark: non-conforming FE method: $\mathcal{V}_h^p \not\subset \mathcal{V}$

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Space discretization (cont.)

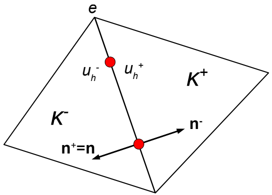
- multiply (1a) by a test function $v_h \in \mathcal{V}_h^p$ and integrate over a cell κ, then integrate by parts:

$$-\int_\kappa u_h \mathbf{c} \cdot \nabla v_h \, d\mathbf{x} + \oint_{\partial\kappa} v_h u_h \mathbf{c} \cdot \mathbf{n} \, ds + \int_\kappa \mu u_h v_h \, d\mathbf{x} = 0$$

- sum over all cells and apply BCs:

$$-\int_{\Omega_h} u_h(\mathbf{c} \cdot \nabla_h v_h - \mu v_h) \, d\mathbf{x} + \sum_{\kappa \in \Omega_h} \oint_{\partial\kappa \backslash \mathcal{E}_D^-} v_h u_h \mathbf{c} \cdot \mathbf{n} ds = -\int_{\mathcal{E}_D^-} v_h u_D \mathbf{c} \cdot \mathbf{n} ds$$

- replace the physical flux $(\mathbf{c} \cdot \mathbf{n})u_h$ by a numerical flux $h(u_h^+, u_h^-, \mathbf{n})$ on $\mathcal{E}_i$;



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Space discretization (cont.)

- introducing $[\![v_h]\!] = v_h^+ - v_h^-$, we get

$$\sum_{\kappa\in\Omega_h} \oint_{\partial\kappa\setminus\mathcal{E}_b} v_h^+ h(u_h^+, u_h^-, \mathbf{n})\, ds = \int_{\mathcal{E}_i} [\![v_h]\!] h(u_h^+, u_h^-, \mathbf{n})\, ds$$

- one obtains the variational form: "find $u_h \in \mathcal{V}_h^p$ s.t.

$$\mathcal{B}_h(u_h, v_h) = \ell_h(v_h), \quad \forall v_h \in \mathcal{V}_h^p\text{''} \tag{2}$$

with bilinear and linear forms:

$$\begin{aligned}
\mathcal{B}_h(u_h, v_h) &= -\int_{\Omega_h} u_h(\mathbf{c}\cdot\nabla_h v_h - \mu v_h)\, d\mathbf{x} + \int_{\mathcal{E}_i} [\![v_h]\!] h(u_h^+, u_h^-, \mathbf{n})\, ds \\
&\quad + \int_{\mathcal{E}_b\setminus\mathcal{E}_D^-} v_h^+ u_h^+ (\mathbf{c}\cdot\mathbf{n})\, ds \\
\ell_h(v_h) &= -\int_{\mathcal{E}_D^-} v_h^+ u_D (\mathbf{c}\cdot\mathbf{n})\, ds
\end{aligned}$$

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Consistency

Problem (2) is said to be consistent if the exact solution $u \in \mathcal{V}$ satisfies

$$\mathcal{B}_h(u, v_h) = \ell_h(v_h), \quad \forall v_h \in \mathcal{V}_h^p$$

Lemma

Problem (2) is consistent iff. the numerical flux is consistent: $h(u, u, \mathbf{n}) = u(\mathbf{c} \cdot \mathbf{n})$.

Proof: integrate by parts the volume integral in $\mathcal{B}_h(u, v_h) - \ell_h(v_h)$ and replace u_h by u, one obtains

$$\begin{aligned} \int_{\Omega_h} v_h \cancel{(\nabla \cdot (\mathbf{c}u) + \mu u)}\, d\mathbf{x} \quad &+ \quad \int_{\mathcal{E}_i} v_h^+ (h(u, u, \mathbf{n}^+) - u(\mathbf{c} \cdot \mathbf{n}^+))\, ds \\ &+ \quad \int_{\mathcal{E}_i} v_h^- (h(u, u, \mathbf{n}^-) - u(\mathbf{c} \cdot \mathbf{n}^-))\, ds \\ &- \quad \cancel{\int_{\mathcal{E}_D^-} u(\mathbf{c} \cdot \mathbf{n}^+) v_h^+\, ds} + \cancel{\int_{\mathcal{E}_D^-} u_D(\mathbf{c} \cdot \mathbf{n}^+) v_h^+\, ds} = 0 \end{aligned}$$

Hence on internal faces we get $h(u, u, \mathbf{n}) = u(\mathbf{c} \cdot \mathbf{n})$. $\square$

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Conservation

Problem (2) is said to be conservative if the approximate solution u_h satisfies

$$\int_{\mathcal{E}_b \backslash \mathcal{E}_D^-} u_h(\mathbf{c}\cdot\mathbf{n}^+)\,ds + \int_{\mathcal{E}_D^-} u_D(\mathbf{c}\cdot\mathbf{n}^+)\,ds + \int_{\Omega_h} \mu u_h d\mathbf{x} = 0 \qquad (3)$$

Lemma

Problem (2) is conservative iff. the numerical flux is conservative: $h(u_h^+, u_h^-, \mathbf{n}^+) = -h(u_h^-, u_h^+, \mathbf{n}^-)$.

Proof: set $v_h \equiv 1$ in (2), one obtains

$$\begin{aligned} -\int_{\Omega_h} u_h \mathbf{c}\cdot \nabla_h 1 - \mu u_h d\mathbf{x} \quad &+ \int_{\mathcal{E}_i} 1\times h(u_h^+, u_h^-, \mathbf{n}^+) + 1\times h(u_h^-, u_h^+, \mathbf{n}^-)\,ds \\ &+ \int_{\mathcal{E}_b\backslash\mathcal{E}_D^-} 1\times u_h^+(\mathbf{c}\cdot\mathbf{n})\,ds \\ &+ \int_{\mathcal{E}_D^-} 1\times u_D(\mathbf{c}\cdot\mathbf{n})\,ds = 0 \end{aligned}$$

then, substract (3) which proves the lemma. □

FE analysis for the upwind flux

- DG-norm:

$$|||v_h|||^2 \quad = \quad \mu_0\|v_h\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\mathcal{E}_b} |\mathbf{c}\cdot\mathbf{n}| v_h^2\, ds + \frac{1}{2}\int_{\mathcal{E}_i} |\mathbf{c}\cdot\mathbf{n}| [\![v_h]\!]^2\, ds, \quad \forall v_h \in \mathcal{V}_h^p$$

- Galerkin orthogonality:

$$\mathcal{B}_h(u - u_h, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p$$

- coercivity:

$$\mathcal{B}_h(v_h, v_h) \geq |||v_h|||^2, \quad \forall v_h \in \mathcal{V}_h^p$$

- continuity:

$$\mathcal{B}_h(z, v_h) \leq |||z|||_\star \times |||v_h|||, \quad \forall v_h \in \mathcal{V}_h^p,\ z \in \mathcal{V} \oplus \mathcal{V}_h^p$$

- stability:

$$|||v_h||| \leq \sup_{w_h \in \mathcal{V}_h^p \setminus \{0\}} \frac{\mathcal{B}_h(v_h, w_h)}{|||w_h|||}, \quad \forall v_h \in \mathcal{V}_h^p$$

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Convergence analysis for the upwind flux

Theorem (a priori error estimate)

Let u_h be solution to problem (2) and $u \in H^{p+1}(\Omega_h)$, then $\exists\ C > 0$ s.t.

$$|||u - u_h||| \leq Ch^{p+\frac{1}{2}} |u|_{H^{p+1}(\Omega_h)}$$

Theorem (exponential convergence)

Let u be elementwise analytic on Ω_h, then $\exists\ C = C(u) > 0$, $d_\kappa > 0$ s.t.

$$|u|_{H^s(\kappa)} \leq C(u) d_\kappa^s s! \sqrt{|meas(\kappa)|}, \quad \forall s \geq 0,$$

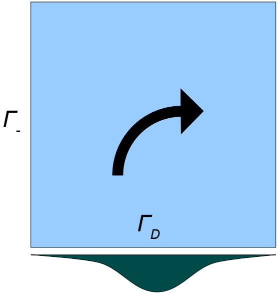
and suppose that u_h is solution to problem (2), then $\exists\ C' = C'(u, \mathbf{c}, \mu_0) > 0$ and $\chi_\kappa > 0$ s.t.

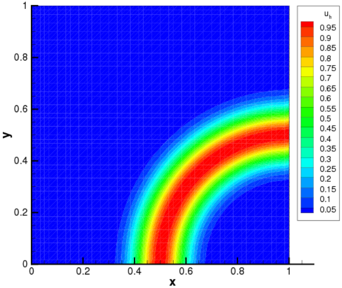
$$|||u - u_h|||^2 \leq C' \sum_{\kappa \in \Omega_h} e^{-2p\left(\chi_\kappa + \frac{|\ln h_\kappa|}{\sqrt{1+d_\kappa^2}}\right)} |meas(\kappa)|$$

[Houston *et al.* '02]

Numerical experiments

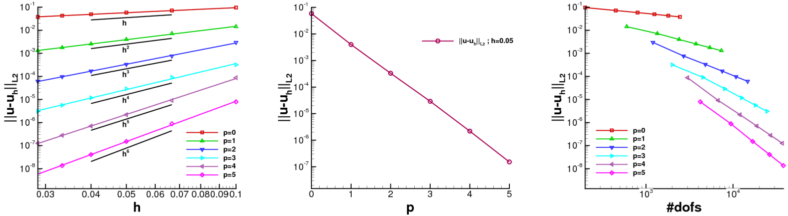
rotational advection $\mathbf{c} = (x, 1-y)^\top$ and $\mu \equiv 0$ with smooth inlet BC





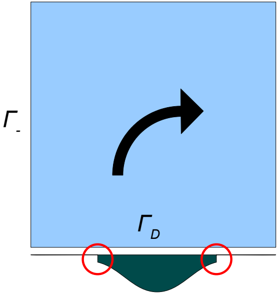
Numerical experiments (cont.)

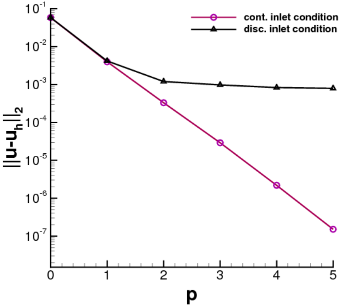
rotational advection $\mathbf{c} = (x, 1-y)^\top$ and $\mu \equiv 0$ with smooth inlet BC (cont.)



Numerical experiments (cont.)

rotational advection $\mathbf{c} = (x, 1-y)^\top$ and $\mu \equiv 0$ with discontinuous inlet BC





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Outline

Nonlinear advection-diffusion equation

model problem

Consider the scalar advection-diffusion equation for $u \in \mathbb{R}$:

$$\nabla \cdot (\mathbf{c}(\mathbf{x})u) - \nabla \cdot (\mathbf{B}(\mathbf{x}, u)\nabla u) = s(\mathbf{x}) \quad \text{in } \Omega \subset \mathbb{R}^d \tag{4}$$

with $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^d$ and $\mathbf{B}(\mathbf{x}, u) \in \mathbb{R}^{d\times d}$ a nonlinear function of u and BCs on $\partial\Omega = \Gamma_D \cup \Gamma_N$:

$$\begin{aligned} u &= u_D \quad \text{on } \Gamma_D \\ \nabla u \cdot \mathbf{n} &= g_N \quad \text{on } \Gamma_N \end{aligned}$$

We solve (4) with a fast time marching method:

$$u_t + \nabla \cdot (\mathbf{c}(\mathbf{x})u) - \nabla \cdot (\mathbf{B}(\mathbf{x}, u)\nabla u) = s(\mathbf{x}) \quad \text{in } \Omega \times (0, \infty) \tag{5}$$

and IC

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{in } \Omega$$

entropy solution

- $\mathcal{U}(u)$ is an entropy function for (5) if
  - $\mathcal{U}_{uu} > 0$
  - $\mathcal{U}$ satisfies

$$\mathbf{B}(\mathbf{x}, u) = \mathcal{U}_{uu}\mathbf{C}(\mathbf{x})$$

  where $\mathbf{C} \in \mathbb{R}^{d\times d}$ is a positive definite matrix

- Introduce the change of variable $u(v)$ s.t.

$$\mathcal{U}_{uu}u_v = 1$$

one obtains the following problem for v:

$$u(v)_t + \nabla\cdot(\mathbf{c}(\mathbf{x})u(v)) - \nabla\cdot(\mathbf{C}(\mathbf{x})\nabla v) = s(\mathbf{x}) \quad \text{in } \Omega\times(0,\infty)$$

[Degond *et al.* '97]

Space discretization

variational form

- Partition of Ω into N simplicies:

$$\Omega_h = \{\kappa\} \quad \text{s.t.} \quad \overline{\Omega} = \bigcup_{\kappa \in \Omega_h} \overline{\kappa}$$

- Function space of discontinuous polynomials:

$$\mathcal{V}_h^p = \{\varphi \in L^2(\Omega_h) : \varphi|_\kappa \in \mathcal{P}_p(\kappa), \ \forall \kappa \in \Omega_h\},$$

with for $d = 2$:

$$\mathcal{P}^p(\kappa) = \{\varphi \in L^2(\kappa) : \varphi(x,y) = \sum_{0 \leq k+l \leq p} \alpha^l x^k y^l\}$$

- We look for a numerical approximation of the solution $v_h \in \mathcal{V}_h^p$:

$$v_h(\mathbf{x}, t) = \sum_{l=1}^{N_p} \phi^l(\mathbf{x}) V_\kappa^l(t) \quad \forall \mathbf{x} \in \kappa, \ t \in (0, \infty)$$

Remark: number of DOFs per discretization element: $N_p = N \times \prod_{i=1}^{d} \frac{p+i}{i}$

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Space discretization

upwind discretization flux

Numerical approximation of the weak formulation: find $v_h \in \mathcal{V}_h^p$ s.t. $\forall \phi \in \mathcal{V}_h^p$

$$\int_{\Omega_h} \phi u(v_h)_t d\mathbf{x} - \int_{\Omega_h} u(v_h)\mathbf{c} \cdot \nabla_h \phi d\mathbf{x} + \sum_{e \in \mathcal{E}_i} \int_e [\![\phi]\!] h_c \, ds$$

$$+ \int_{\Omega_h} (\mathbf{C}\theta_h) \cdot \nabla_h \phi d\mathbf{x} - \sum_{e \in \mathcal{E}_i} \int_e [\![\phi]\!] h_v \, ds - \int_{\Omega_h} \phi s(\mathbf{x}) d\mathbf{x} = 0$$

Upwind discretization flux

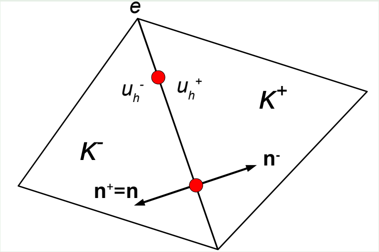
$$h_c = \{u(v_h)\mathbf{c}\} \cdot \mathbf{n} + \frac{\alpha}{2}[\![v_h]\!]$$

$$\alpha = \max\{|u_v(v)\mathbf{c} \cdot \mathbf{n}| : v = v_h^{\pm}\}$$

Jump and average operators

$$[\![v_h]\!] = v_h^+ - v_h^-$$

$$\{v_h\} = \frac{1}{2}(v_h^+ + v_h^-)$$



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Space discretization

BR2 scheme

Numerical approximation of the weak formulation: find $v_h \in \mathcal{V}_h^p$ s.t. $\forall \phi \in \mathcal{V}_h^p$

$$\int_{\Omega_h} \phi u(v_h)_t dx - \int_{\Omega_h} u(v_h)\mathbf{c} \cdot \nabla_h \phi d\mathbf{x} + \sum_{e \in \mathcal{E}_i} \int_e [\![\phi]\!] h_c \, ds$$

$$+ \int_{\Omega_h} (\mathbf{C}\boldsymbol{\theta}_h) \cdot \nabla_h \phi d\mathbf{x} - \sum_{e \in \mathcal{E}_i} \int_e [\![\phi]\!] h_v \, ds - \int_{\Omega_h} \phi s(\mathbf{x}) d\mathbf{x} = 0$$

lifting operators

$$\boldsymbol{\theta}_h = \nabla_h v_h + \mathbf{R_h}$$

$$h_v = \mathbf{C}\{\nabla_h v_h + \mathbf{r_h^e}\} \cdot \mathbf{n}$$

with

$$\mathbf{R_h} \triangleq \sum_{e \in \partial\kappa} \mathbf{r_h^e}$$

$$\int_{\kappa^+ \cup \kappa^-} \phi \mathbf{r_h^e} \, d\mathbf{x} = -\int_e \{\phi\}[\![v_h]\!]\mathbf{n} \, ds$$

[Bassi *et al.* '97]

Implicit-explicit time discretization
backward Euler scheme

- Convective terms: explicit time discretization
- Diffusive terms: backward Euler scheme

$$\mathbf{A}(\mathbf{V}^{(n+1)} - \mathbf{V}^{(n)}) + \mathbf{R}(\mathbf{V}^{(n)}) = 0$$

with $\mathbf{V}^{(n)} = \mathbf{V}(n\Delta t)$ and

- residual vector:

$$\mathbf{R}(\mathbf{V}^{(n)}) = \mathbf{L_c}(\mathbf{V}^{(n)}) + \mathbf{L_v}\mathbf{V}^{(n)} + \mathbf{L_s}$$

- implicit matrix:

$$\mathbf{A} = \frac{1}{\Delta t}\mathbf{M}^{(n)} + \mathbf{L_v}$$

- mass matrix $\mathbf{M}^{(n)} = \text{diag}(\mathbf{M_1}^{(n)}, \dots, \mathbf{M_N}^{(n)})$:

$$(\mathbf{M_j}^{(n)})_{kl} = \int_{\kappa_j} \phi^k \phi^l u_v(v_h^{(n)}) dx, \quad 1 \leq j \leq N, \quad 1 \leq k, l \leq N_p$$

Algorithm simplification
evaluation cost reduction (1/2)

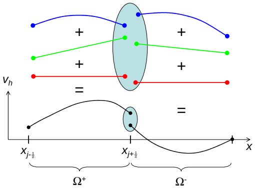
Basics:

- implicit matrix coefficients represent coupling between DOFs
- coupling appears through numerical fluxes

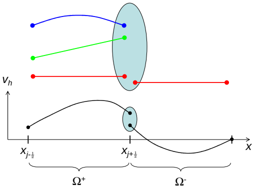
Lowering coupling between modes by:

1. low order problem resolution: modes s.t. $0 \leq q \leq p_s$
2. reconstruction of higher modes s.t. $p_s < q \leq p$

1D example for $p = 2$ and $p_s = 0$:



(a) full coupling



(b) partial coupling

Algorithm simplification
evaluation cost reduction (2/2)

Resolution algorithm:

1. low order problem resolution:

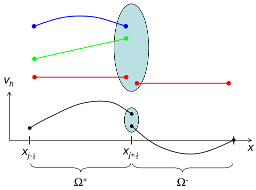
$$\tilde{\mathbf{A}}(\tilde{\mathbf{V}}^{(n+1)} - \tilde{\mathbf{V}}^{(n)}) = -\tilde{\mathbf{R}}(\mathbf{V}^{(n)})$$

   with $\tilde{\mathbf{V}} = (\mathbf{V}^0, \dots, \mathbf{V}^{p_s})$

2. local reconstruction of higher modes:

$$\mathbf{V}^{q^{(n+1)}} = \mathbf{V}^q(\tilde{\mathbf{V}}^{(n+1)}, \mathbf{V}^{q^{(n)}}) \quad , \quad p_s < q \leq p$$

1D example for $p = 2$ and $p_s = 0$:



Algorithms comparison
operation count

- FULL method: $\mathbf{A}$ is a square matrix of size $N \times N_p$:
  - matrix inversion is a $\mathcal{O}(N^2 N_p^2)$ process
- SIMP method: $\tilde{\mathbf{A}}$ is a square matrix of size $N \times N_{p_s}$:
  - matrix inversion is a $\mathcal{O}(N^2 N_{p_s}^2)$ process
  - higher DOFs reconstruction is a $\mathcal{O}(2NN_p(N_p - N_{p_s}))$ process

FLOPs reduction for matrix inversion:

$$\frac{N^2 N_p^2}{NN_{p_s}^2 + 2NN_p(N_p - N_{p_s})} \sim \left(\frac{N_p}{N_{p_s}}\right)^2 \qquad \text{when } N \text{ large}$$

	p	1	2	2	3	3	3	4	4	4	4
	p_s	0	0	1	0	1	2	0	1	2	3
$\left(\frac{N_p}{N_{p_s}}\right)^2$	1D	4	9	2.25	16	4	1.78	25	6.25	2.78	1.56
	2D	9	36	4	100	11.1	2.78	225	25	6.25	2.25

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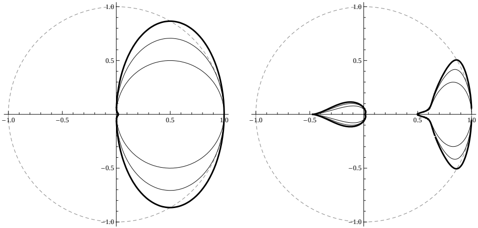
Algorithms comparison

Von Neumann analysis for the 1D linear advection-diffusion equation with periodic BCs

Amplification matrix eigenspectra ($p = 1$; $\sigma = \frac{c\Delta t}{h} = 10$, 20 and 30; $Re_h = \frac{ch}{\nu} = 0.1$):



(a) FULL (b) SIMP0

FULL: full implicit matrix
SIMP0: simplified implicit matrix with $p_s = 0$

Necessary condition of stability for both methods:

$$\sigma Re_h \leq 2 \quad \text{and} \quad Re_h \leq 6$$

Renac, Marmignon & Coquel, Time implicit HO DG method with reduced evaluation cost, submitted to SIAM J. Sci. Comput., Dec. 2010 (accepted).

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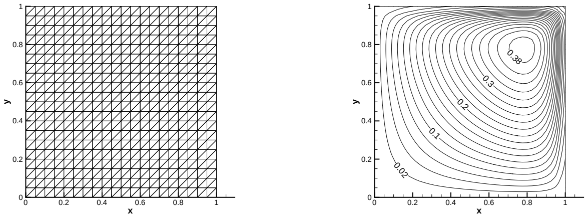
Outline

2D steady convection-diffusion equation

model problem

$$\begin{aligned} \nabla\cdot(\mathbf{c}u) - \nabla\cdot(\mathbf{B}(u)\nabla u) &= s(\mathbf{x}) \quad \text{in } \Omega=(0,1)^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

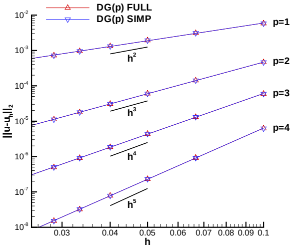
with $\mathbf{c}=(1,1)^\top$, $\mathbf{B}(u)=\nu(e^u-1)\mathbf{I}$ with $\nu>0$



2D steady convection-diffusion equation

error analysis

$1 \leq p \leq 4;\ \sigma = 10;\ Re_h = 0.1$



FULL: full implicit matrix

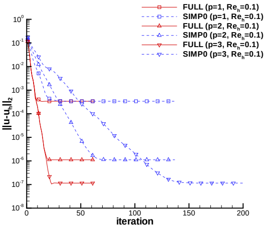
SIMP: simplified implicit matrix

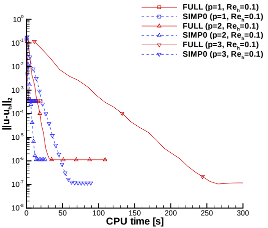
[Arnold *et al.* '02.]

2D steady convection-diffusion equation

convergence analysis (1/2)

$1 \leq p \leq 3$; $\sigma = 10$; $Re_h = 0.1$





FULL: full implicit matrix

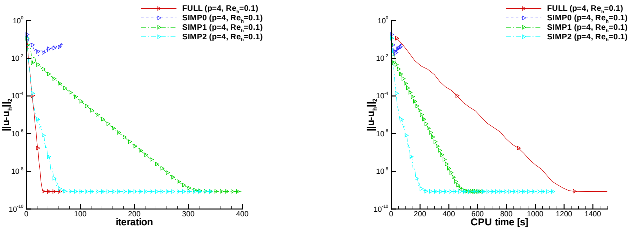
SIMP0: simplified implicit matrix with $p_s = 0$

$$speedup = \frac{\text{CPU time(FULL)}}{\text{CPU time(SIMP)}}$$

p	1	2	3
speedup	1.1	2.6	4.0

2D steady convection-diffusion equation
convergence analysis (2/2)

$p = 4;\ \sigma = 10;\ Re_h = 0.1$



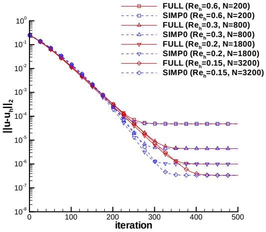
FULL: full implicit matrix
SIMP0: simplified implicit matrix with $p_s = 0$
SIMP1: simplified implicit matrix with $p_s = 1$
SIMP2: simplified implicit matrix with $p_s = 2$

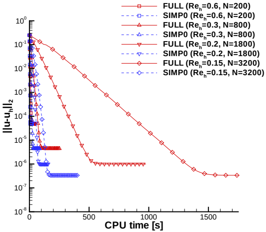
p / p_s	4 / 1	4 / 2
speedup	2.3	4.8

2D steady convection-diffusion equation

mesh size effects

$p = 3;\ \Delta t = 6 \times 10^{-3}$





FULL: full implicit matrix

SIMP0: simplified implicit matrix with $p_s = 0$

p	3	3	3	3
N	200	800	1800	3200
speedup	2.0	3.0	6.9	9.4

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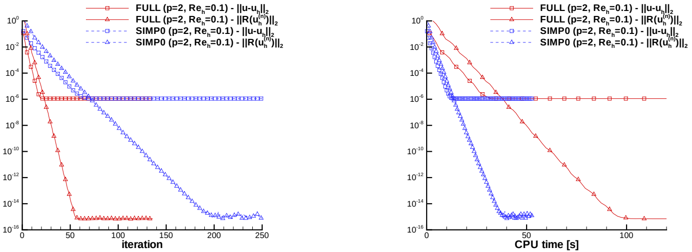
2D steady convection-diffusion equation

speedup of the residuals



FULL: full implicit matrix
SIMP0: simplified implicit matrix with $p_s = 0$

p	1	2	3	4
p_s	0	0	0	2
speedup	1.1	2.6	4.0	4.8
speedup(*res.*)	1.2	2.5	4.5	5.2

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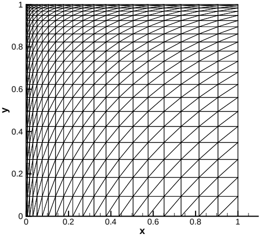
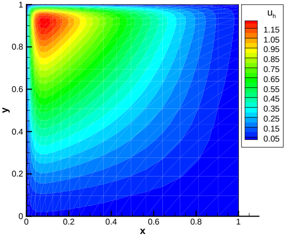
2D steady convection-diffusion equation
effects of mesh aspect ratio (AR)



(a) $AR = 16,\ N = 800$



(b) $p = 4$

AR	8	16	32	64	128	8	16	32	64	128
p	2	2	2	2	2	4	4	4	4	4
p_s	1	1	0	0	0	3	3	3	3	2
speedup	1.6	1.6	2.2	2.1	2.0	2.8	2.9	2.9	3.2	4.1

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2D unsteady convection-diffusion equation
model problem

$$
\begin{aligned}
u_t + \nabla \cdot (\mathbf{c}u) - \nabla \cdot (\mathbf{B}(u,\mathbf{x})\nabla u) &= s(\mathbf{x},t) \quad \text{in } \Omega \times (0,T] \\
u(x,y,0) &= u_0(\mathbf{x}) \quad \text{in } \Omega \\
u(0,y,t) &= u(1,y,t) \quad \forall y \in [0,1], t \in (0,T] \\
u(x,0,t) &= u(x,1,t) \quad \forall x \in [0,1], t \in (0,T]
\end{aligned}
$$

with $\mathbf{c} = (c,0)^\top$; $\mathbf{B}(u,\mathbf{x}) = \nu(e^u - 1)\mathbf{C}(\mathbf{x})$, $\nu > 0$ and $\mathbf{C}(\mathbf{x}) = \frac{1}{4}(3 + \cos \pi y)\mathbf{I}$

Exact solution:

$$
u(x,y,t) = \exp\left(e^{\frac{(x-\frac{1}{2}+cT)^2+(y-\frac{1}{2})^2}{R^2} - 4\pi\nu t} \right) - 1
$$

2D unsteady convection-diffusion equation

pseudo-time stepping method

Add a pseudo-time derivative:

$$v_\tau + u(v)_t + \nabla \cdot (\mathbf{c}u(v)) - \nabla \cdot (\mathbf{C}(\mathbf{x})\nabla v) = s(\mathbf{x}, t)$$

Space-time discretization:

$$\mathbf{A}\Delta\mathbf{V}^{(n+1,m+1)} = -\mathbf{R}(\mathbf{V}^{(n+1,m)}) - \frac{1}{\Delta t}\left(\mathbf{U}(v_h^{(n+1,m)}) - \mathbf{U}(v_h^{(n)})\right)$$

where $\mathbf{V}^{(n+1)} = \lim_{m\to\infty} \mathbf{V}^{(n+1,m)}$

Implicit matrix:

$$\mathbf{A} = \frac{1}{\Delta\tau}\mathbf{N} + \frac{1}{\Delta t}\mathbf{M}^{(n+1,m)} + \mathbf{L_v}$$

Diagonal mass matrix $\mathbf{N} = \mathrm{diag}(\mathbf{N_1}, \dots, \mathbf{N_N})$:

$$(\mathbf{N_j})_{kl} = \int_\kappa \phi^k \phi^l d\mathbf{x}$$

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2D unsteady convection-diffusion equation

error analysis

$1 \leq p \leq 3$; $Re_h = 0.1$; $T = 2.5$

parameters		FULL method		SIMP method		
p	σ	$\|u - u_h\|_2$	order	p_s	$\|u - u_h\|_2$	order
1	10.0	$0.34570E-02$	–	0	$0.34568E-02$	–
1	5.0	$0.15807E-02$	1.13	0	$0.15806E-02$	1.13
1	2.5	$0.69140E-03$	1.19	0	$0.69137E-03$	1.19
1	1.25	$0.27143E-03$	1.35	0	$0.27144E-03$	1.35
2	10.0	$0.34570E-02$	–	0	$0.34567E-02$	–
2	5.0	$0.15807E-02$	1.13	0	$0.15806E-02$	1.13
2	2.5	$0.69142E-03$	1.19	0	$0.69147E-03$	1.19
2	1.25	$0.27147E-03$	1.35	0	$0.27174E-03$	1.35
3	10.0	$0.34570E-02$	–	0	$0.34578E-02$	–
3	5.0	$0.15807E-02$	1.13	0	$0.15810E-02$	1.13
3	2.5	$0.69142E-03$	1.19	0	$0.69173E-03$	1.19
3	1.25	$0.27147E-03$	1.35	0	$0.27171E-03$	1.35

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2D unsteady convection-diffusion equation

convergence acceleration ($\sigma = 1$; $T = 2.5$)

		$Re_h = 2 \times 10^{-3}$		$Re_h = 10^{-2}$		$Re_h = 10^{-1}$		$Re_h = 1$	
p	N	p_s	*speedup*	p_s	*speedup*	p_s	*speedup*	p_s	*speedup*
1	200	0	2.1	0	1.5	0	1.6	0	1.6
1	800	0	1.8	0	1.4	0	1.3	0	1.2
1	1800	0	1.7	0	1.2	0	1.3	0	1.1
1	3200	0	1.5	0	1.2	0	1.2	0	1.1
2	200	0	1.6	0	1.2	0	1.2	1	1.3
2	800	0	1.4	0	1.2	0	1.3	1	1.4
2	1800	0	1.6	0	1.3	0	1.2	1	1.3
2	3200	0	2.5	0	1.9	0	1.7	1	1.4
3	200	0	2.2	1	1.7	1	1.6	1	1.5
3	800	0	2.7	1	1.8	1	1.8	1	1.4
3	1800	0	5.2	1	3.7	1	2.8	1	1.9
3	3200	0	11.1	1	4.0	1	2.8	1	1.8
4	200	2	1.4	2	1.3	2	1.6	2	1.7
4	800	2	4.5	2	4.4	2	5.6	2	3.9
4	1800	2	7.1	2	7.4	2	7.3	2	9.4
4	3200	2	5.6	2	13.3	2	5.7	2	3.9

Outline

Summary and outlook

- Derived a fast implicit-explicit time discretization for high-order DG method:
  - partial mode uncoupling in neighbouring elements
  - implicit treatment for low order modes
  - local reconstruction of high order modes
- Application to nonlinear 2D scalar equations:
  - convergence acceleration
  - speedup increases with p and N
  - similar results for steady and unsteady flow problems
- Outlook:
  - independent of the DG method (successfully applied to BR1 scheme)
  - extension to systems of conservation laws

[Bassi & Rebay, '97], [Dairay, MSc Thesis, '10]

Thank you for your attention

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