Conservation laws with a non-local flow
Application to Pedestrian traffic.

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Conservation Laws:

$$\partial_t u + \operatorname{div} f(t,x,u) = F(t,x,u)\,, \qquad\qquad u(0) = u_0\,,$$

f is the flow, F is the source,
for scalar equations: $u \in \mathbb{R}$,
time $t \in [0, T]$, space $x \in \mathbb{R}^N$.

Non-local flow

$$\partial_t u + \operatorname{div} f(x, u, u * \eta) = 0\,, \qquad\qquad u(0) = u_0\,,$$

where η is a smooth convolution kernel.

Typical example: Let $\rho(t,x)$ be the density of pedestrian at time t in position $x \in \mathbb{R}^N$. We consider

$$\partial_t \rho + \mathrm{Div}(\rho V(x, \rho, \rho * \eta)) = 0\,; \quad \rho_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})\,.$$

Goal: Existence and uniqueness of solutions.

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Goal: Existence and uniqueness of solutions.

Idea: Fixed point on the nonlocal term:
Study of

$$\mathcal{Q} : w \mapsto \rho$$

where ρ is the solution of

$$\partial_t \rho + \mathrm{Div}(\rho V(x, \rho, w * \eta)) = 0\,; \quad \rho_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})\,.$$

$$\partial_t \rho + \mathrm{div}(\rho\, V) = 0$$

- Coscia & Canavesio $V(\rho) = f(\rho, \nabla\rho)\, \boldsymbol{\nu}(x)$,
- Maury, Roudneff-Chupin & Santambrogio $V = P_{C_\rho} U$ where U is the preferred speed and P_{C_ρ} is the projection in $\mathbf{L}^2$ on admissible states.
- N. Bellomo & C. Dogbé ; P. Degond

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho V) &= 0 \\ \partial_t V + (V \cdot \nabla) V &= F(\rho, \nabla\rho, V) \end{cases}$$

- Hughes ; Di Francesco, Markowich, Pietschmann & Wolfram

$$V = f^2(\rho) |\nabla\varphi|\,,$$

with $|\nabla\varphi| = \frac{1}{f(\rho)}$ or $-\varepsilon\Delta\varphi + |\nabla\varphi|^2 = \frac{1}{(f(\rho)+\varepsilon)^2}$.

- Piccoli & Tosin first order model, ρ_t is a measure such that

$$V = \boldsymbol{\nu}(x) + \int \vartheta\left((y-x)\cdot\boldsymbol{\nu}(x)\right) \eta(x-y)\mathrm{d}\rho_t(y)\,.$$

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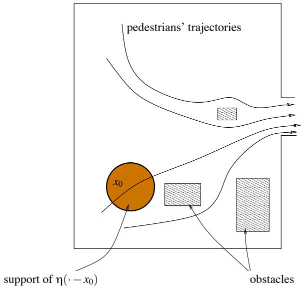
Let $\rho(t,x)$ be the density of pedestrians at time t and position $x \in \mathbb{R}^N$. We consider

$$\partial_t \rho + \mathrm{Div}\left(\rho V(x, \rho, \rho * \eta)\right) = 0\,; \quad \rho_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})\,,$$

with $V(x,\rho) = v(\eta *_x \rho)\,\vec{\boldsymbol{\nu}}(x)$.



Conservation of the regularity.

No uniform a priori bound in $\mathbf{L}^\infty$.
[Colombo, Herty, Mercier, Esaim-Cocv, 2010]

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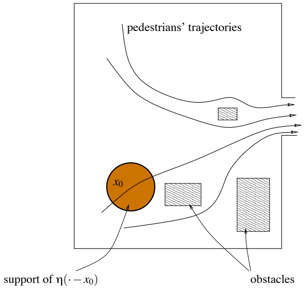

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Let $\rho(t, x)$ be the density of pedestrians at time t and position $x \in \mathbb{R}^{\boldsymbol{N}}$. We consider

$$V = v(\rho)\Big(\boldsymbol{\nu}(x) - \frac{\nabla(\rho * \eta)}{\sqrt{1 + \left\|\nabla(\rho * \eta)\right\|^2}}\Big).$$

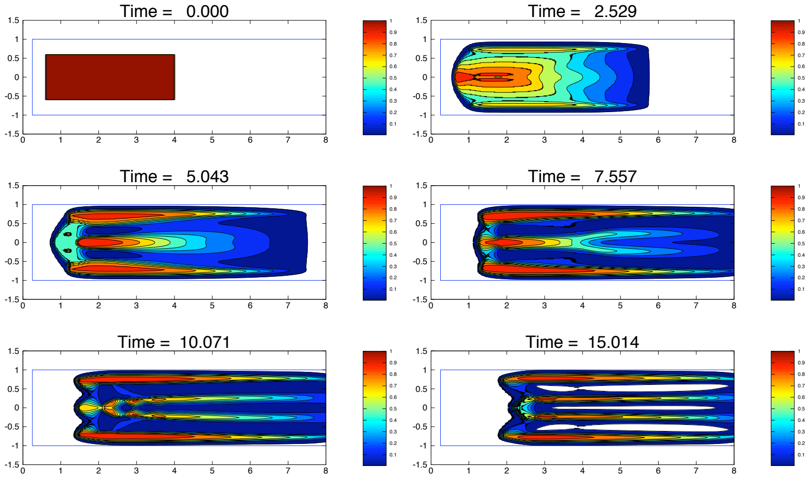
Replacing, we get

$$\partial_t \rho + \mathrm{Div}\left[\rho v(\rho)\Big(\boldsymbol{\nu}(x) - \frac{\nabla(\rho * \eta)}{\sqrt{1 + \left\|\nabla(\rho * \eta)\right\|^2}}\Big)\right] = 0\,;$$

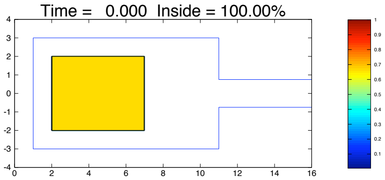
with $\rho_0 \in (\mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^{\boldsymbol{N}}; \mathbb{R})$.

Uniform bound in $\mathbf{L}^\infty$: $\rho_0 \in [0, 1]$ implies $\rho(t) \in [0, 1]$.
How can we tackle the walls and obstacles ?

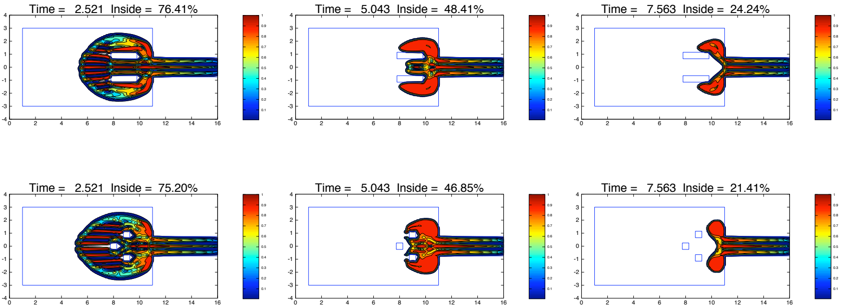
[Colombo, Garavello, Lécureux-Mercier]

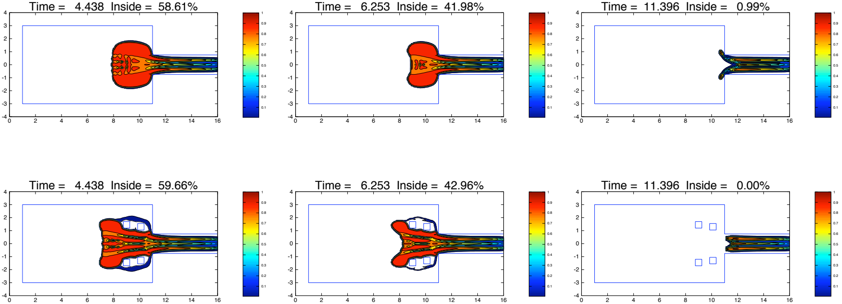

Time = 0.000
Time = 2.529
Time = 5.043
Time = 7.557
Time = 10.071
Time = 15.014

Orderly crowd



Orderly crowd




Time = 4.438 Inside = 58.61%
Time = 6.253 Inside = 41.98%
Time = 11.396 Inside = 0.99%
Time = 4.438 Inside = 59.66%
Time = 6.253 Inside = 42.96%
Time = 11.396 Inside = 0.00%

Several Densities.
ρ_1 and ρ_2 have two different goals,
ρ_1 is repelled by ρ_2,
ρ_2 is repelled by ρ_1.

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}\left(\rho_1 \, v_1(\rho_1)\left(\nu_1(x) - \dfrac{\nabla \rho_2 * \eta_2}{\sqrt{1 + \left\|\nabla \rho_2 * \eta_2\right\|^2}}\right)\right) & = & 0\,, \\ \partial_t \rho_2 + \operatorname{div}\left(\rho_2 \, v_2(\rho_2)\left(\nu_2(x) - \dfrac{\nabla \rho_1 * \eta_1}{\sqrt{1 + \left\|\nabla \rho_1 * \eta_1\right\|^2}}\right)\right) & = & 0\,. \end{cases}$$

Remark: The interaction between the equations is only in the nonlocal term.

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Remark: The interaction between the equations is only in the nonlocal term.

Let $\rho \in \mathbb{R}_+$ be the density of the group and $p \in \mathbb{R}^k$ the position of an isolated agent (*e.g.* a leader or a predator). We describe the interaction by the coupling:

$$\begin{cases} \partial_t \rho + \operatorname{div}\Big(\rho\, V\left(t, x, \rho, p(t)\right)\Big) = 0 & (t,x) \quad \in \quad \mathbb{R}^+ \times \mathbb{R}^N \\ \dot{p} = \varphi\left(t, p, \left(A\rho(t)\right)\left(p(t)\right)\right) & \end{cases}$$

with initial conditions

$$\rho(0,x) = \bar{\rho}(x)\,, \qquad\qquad p(0) = \bar{p}\,.$$

[Colombo, Lécureux-Mercier, to appear on J. Nonlinear Sciences]

Examples:

- Followers / Leader

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\,(p(t) - x) e^{-\|p-x\|}\right) = 0 \\ \dot{p} = (1 + \rho * \eta(p(t)))\, \vec{\psi}(t) \end{cases}$$

- Sheeps / Dogs

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\left(\vec{v}_r(x) + \sum_{i=1}^{n} (x - p_i) e^{-\|p_i - x\|}\right)\right) = 0 \\ \dot{p} = \dfrac{\rho * \nabla \eta^{\perp}}{\sqrt{1 + \|\rho * \nabla \eta\|^2}} \end{cases}$$

- Preys / Predator

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\left(1 + e^{-\|x - p(t)\|}\,(x - p(t))\right)\right) = 0 \\ \dfrac{\mathrm{d}^2 p}{\mathrm{d}t^2} = \rho *_x \nabla \eta(p(t)) \end{cases}$$

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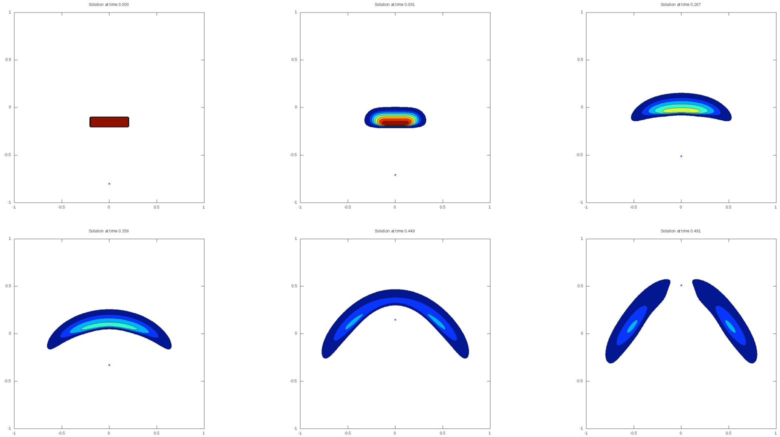
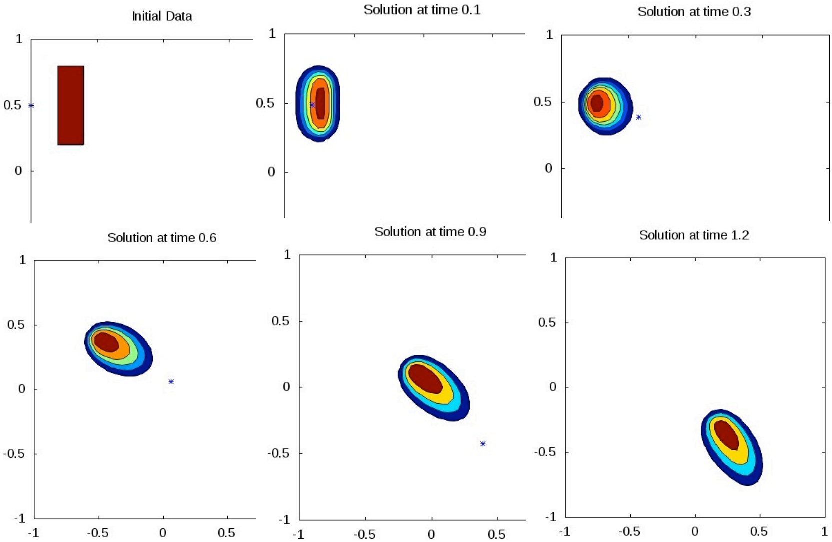
Multi-population

Interaction group / isolated agent: Predators and preys.




Initial Data
Solution at time 0.1
Solution at time 0.3
Solution at time 0.6
Solution at time 0.9
Solution at time 1.2
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Let

$$\begin{array}{rcl} \partial_t u + \mathrm{Div}\, f(t,x,u) = F(t,x,u) & \qquad (t,x) & \in \quad \mathbb{R}_+^* \times \mathbb{R}^N \\ \partial_t v + \mathrm{Div}\, g(t,x,v) = G(t,x,v) & & \end{array}$$

with $u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$, $f \in \mathcal{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$,
$F \in \mathcal{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$.

Goal :

- Estimate on the total variation of the solution:
- Stability $\mathbf{L}^1$ of the solution when (f, F) vary: how can we estimate $(u - v)(t)$ by $u_0 - v_0$, $f - g$, $F - G$.

Method: doubling variables method (Kružkov).

Theorem (Kružkov, 1970, Mat. Sb. (N.S.))

Let $T, U > 0$, *let* $\Omega_T^U = [0, T] \times \mathbb{R}^N \times [-U, U]$. *We consider the equation*

$$\partial_t u + \operatorname{div} f(t, x, u) = F(t, x, u),$$

with initial condition $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N)$. *Under the hypothesis*

$$\textbf{(K)} \quad \left\{ \begin{array}{lll} f \in \mathcal{C}^2, & F \in \mathcal{C}^1, & \forall U > 0\,,\ \partial_u f \in \mathbf{L}^\infty(\Omega_T^U), \\ F - \operatorname{div} f \in \mathbf{L}^\infty(\Omega_T^U), & & \partial_u (F - \operatorname{div} f) \in \mathbf{L}^\infty(\Omega_T^U) \end{array} \right\}$$

there exists a unique weak entropy solution $u \in \mathbf{L}^\infty([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ *continuous from the right in time.*

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Theorem (Lucier, 1986, Math. Comp.)

If $f, g : \mathbb{R} \to \mathbb{R}^N$ *are globally lipschitz, then* $\exists C > 0$ *such that* $\forall u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ *initial conditions for*

$$\partial_t u + \operatorname{div} f(u) = 0\,, \qquad \partial_t v + \operatorname{div} g(v) = 0\,.$$

If furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$, *we have* $\forall t \geq 0$,

$$\left\|(u - v)(t)\right\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + C\, t\, \mathrm{TV}(v_0)\ \mathbf{Lip}\,(f - g)\,.$$

Definition: For $u \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R})$ we denote

$$\begin{aligned}
\mathrm{TV}(u) &= \sup\left\{\int_{\mathbb{R}^N} u \,\mathrm{div}\,\Psi\,;\quad \Psi \in \mathcal{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,,\quad \|\Psi\|_{\mathbf{L}^\infty} \leq 1\right\}\;;\\
&\text{and}\\
\mathbf{BV}(\mathbb{R}^N;\mathbb{R}) &= \left\{u \in \mathbf{L}^1_{\mathbf{loc}};\, \mathrm{TV}(u) < \infty\right\}\;.
\end{aligned}$$

If $u \in \mathcal{C}^1 \cap \mathbf{W}^{1,1}$ then $\mathrm{TV}(u) = \|\nabla u\|_{\mathbf{L}^1}$.

When f and F are not depending on u, we have

$$u_0 \in \mathbf{L}^\infty \cap \mathbf{BV} \Rightarrow \forall t \geq 0\,, \quad u(t) \in \mathbf{L}^\infty \cap \mathbf{BV}$$

and, with $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_T^M)}$,

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0) e^{\gamma t}\,.$$

Goal: general estimate on the total variation.

Theorem (TV — Mercier, 2010)

We assume that (f, F) *satisfies* ***(K) + (TV)****. Soit* $U = \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}$ *and*

$$\kappa_0 = (2N+1)\|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_T^U)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_T^U)}.$$

If $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$*, then* $\forall t \in [0, T]\,,\ u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ *and*

$$\begin{aligned}\mathrm{TV}(u(t)) \leq& \mathrm{TV}(u_0) e^{\kappa_0 t} \\ &+ N W_N \int_0^t e^{\kappa_0 (t-\tau)} \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div}\, f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty([-U,U])} \mathrm{d}x \, \mathrm{d}\tau\,.\end{aligned}$$

Let $\Omega_T^U = [0,T] \times \mathbb{R}^N \times [-U,U]$ and

$$
\textbf{(TV)} \quad \begin{cases}
\text{for all } U,\ T > 0\,, & \nabla \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^{N\times N})\,, \qquad \partial_u F \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R})\,, \\
\displaystyle\int_0^T \int_{\mathbb{R}^N} \left\| \nabla (F - \operatorname{div} f)(t,x,\cdot) \right\|_{\mathbf{L}^\infty([-U,U];\mathbb{R}^N)} \mathrm{d}x \, \mathrm{d}t < \infty\,. &
\end{cases}
$$

[Colombo, Mercier, Rosini, Comm. in Math. Sc., 2009]

[Lécureux-Mercier, to appear on JHDE]

Theorem (Flow/Source — Mercier, 2010)

Assume that $(f, F), (g, G)$ *satisfy* ***(K)****,* (f, F) *satisfies* ***(TV)*** *and* $(f-g, F-G)$ *satisfies* ***(FS)****. Let* $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$*. Let* u *and* v *be the associated solutions to* (f, F) *and* (g, G) *with initial conditions* u_0 *and* v_0*.*
Let $V = \max(\|u\|_{\mathbf{L}^\infty}, \|v\|_{\mathbf{L}^\infty})$ *and* $\kappa = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_T^V)}$*. Then* $\forall t \in [0, T]$*:*

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq e^{\kappa t}\|u_0 - v_0\|_{\mathbf{L}^1} + \left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty(\Omega_T^V)} \left[\frac{e^{\kappa_0 t} - e^{\kappa t}}{\kappa_0 - \kappa} \mathrm{TV}(u_0) \right.$$

$$\left. + N W_N \int_0^t \frac{e^{\kappa_0 (t-\tau)} - e^{\kappa (t-\tau)}}{\kappa_0 - \kappa} \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div}\, f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty([-V,V])} \mathrm{d}x \mathrm{d}\tau \right]$$

$$+ \int_0^t e^{\kappa (t-\tau)} \int_{\mathbb{R}^N} \left\|((F-G) - \mathrm{div}\,(f-g))(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty([-V,V])} \mathrm{d}x \mathrm{d}\tau \,.$$

$$
\textbf{(FS)} \begin{cases} \text{for all } U,\ T > 0\,, \\ \displaystyle\int_0^T \int_{\mathbb{R}^N} \left\| (F - \operatorname{div} f)(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-U,U];\mathbb{R})} \mathrm{d}x\, \mathrm{d}t < +\infty\,. \end{cases}
$$

Particular cases:

- $f(u), g(u), F = G = 0 :\ \kappa_0 = \kappa = 0$ and

$$
\left\| (u - v)(t) \right\|_{\mathbf{L}^1} \leq \| u_0 - v_0 \|_{\mathbf{L}^1} + t\, \mathrm{TV}(u_0) \left\| \partial_u (f - g) \right\|_{\mathbf{L}^\infty(\Omega_T^V)}
$$

- $f(t, x), F(t, x) :\ \kappa_0 = \kappa = 0$ and

$$
\begin{aligned}
\left\| (u - v)(t) \right\|_{\mathbf{L}^1} \leq & \| u_0 - v_0 \|_{\mathbf{L}^1} \\
& + \int_0^t \int_{\mathbb{R}^N} \left| ((F - G) - \operatorname{div}(f - g))(\tau, x) \right| \mathrm{d}x \mathrm{d}\tau\,.
\end{aligned}
$$

Proof: Doubling variables method + estimate on the total variation. Proof

Continuity equation with a non-local flow:

$$\partial_t \rho + \operatorname{div}\left(\rho V(x, \rho(t))\right) = 0\,, \qquad \rho(0,\cdot) = \rho_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}\,,$$

where $V(x) : \mathbb{R}^N \times \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ is a nonlocal regularising functional.

If $v : \mathbb{R} \to \mathbb{R}$ is smooth: $V(x,\rho) = v(\rho * \eta)\,\boldsymbol{\nu}(x)$ for a model of pedestrian traffic in panic.

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Goals:

- Existence and uniqueness of a weak entropy solution ?
- Gâteaux-Differentiability with respect to initial conditions ?

Theorem (Colombo, Herty, Mercier, Esaim-Cocv, 2010)

*If V satisfies **(V1)**, $\rho_0 \in \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}^N, [0, \alpha])$, $\beta > \alpha$, then there exists $T(\alpha, \beta) > 0$ and a unique weak entropy solution*

$$\rho \in \mathcal{C}^0([0, T(\alpha, \beta)]; \mathbf{L^1}(\mathbb{R}^N, [0, \beta]))\,.$$

We denote $S_t \rho_0 = \rho(t, \cdot)$.

(V1) There exists $C \in \mathbf{L}^{\infty}_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall \rho \in \mathbf{L}^{\mathbf{1}}(\mathbb{R}^N;\mathbb{R})$

$$V(\rho) \in \mathbf{L}^{\infty}, \qquad \left\|\nabla_x V(\rho)\right\|_{\mathbf{L}^{\infty}} \leq C(\|\rho\|_{\mathbf{L}^{\infty}}),$$
$$\left\|\nabla_x V(\rho)\right\|_{\mathbf{L}^{1}} \leq C(\|\rho\|_{\mathbf{L}^{\infty}}), \qquad \left\|\nabla_x^2 V(\rho)\right\|_{\mathbf{L}^{1}} \leq C(\|\rho\|_{\mathbf{L}^{\infty}}),$$

and $\forall \rho_1, \rho_2 \in \mathbf{L}^{\mathbf{1}}(\mathbb{R}^N;\mathbb{R})$

$$\left\|V(\rho_1) - V(\rho_2)\right\|_{\mathbf{L}^{\infty}} \leq C(\|\rho_1\|_{\mathbf{L}^{\infty}})\|\rho_1 - \rho_2\|_{\mathbf{L}^{1}},$$
$$\left\|\nabla_x(V(\rho_1) - V(\rho_2))\right\|_{\mathbf{L}^{1}} \leq C(\|\rho_1\|_{\mathbf{L}^{\infty}})\|\rho_1 - \rho_2\|_{\mathbf{L}^{1}}.$$

Let $\beta > \alpha > 0$ and $T \leq T_* = \frac{\ln(\beta/\alpha)}{C(\beta)}$. Let us introduce the space

$$X_\alpha = \mathbf{L}^1(\mathbb{R}^N; [0, \alpha]) .$$

Let $w \in \mathcal{X}_\beta = \mathcal{C}^0([0, T[, X_\beta)$ and $\rho \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$ be the solution of

$$\partial_t \rho + \mathrm{Div}(\rho V(x, w)) = 0 , \quad \rho(0, \cdot) = \rho_0 \in X_\alpha .$$

We introduce the application

$$\mathcal{Q} : w \in \mathcal{X}_\beta \to \rho \in \mathcal{X}_\beta .$$

For w_1, w_2, we get by Theorem (Flow/Source)

$$\left\|\mathcal{Q}(w_1) - \mathcal{Q}(w_2)\right\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq f(T)\|w_1 - w_2\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)},$$

where f is increasing, $f(0) = 0$ and $f \to_{T\to\infty} \infty$.
Then, we apply the fixed point theorem.

Iterating, we obtain existence until T_* and then until

$$T_{ex} = \sup\{\sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})}\, ;\, (\alpha_n)_n \text{ strictly increasing}\, ,\, \alpha_0 = \|u_0\|_{\mathbf{L}^\infty}\}\,.$$

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$$\begin{aligned} \partial_t \rho_i + \mathrm{div}(\rho_i V_i(x, \rho_1 * \eta_{i,1}, \ldots, \rho_k * \eta_{i,k})) = 0\,, \qquad & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ \rho_i(0) = \bar{\rho}_i\,, \qquad & i \in \{1, \ldots, k\}\,. \end{aligned}$$

[G. Crippa and M. Lécureux-Mercier, in preparation]

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Remark: There is a coupling only through the nonlocal term.

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Remark: There is a coupling only through the nonlocal term.

Goal: Existence and uniqueness of measure solutions for the Cauchy problem.

Fixing the nonlocal term, we obtain a system of decoupled continuity equations:

$$\begin{cases} \partial_t \rho_1 + \mathrm{div}(\rho_1 \, b_1(t,x)) = 0 \,, \\ \dots \\ \partial_t \rho_k + \mathrm{div}(\rho_k \, b_k(t,x)) = 0 \,, \end{cases}$$

where $b_1, \dots, b_k$ are regular with respect to x.

Pedestrian traffic with multi-population.

$$V_1(x, \rho_1 * \eta, \rho_2 * \zeta) = v\left(\frac{1}{3}\rho_1 * \eta + \frac{2}{3}\rho_2 * \zeta\right)\boldsymbol{e_1}\,, \qquad ,$$

$$V_2(x, \rho_1 * \eta, \rho_2 * \zeta) = -v\left(\frac{2}{3}\rho_1 * \eta + \frac{1}{3}\rho_2 * \zeta\right)\boldsymbol{e_1}\,,$$

with e_1 a fixed vector of $\mathbb{R}^2$.

Interaction continuum / individuals. Let us assume that $\rho_1 \in \mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^1(\mathbb{R}^d, \mathbb{R}_+))$ and $\rho_2 = \delta_{p(t)}$ is a Dirac measure.

We have

$$\begin{cases} \partial_t \rho_1 + \operatorname{div}\Big(\rho_1 \, V_1\left(x, \, \rho_1 * \eta_1(x), \, \eta_2(x - p(t))\right)\Big) = 0\,, \\ \dot{p}(t) = V_2\Big(p(t), \, \rho_1 * \eta_3\left(p(t)\right) \, , \eta_4(0)\Big)\,. \end{cases}$$

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Theorem

Let $\bar{\rho} \in \mathcal{M}^+(\mathbb{R}^d, \mathbb{R}^k)$. Let us assume that $V \in \mathbf{L}^\infty \cap \mathrm{Lip}(\mathbb{R}^d \times \mathbb{R}^k, \mathbb{R}^{d\times k})$ and that $\eta \in \mathbf{L}^\infty \cap \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^{k\times k})$. Then there exists a unique measure solution to our system with initial condition $\bar{\rho}$.
Furthermore, this measure solution is also a Lagrangian solution.

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Corollary

If $\bar{\rho} \in \mathbf{L}^1(\mathbb{R}^d, (\mathbb{R}^+)^k)$ then $\rho \in \mathbf{L}^\infty(\mathbb{R}^+, \mathbf{L}^1(\mathbb{R}^d, (\mathbb{R}^+)^k))$ and we have

$$\left\|\rho(t)\right\|_{\mathbf{L}^1} = \|\bar{\rho}\|_{\mathbf{L}^1}.$$

If moreover $\bar{\rho} \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^d, (\mathbb{R}^+)^k)$, then $\rho(t) \in \mathbf{L}^\infty$ for all $t \geq 0$ and we have the estimate

$$\left\|\rho(t)\right\|_{\mathbf{L}^\infty} \leq \|\bar{\rho}\|_{\mathbf{L}^\infty} e^{Ct},$$

where C depends on $\|\bar{\rho}\|_{\mathcal{M}}$, V and η.

Def: ρ is a measure solution if, $\forall \varphi \in \mathcal{C}_c^\infty(]-\infty, T] \times \mathbb{R}^d, \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}^d} \left[\partial_t \varphi + V_i(x, \rho * \eta_i) \cdot \nabla \varphi\right] \mathrm{d}\rho_t^i(x)\,\mathrm{d}t + \int_{\mathbb{R}^d} \varphi(0,x)\,\mathrm{d}\bar{\rho}_i(x) = 0\,.$$

Def: $\rho \in \mathbf{L}^\infty([0,T], \mathcal{M}^+(\mathbb{R}^d)^k)$ is a Lagrangian solution with initial condition $\bar{\rho} \in \mathcal{M}^+(\mathbb{R}^d)^k$ if there exists an ODE flow $X^i : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, solution of

$$\begin{cases} \dfrac{\mathrm{d}X^i}{\mathrm{d}t}(t,x) = & V_i(X^i(t,x), \rho_t * \eta^i(X^i(t,x)))\,, \\ X^i(0,x) = & x\,; \end{cases}$$

and such that $\rho_t^i = X^i_{t\,\sharp}\bar{\rho}^i$ where $X_t^i : \mathbb{R}^d \to \mathbb{R}^d$ is the map defined as $X_t^i(x) = X^i(t,x)$ for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Proposition

A Lagrangian solution is always a weak measure solution.

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Proposition

A Lagrangian solution is always a weak measure solution.

Wasserstein distance.
Let μ, $\nu \in \mathcal{P}$,

$$\Xi(\mu,\nu) = \left\{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \mathbb{P}_{x\sharp}\gamma = \mu \text{ and } \mathbb{P}_{y\sharp}\gamma = \nu\right\}.$$

The Wasserstein distance of order one between μ and ν is

$$W_1(\mu,\nu) = \inf_{\gamma\in\Xi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|\,\mathrm{d}\gamma(x,y)\ .$$

Let $\rho = (\rho^1,\ldots,\rho^k)$, $\sigma = (\sigma^1,\ldots,\sigma^k) \in \mathcal{P}(\mathbb{R}^d)^k$. The Wasserstein distance of order one between ρ and σ, denoted $\mathcal{W}_1(\rho,\sigma)$, as

$$\mathcal{W}_1(\rho,\sigma) = \sum_{i=1}^{k} W_1(\rho^i,\sigma^i)\,.$$

Proposition

Let $\bar{\rho}, \bar{\sigma} \in \mathcal{P}(\mathbb{R}^d)$. *Let* $r, s \in \mathcal{C}^0([0,T], \mathcal{P}(\mathbb{R}^d))$.
Let $V \in \mathbf{L}^\infty \cap \mathrm{Lip}(\mathbb{R}^d \times \mathbb{R}^k, \mathbb{R}^d)$, $\eta, \nu \in \mathbf{L}^\infty \cap \mathrm{Lip}(\mathbb{R}^d, \mathbb{R})$. *If* ρ *and* σ *are Lagrangian solutions of*

$$\begin{aligned} &\partial_t \rho + \mathrm{div}(\rho V(x, r * \eta)) = 0\,, \qquad && \rho(0,\cdot) = \bar{\rho}\,, \\ &\partial_t \sigma + \mathrm{div}(\sigma V(x, s * \eta)) = 0\,, \qquad && \sigma(0,\cdot) = \bar{\sigma}\,. \end{aligned}$$

We have the estimate:

$$\mathcal{W}_1(\rho_T, \sigma_T) \leq e^{CT} \mathcal{W}_1(\bar{\rho}, \bar{\sigma}) + T\, e^{CT}\, C' \sup_{t \in [0,T]} \mathcal{W}_1(r_t, s_t)\,,$$

where $C = \mathrm{Lip}_x(V) + \mathrm{Lip}_r(V)\mathrm{Lip}(\eta)\|\bar{\rho}\|_{\mathcal{M}} + \mathrm{Lip}_r(V)\mathrm{Lip}(\eta)\|\bar{\rho}\|_{\mathcal{M}}$ *and* $C' = \mathrm{Lip}_r(V)\mathrm{Lip}(\eta)\|\bar{\rho}\|_{\mathcal{M}}$.

Proof

Let $r \in \mathbf{L}^\infty([0,T], \mathcal{P}(\mathbb{R}^d)^k)$.
Define $b_i(t,x) = V_i(x, r_t * \eta_i) \in \mathbf{L}^\infty([0,T], \mathbf{W}^{1,\infty}(\mathbb{R}^d)^k)$. We consider

$$\partial_t \rho_i + \operatorname{div}\left(\rho_i b_i(t,x)\right) = 0$$

Let $\rho_i = X_{t\sharp}\bar{\rho}$ be the Lagrangian solution and define

$$\mathcal{T} : r \in \mathbf{L}^\infty([0,T], \mathcal{P}(\mathbb{R}^d)^k) \mapsto \rho \in \mathbf{L}^\infty([0,T], \mathcal{P}(\mathbb{R}^d)^k)$$

.

The stability estimate allows us to apply Banach fixed point Theorem for T small enough.

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The stability estimate allows us to apply Banach fixed point Theorem for T small enough.

Let ρ be a measure solution. Let $b = V(x, \rho * \eta)$.
Denote σ the Lagrangian solution associated to $\partial_t \sigma + \operatorname{div}(\sigma b) = 0$ with $\sigma(0) = \bar{\rho}$.

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Let $\delta = \rho - \sigma$. Then δ is the measure solution of

$$\partial_t \delta + \mathrm{div}(\delta b) = 0\,, \qquad \delta(0) = 0\,.$$

Hence $\delta \equiv 0$ so that $\rho = \sigma$.

Kružkov: Existence and uniqueness of weak entropy solution for

$$\partial_t \rho + \mathrm{div}(\rho V(x, \rho, \rho * \eta)) = 0\,.$$

Uniform bound in $\mathbf{L}^\infty$. Strong hypotheses: $V \in \mathcal{C}^2 \cap \mathbf{W}^{2,1} \cap \mathbf{W}^{2,\infty}$. Entropy solutions.

Optimal Transport:

$$\partial_t \rho + \mathrm{div}(\rho V(x, \rho * \eta)) = 0\,.$$

Exponential bound in time in $\mathbf{L}^\infty$. Weaker hypotheses $V \in \mathbf{Lip} \cap \mathbf{L}^\infty$. Measure solutions = Lagrangian solutions.

Other issues: Taking into account the geometry of the room: boundary problem. Finding the best geometry ?

1. Pedestrian Traffic Modelization
   - Macroscopic models
   - One-Population model
   - Multi-population
2. Using the Kružkov theory
   - Stability $\mathbf{L}^1$ with respect to flow and source
   - Scalar conservation law with a non-local flow
3. Using the Optimal transport theory
   - Introduction
   - Main result
   - Proof
4. Conclusion

Let X, Y be the ODE flows associated to ρ, σ. Let $\gamma_0 \in \Xi(\bar{\rho}, \bar{\sigma})$. Define

$$X_t \bowtie Y_t : (x, y) \mapsto (X(x), Y(y)) .$$

Then $\gamma_t = (X_t \bowtie Y_t)_\sharp \gamma_0 \in \Xi(\rho_t, \sigma_t)$. We estimate

$$Q(t) = \int_{\mathbb{R} \times \mathbb{R}^d} |x - y| \, \mathrm{d}\gamma_t(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| X_t(x) - Y_t(y) \right| \mathrm{d}\gamma_0(x, y) \ .$$

Then

$$Q'(t) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| V(X_t(x), r_t * \eta(X_t(x))) - V(Y_t(y), s_t * \eta(Y_t(x))) \right| \mathrm{d}\gamma_0(x, y) .$$

By triangular inequality,

$$
\begin{aligned}
Q'(t) \leq & \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| V(X_t(x), r_t * \eta(X_t(x))) - V(Y_t(y), r_t * \eta(X_t(x))) \right| \mathrm{d}\gamma_0(x, y) \\
& + \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| V(Y_t(y), r_t * \eta(X_t(x))) - V(Y_t(y), r_t * \eta(Y_t(y))) \right| \mathrm{d}\gamma_0(x, y) \\
& + \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| V(Y_t(y), r_t * \eta(Y_t(y))) - V(Y_t(y), s_t * \eta(Y_t(y))) \right| \mathrm{d}\gamma_0(x, y) \\
\leq & (\mathrm{Lip}_x(V) + \mathrm{Lip}_r(V)\mathrm{Lip}(r_t * \eta)) Q(t) \\
& + \mathrm{Lip}_r(V) \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| (r_t - s_t) * \eta(Y_t(y)) \right| \mathrm{d}\gamma_0(x, y) \ .
\end{aligned}
$$

Note besides that

$$
(r_t^i - s_t^i) * \eta(z) = \int_{\mathbb{R}^d} \eta(z - \zeta)(\mathrm{d}r_t^i(\zeta) - ds_t^i(\zeta)) \leq \mathrm{Lip}(\eta) W_1(r_t^i, s_t^i) \,.
$$

Integrating, we get

$$Q(t) \leq Q(0)e^{Ct} + tC' \, e^{Ct} \sup_{\tau} \mathcal{W}_1(r_t, s_t) \, .$$

where $C = \mathrm{Lip}_x(V) + \mathrm{Lip}_r(V)\|r_t\|_{\mathcal{M}}\mathrm{Lip}(\eta)$, $C' = \mathrm{Lip}_r(V)\mathrm{Lip}(\eta)\|\bar{\rho}\|_{\mathcal{M}}$.
We conclude taking γ_0 in an optimal way and using the inequality

$$\mathcal{W}_1(\rho_t, \sigma_t) \leq Q(t) \, .$$

Back

Proposition

Soit $\mu \in \mathcal{C}_c^\infty(\mathbb{R}_+;\mathbb{R}_+)$ tel que $\|\mu\|_{\mathbf{L}^1} = 1$ et $\mu' < 0$ sur $\mathbb{R}_+^*$. On définit $\mu_\lambda(x) = \frac{1}{\lambda^N}\mu\left(\frac{\|x\|}{\lambda}\right)$. Si il existe $C_0 > 0$ tel que $\forall\lambda > 0$,

$$\mathcal{I}(\lambda) = \frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|u(x+y)-u(x)\right|\mu_\lambda(y)\mathrm{d}x\mathrm{d}y \leq C_0,$$

*alors $u \in$ **BV** et*

$$C_1\,\mathrm{TV}(u) = \lim_{\lambda\to 0}\mathcal{I}(\lambda) \leq C_0.$$

avec $C_1 = \int_{\mathbb{R}^N}|y_1|\mu(\|y\|)\mathrm{d}y$.

On cherche à estimer

$$\mathcal{F}(T,\lambda) = \int_0^T\int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-t))}\left|u(x+y)-u(x)\right|\mu_\lambda(y)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Proposition

Soit $\mu \in \mathcal{C}_c^\infty(\mathbb{R}_+;\mathbb{R}_+)$ *tel que* $\|\mu\|_{\mathbf{L}^1} = 1$ *et* $\mu' < 0$ *sur* $\mathbb{R}_+^*$. *On définit* $\mu_\lambda(x) = \frac{1}{\lambda^N}\mu\left(\frac{\|x\|}{\lambda}\right)$. *Si il existe* $C_0 > 0$ *tel que* $\forall \lambda > 0$,

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alors $u \in \mathbf{BV}$ *et*

$$C_1\,\mathrm{TV}(u) = \lim_{\lambda\to 0}\mathcal{I}(\lambda) \leq C_0.$$

avec $C_1 = \int_{\mathbb{R}^N}|y_1|\mu(\|y\|)\mathrm{d}y$.

On cherche à estimer

$$\mathcal{F}(T,\lambda) = \int_0^T\int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-t))}\left|u(x+y)-u(x)\right|\mu_\lambda(y)\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

Par la méthode de doublement des variables [?], on obtient l'estimation :

$$\partial_T \mathcal{F}(T,\lambda) \leq \partial_T \mathcal{F}(0,\lambda) + C\lambda \left(\partial_\lambda \mathcal{F}(T,\lambda) + \frac{N}{\lambda} \mathcal{F}(T,\lambda) \right) + C' \mathcal{F}(T,\lambda) + \lambda \int_0^T A(t) \mathrm{d}t \,,$$

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où

$$A(t) = M_1\int_{\mathbb{R}^{\boldsymbol{N}}} \left\|\nabla(F - \operatorname{div} f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}\boldsymbol{u})}, \qquad M_1 = \int_{\mathbb{R}^{\boldsymbol{N}}} \|y\|\mu(\|y\|)\mathrm{d}y,$$

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On utilise $\frac{1}{\lambda}\partial_T \mathcal{F}(0,\lambda) \leq M_1 \mathrm{TV}(u_0)$, puis on intègre en temps et on divise par $CT\lambda$:

$$0 \leq \frac{M_1}{C}\mathrm{TV}(u_0) + \partial_\lambda \mathcal{F}(T,\lambda) + \frac{\alpha(T)}{\lambda}\mathcal{F}(T,\lambda) + \frac{1}{C}\int_0^T A(t)\mathrm{d}t\,,$$

où

$$\alpha(T) = N + \frac{C'}{C} - \frac{1}{C\,T} \to_{T\to 0} -\infty\,.$$

On choisit alors T de sorte que $\alpha < -1$ et on intègre sur $[\lambda, +\infty[$.

On obtient

$$\mathcal{F}(T,\lambda) \leq \frac{\lambda}{-\alpha-1}\frac{M_1}{C}\,\mathrm{TV}(u_0) + \frac{\lambda}{C(-\alpha-1)}\int_0^T A(t)\mathrm{d}t\,.$$

D'autre part, on remarque que

$$\partial_\lambda \mathcal{F}(T,\lambda) + \frac{N}{\lambda}\mathcal{F} \leq \frac{K}{2\lambda}\mathcal{F}(T,2\lambda)\,.$$

En reportant, on a finalement,

$$\frac{1}{\lambda}\partial_T \mathcal{F}(T,\lambda) \leq \frac{1}{\lambda}\partial_T \mathcal{F}(0,\lambda) + \frac{CK+C'}{(-\alpha-1)C}\left(M_1 \mathrm{TV}(u_0) + \int_0^T A(t)\mathrm{d}t\right) + \int_0^T A(t)\mathrm{d}t\,.$$

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On note $u = u(t,x)$ et $v = u(s,y)$ pour $(t,x),(s,y) \in \mathbb{R}_+^* \times \mathbb{R}^N$. Pour tous $k, l \in \mathbb{R}$ et toute fonction-test $\varphi = \varphi(t,x,s,y) \in \mathcal{C}_c^1\left((\mathbb{R}_+^* \times \mathbb{R}^N)^2; \mathbb{R}_+\right)$, on a

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\Big[(u-k)\,\partial_t\varphi + \big(f(t,x,u) - f(t,x,k)\big)\,\nabla_x\varphi + \big(F(t,x,u) - \mathrm{div} f(t,x,k)\big)\,\varphi\Big] \times \mathrm{sign}(u-k)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{1}$$

et

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\Big[(v-l)\,\partial_s\varphi + \big(f(s,y,v) - f(s,y,l)\big)\,\nabla_y\varphi + (F(s,y,v) - \mathrm{div} f(s,y,l))\varphi\Big] \times \mathrm{sign}(v-l)\,\mathrm{d}y\,\mathrm{d}s \geq 0 \tag{2}$$

On choisit $k = v(s,y)$ dans (1) et on intègre en (s,y). De même, on prend $l = u(t,x)$ dans (2) et on intègre en (t,x).
De plus, on choisit $\varphi(t,x,s,y) = \Psi(t-s,x-y)\Phi(t,x)$ et on somme

$$\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \mathrm{sign}(u-v)\Big[(u-v)\,\Psi\,\partial_t\Phi + \big(f(t,x,u)-f(t,x,v)\big)\cdot(\nabla\Phi)\,\Psi$$
$$+\big(f(s,y,v)-f(s,y,u)-f(t,x,v)+f(t,x,u)\big)\cdot(\nabla\Psi)\,\Phi$$
$$+\big(F(t,x,u)-F(s,y,v)+\mathrm{div} f(s,y,u)-\mathrm{div} f(t,x,v)\big)\,\varphi\Big]\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \quad \geq 0. \tag{3}$$

On prend ensuite

$$\Psi(t-s,x-y) = \frac{1}{\lambda^N}\mu\left(\frac{x-y}{\lambda}\right)\frac{1}{\eta}\nu\left(\frac{t-s}{\eta}\right),$$

tandis que $\Phi \to \mathbf{1}_{[0,T]\times B(x_0,R+M(T-t))}$ lorsque θ, $\varepsilon \to 0$.
On fait des estimations et on fait tendre $\eta,\theta,\varepsilon \to 0$.

Retour

1. Pedestrian Traffic Modelization
   - Macroscopic models
   - One-Population model
   - Multi-population
2. Using the Kružkov theory
   - Stability $\mathbf{L}^1$ with respect to flow and source
   - Scalar conservation law with a non-local flow
3. Using the Optimal transport theory
   - Introduction
   - Main result
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4. Conclusion