# Detection and localization of defects for sensor array imaging in noisy environments

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PEPS Interactions Mathématiques-Industrie - project Imagelec (Imagerie électromagnétique à partir de signaux fortement bruités).

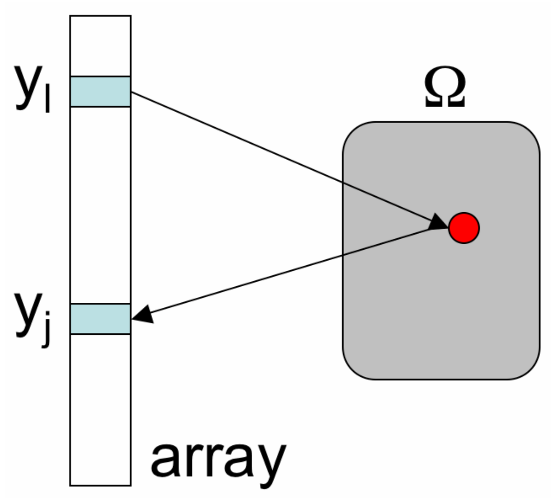
Consider a simple imaging problem: detection and localization of a point reflector.
$\rightarrow$ "All" methods work.
Add noise.
$\rightarrow$ Which method is best ?

## The array response matrix (in an ideal world)

- Time-harmonic waves emitted by point sources and recorded by point sensors

- Array of $n$ elements $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$.

- $\hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l)$ = field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits a time-harmonic signal at frequency $\omega$.



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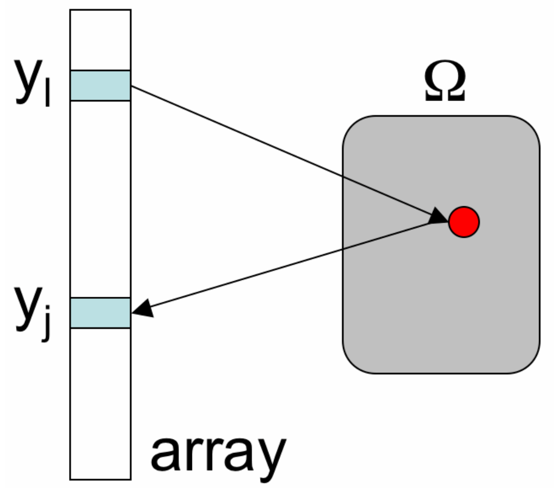



- Array of $n$ elements $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$.
- $\hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l)$ = field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits a time-harmonic signal at frequency $\omega$.
- Response matrix $\mathbf{A}_0 = ((A_0)_{jl})_{j,l=1,\ldots,n}$ in a known environment:

$$(A_0)_{jl} = \hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l) - \hat{G}_0(\boldsymbol{y}_j, \boldsymbol{y}_l)$$

where

- $\hat{u}(\boldsymbol{y}_j, \boldsymbol{y}_l)$ the field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits.
- $\hat{G}_0(\boldsymbol{y}_j, \boldsymbol{y}_l)$ is the incident field (Green's function of the background medium).

# A simple model: a point reflector in a constant background (1/2)

Scalar wave equation: $\Delta_{\boldsymbol{x}} \hat{u}(\boldsymbol{x},\boldsymbol{y}) + \frac{\omega^2}{c^2(\boldsymbol{x})} \hat{u}(\boldsymbol{x},\boldsymbol{y}) = -\delta_{\boldsymbol{y}}(\boldsymbol{x})$.

In the presence of a localized reflector in the search domain $\Omega$:

$$\frac{1}{c^2(\boldsymbol{x})} = \frac{1}{c_0^2}\big(1+\sigma_{\mathrm{r}} V_{\mathrm{ref}}(\boldsymbol{x})\big)$$

- $c_0$ is the known background speed (supposed constant),
- the local variation $V_{\mathrm{ref}}(\boldsymbol{x}) = \mathbf{1}_{\Omega_{\mathrm{ref}}}(\boldsymbol{x}-\boldsymbol{x}_{\mathrm{ref}})$ represents the reflector at $\boldsymbol{x}_{\mathrm{ref}}$, where $\Omega_{\mathrm{ref}}$ is a compactly supported domain with volume $l_{\mathrm{ref}}^3$.

• Response matrix $\mathbf{A}_0 = ((A_0)_{jl})_{j,l=1,\dots,n}$:

$$(A_0)_{jl} = \hat{u}(\boldsymbol{y}_j,\boldsymbol{y}_l) - \hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l)$$

- $\hat{u}(\boldsymbol{y}_j,\boldsymbol{y}_l)$ the field recorded by the sensor at $\boldsymbol{y}_j$ when the sensor at $\boldsymbol{y}_l$ emits:

$$\Delta_{\boldsymbol{x}} \hat{u}(\boldsymbol{x},\boldsymbol{y}_l) + \frac{\omega^2}{c_0^2}\big(1+\sigma_{\mathrm{r}} V_{\mathrm{ref}}(\boldsymbol{x})\big)\hat{u}(\boldsymbol{x},\boldsymbol{y}_l) = -\delta_{\boldsymbol{y}_l}(\boldsymbol{x})$$

- $\hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l)$ is the incident field:

$$\hat{G}_0(\boldsymbol{y}_j,\boldsymbol{y}_l) = \frac{e^{i\frac{\omega}{c_0}|\boldsymbol{y}_j-\boldsymbol{y}_l|}}{4\pi|\boldsymbol{y}_j-\boldsymbol{y}_l|}$$

# A simple model: a point reflector in a constant background (2/2)

- If the reflector is small, the response matrix has the form (Born approximation):

$$(A_0)_{jl} = \hat{G}_0(\boldsymbol{x}_j, \boldsymbol{x}_{\mathrm{ref}}) \frac{\omega^2}{c_0^2} \sigma_{\mathrm{r}} l_{\mathrm{ref}}^3 \hat{G}_0(\boldsymbol{x}_{\mathrm{ref}}, \boldsymbol{x}_l)$$

or equivalently:

$$\mathbf{A}_0 = \sigma_{\mathrm{ref}} \boldsymbol{g}(\boldsymbol{x}_{\mathrm{ref}}) \boldsymbol{g}(\boldsymbol{x}_{\mathrm{ref}})^T, \text{ with } \quad \sigma_{\mathrm{ref}} = \frac{\omega^2}{c_0^2} \sigma_{\mathrm{r}} l_{\mathrm{ref}}^3 \Big( \sum_{l=1}^{n} |\hat{G}_0(\boldsymbol{x}_{\mathrm{ref}}, \boldsymbol{x}_l)|^2 \Big)$$

- $\sigma_{\mathrm{r}}$ is the scattering amplitude of the reflector,
- $l_{\mathrm{ref}}^3$ is the volume of the reflector,
- $\boldsymbol{g}(\boldsymbol{x})$ is the normalized vector of Green's functions from the array to the point $\boldsymbol{x}$:

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{1}{\big( \sum_{l=1}^{n} |\hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_l)|^2 \big)^{1/2}} \big( \hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_j) \big)_{j=1,\dots,n}$$

The matrix $\mathbf{A}_0$ has rank 1 and its unique non-zero singular value is $\sigma_{\mathrm{ref}}$.

Remark: other possible models: perfectly conducting crack, small inclusion, ...

Question 1: What is the structure of the (SVD of the) measured matrix $\mathbf{A}$ in the presence of noise ? When do we sound the alarm (presence of a reflector) ?

# Imaging functionals (1/3)

Let $\mathbf{A}$ be the (symmetrized) measured response matrix. Perform a SVD:

$$\mathbf{A} = \sum_{l=1}^{n} \sigma^{(l)} \boldsymbol{v}^{(l)} (\boldsymbol{v}^{(l)})^T$$

Remember that the unperturbed matrix is $\mathbf{A}_0 = \sigma_{\rm ref} \boldsymbol{g}(\boldsymbol{x}_{\rm ref}) \boldsymbol{g}(\boldsymbol{x}_{\rm ref})^T$.

- MUSIC functional ($\mathbf{P}$ =projection on image space of $\mathbf{A}$=projection on span($\boldsymbol{v}^{(1)}$)):

$$\mathcal{I}_{\rm MUSIC}(\boldsymbol{x}) = \big\| (\mathbf{I} - \mathbf{P}) \boldsymbol{g}(\boldsymbol{x}) \big\|^{-1} = \Big\| \boldsymbol{g}(\boldsymbol{x}) - \Big( \overline{\boldsymbol{v}^{(1)}}^T \boldsymbol{g}(\boldsymbol{x}) \Big) \boldsymbol{v}^{(1)} \Big\|^{-1}$$

- Reverse-Time migration functional:

$$\mathcal{I}_{\rm RT}(\boldsymbol{x}) = \overline{\boldsymbol{g}(\boldsymbol{x})}^T \mathbf{A} \overline{\boldsymbol{g}(\boldsymbol{x})}$$

- Kirchhoff Migration functional:

$$\mathcal{I}_{\rm KM}(\boldsymbol{x}) = \overline{\boldsymbol{d}(\boldsymbol{x})}^T \mathbf{A} \overline{\boldsymbol{d}(\boldsymbol{x})}$$

where

$$\boldsymbol{d}(\boldsymbol{x}) = \frac{1}{\sqrt{n}} \Big( \exp(i \frac{\omega}{c_0} |\boldsymbol{x} - \boldsymbol{x}_j|) \Big)_{j=1,\ldots,n}$$

# Imaging functionals (2/3)

A general imaging functional can be obtained using a weighted-subspace migration:

$$\begin{aligned}\mathcal{I}_{\mathrm{SM}}(\boldsymbol{x},\boldsymbol{w}) &= \overline{\boldsymbol{g}(\boldsymbol{x})}^T \Big[\sum_{l=1}^{n} w_l(\boldsymbol{x})\boldsymbol{v}^{(l)}(\boldsymbol{v}^{(l)})^T\Big]\overline{\boldsymbol{g}(\boldsymbol{x})} \\ &= \sum_{l=1}^{n} w_l(\boldsymbol{x})\big(\overline{\boldsymbol{g}(\boldsymbol{x})}^T\boldsymbol{v}^{(l)}\big)^2\end{aligned}$$

where $\boldsymbol{w}(\boldsymbol{x}) = (w_l(\boldsymbol{x}))_{l=1,\dots,n}$ are (complex) weights.

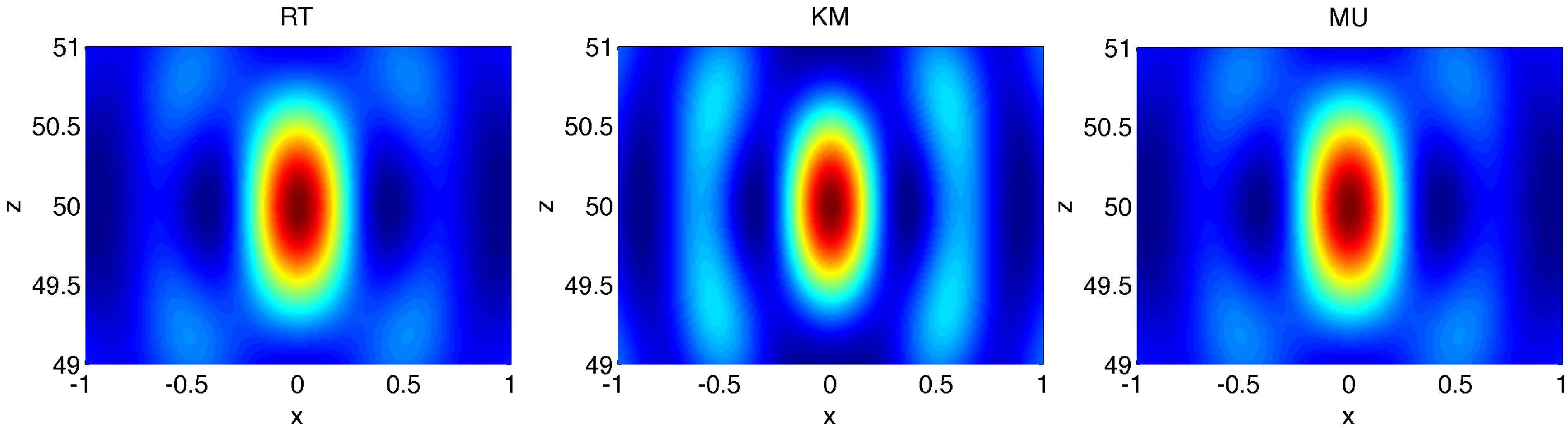
- Denote $w_l^{(1)}(\boldsymbol{x}) = \sigma^{(l)}$. Then $\mathcal{I}_{\mathrm{SM}}(x,\boldsymbol{w}^{(1)})$ corresponds to Reverse-Time migration:

$$\mathcal{I}_{\mathrm{SM}}(\boldsymbol{x},\boldsymbol{w}^{(1)}) = \mathcal{I}_{\mathrm{RT}}(\boldsymbol{x})$$

- Denote $w_l^{(2)}(\boldsymbol{x}) = \exp\Big(-i2\arg\big(\overline{\boldsymbol{g}(\boldsymbol{x})}^T\boldsymbol{v}^{(1)}\big)\Big)\mathbf{1}_1(l)$. Then $\mathcal{I}_{\mathrm{SM}}(x,\boldsymbol{w}^{(2)})$ corresponds to MUSIC:

$$\begin{aligned}\mathcal{I}_{\mathrm{MUSIC}}(\boldsymbol{x}) &= \big\|\boldsymbol{g}(\boldsymbol{x}) - \big(\overline{\boldsymbol{v}^{(1)}}^T\boldsymbol{g}(\boldsymbol{x})\big)\boldsymbol{v}^{(1)}\big\|^{-1} = \big(1-\big|\overline{\boldsymbol{g}(\boldsymbol{x})}^T\boldsymbol{v}^{(1)}\big|^2\big)^{-1/2} \\ &= \big(1-\mathcal{I}_{\mathrm{SM}}(\boldsymbol{x},\boldsymbol{w}^{(2)})\big)^{-1/2}\end{aligned}$$

## Imaging functionals (3/3)




Imaging functionals for a point reflector in the absence of noise.
Left: Reverse-Time migration, center: Kirchhoff Migration, right: MUSIC (or more exactly, $1 - \mathcal{I}_{\mathrm{MUSIC}}(\boldsymbol{x})^{-2}$).

Question 2: What is the best imaging functional to localize a reflector in the presence of noise ?

## A point reflector in a noisy environment: the response matrix

- In the presence of a reflector and noise, the response matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{W}$$

- The matrix $\mathbf{A}_0$ is the rank-one matrix that corresponds to the reflector.
- The matrix $\mathbf{W}$ models noise.

*Assume that:*
there is additive measurement noise (and symmetrize the matrix, since the unperturbed response matrix is symmetric).

*Then:*
The matrix $\mathbf{W}$ is complex symmetric Gaussian, i.e., $W_{jl} = W_{lj}$ and $W_{jl}$, $j \leq l$ obey independent complex Gaussian random variables with mean zero and variance $\delta^2$ off the diagonal ($j \neq l$) and $2\delta^2$ on the diagonal ($j = l$).

## The response matrix without reflector and with noise

- Denote

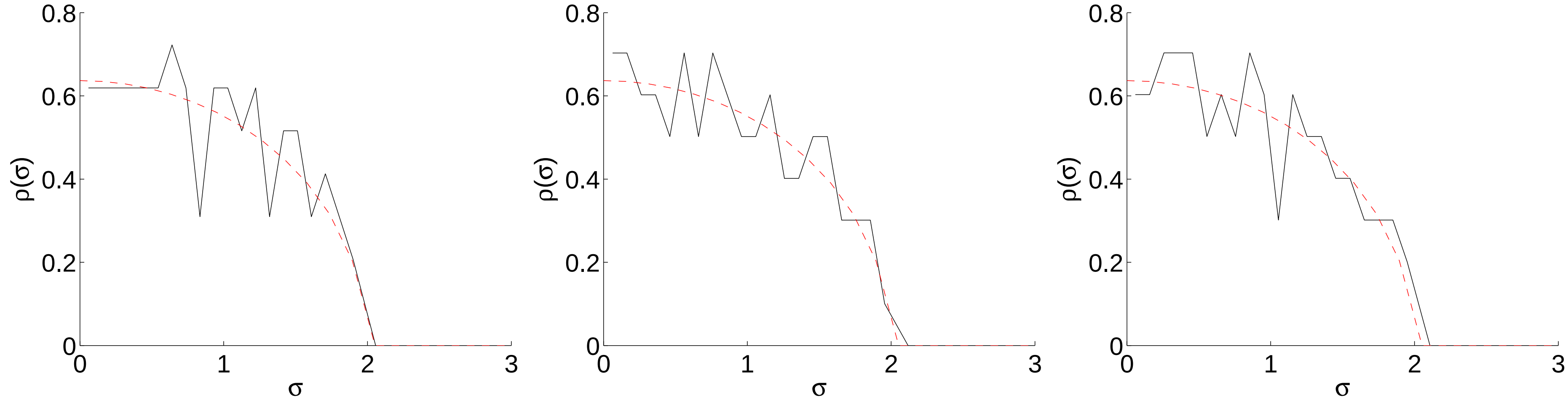
- $\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \sigma_3^{(n)} \geq \cdots \geq \sigma_n^{(n)}$ the singular values of the response matrix $\mathbf{A}$.
- $\sigma_{\rm noise} = \sqrt{n}\delta$ where $\delta^2$ is the variance of the entries of the matrix.

In the regime $n \gg 1$ (number of sensors $\gg 1$):

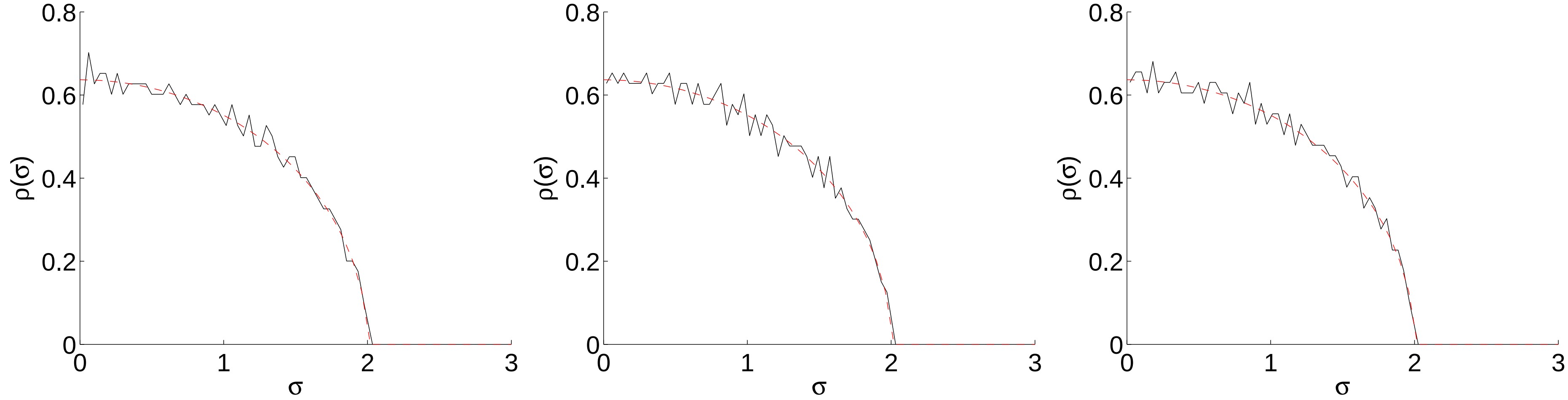
a) The singular values $(\sigma_j^{(n)})_{j=1,\ldots,n}$ follow a quarter-circle distribution:

$$\frac{1}{n}{\rm Card}\big(j = 1,\ldots,n\,,\,\sigma_j^{(n)} \in [a,b]\big) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sigma_{\rm noise}} \int_a^b \rho_{\rm qc}\big(\frac{\sigma}{\sigma_{\rm noise}}\big) d\sigma$$

$$\rho_{\rm qc}(\sigma) = \begin{cases} \frac{1}{\pi}\sqrt{4-\sigma^2} & \text{if } 0 < \sigma \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$




Histogram of the singular values for three realizations of the random matrix ($n = 100$, $\sigma_{\text{noise}} = 1$).




Histogram of the singular values for three realizations of the random matrix ($n = 1000$, $\sigma_{\text{noise}} = 1$).

## The response matrix without reflector and with noise

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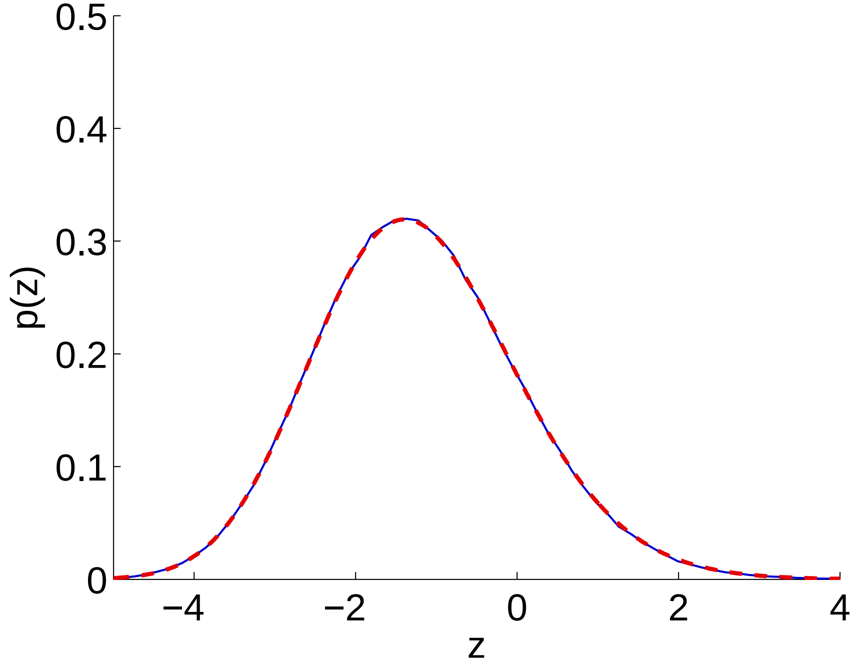
In the regime $n \gg 1$ (number of sensors $\gg 1$):

b) The largest singular value satisfies

$$\sigma_1^{(n)} = \sigma_{\text{noise}}\big[2 + 2^{-2/3} n^{-2/3} Z_1 + o(n^{-2/3})\big]$$

where $Z_1$ follows a type 1 Tracy-Widom distribution ($\mathbb{E}[Z_1] \simeq -1.21$, $\text{Var}(Z_1) \simeq 1.61$).




Histogram of $2^{2/3}n^{2/3}(\sigma_1^{(n)}/\sigma_{\rm noise}-2)$ obtained from MC simulations with $n=50$ in the absence of reflector (solid) and compared with the theoretical type 1 Tracy-Widom distribution (dashed).

## The response matrix with a point reflector and with noise

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In the regime $n \gg 1$ (number of sensors $\gg 1$):

a) The second singular value satisfies $\sigma_2^{(n)} \simeq 2\sigma_{\text{noise}}[1 + O(n^{-2/3})]$,
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b1) If $\sigma_{\rm noise} < \sigma_{\rm ref}$, then the largest singular value satisfies

$$\sigma_1^{(n)} = \sigma_{\rm ref}\left[1 + \sigma_{\rm noise}^2\sigma_{\rm ref}^{-2} + n^{-1/2}\sigma_{\rm noise}\sigma_{\rm ref}^{-1}\left(1 - \sigma_{\rm noise}^2\sigma_{\rm ref}^{-2}\right)^{1/2} Z_0 + o(n^{1/2})\right]$$

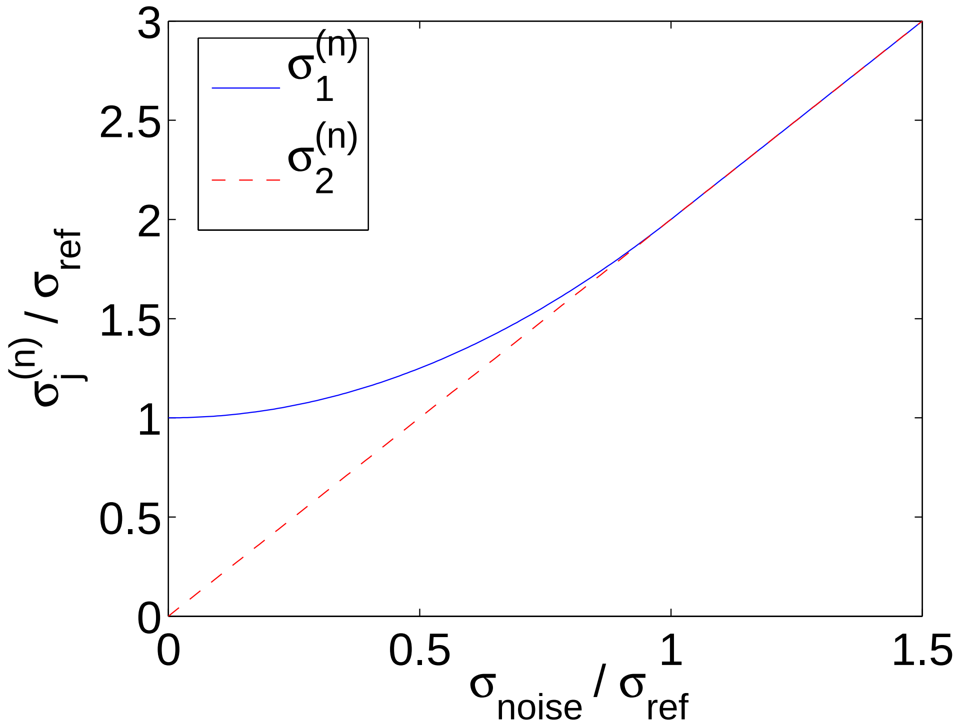
where $Z_0$ follows a standard Gaussian distribution.

b2) If $\sigma_{\rm noise} > \sigma_{\rm ref}$, then the largest singular value satisfies

$$\sigma_1^{(n)} = \sigma_{\rm noise}\left[2 + 2^{-2/3}n^{-2/3}Z_1 + o(n^{-2/3})\right]$$

where $Z_1$ follows a type 1 Tracy-Widom distribution ($\mathbb{E}[Z_1] \simeq -1.21$, ${\rm Var}(Z_1) \simeq 1.61$).

Proof: random matrix theory.




Mean first and second singular values.

Consider

$$R := \frac{\sigma_1^{(n)}}{\left(\frac{1}{n-2}\sum_{j=2}^{n}(\sigma_j^{(n)})^2\right)^{1/2}}$$

In the regime $n \gg 1$ (number of sensors $\gg 1$)

- In the absence of reflector,

$$R \overset{dist.}{=} 2 + \frac{1}{2^{2/3}n^{2/3}} Z_1$$

where $Z_1$ follows a type 1 Tracy-Widom distribution.

- In the presence of a reflector, if $\sigma_{\rm ref} > \sigma_{\rm noise}$, then

$$R \overset{dist.}{=} \frac{\sigma_{\rm ref}}{\sigma_{\rm noise}} + \frac{\sigma_{\rm noise}}{\sigma_{\rm ref}} + \frac{1}{\sqrt{n}}\sqrt{1 - \sigma_{\rm noise}^2 \sigma_{\rm ref}^{-2}}\, Z_0$$

where $Z_0$ follows a standard Gaussian distribution.

If $\sigma_{\rm ref} < \sigma_{\rm noise}$ then $R \overset{dist.}{=} 2 + \frac{1}{2^{2/3}n^{2/3}} Z_1$, as in the absence of a reflector !

• *Detection test with level $r$: If the data gives $R$, then sound the alarm if $R > r$.*

• The false alarm rate (FAR) is the probability to sound the alarm when there is no reflector:

$$\text{FAR} = \mathbb{P}(R > r | \text{ without reflector })$$

The probability of detection (POD) is the probability to sound the alarm when there is a reflector:

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- Fix $\alpha \in (0, 1)$. Choose

$$r_\alpha = 2 + \frac{1}{2^{2/3} n^{2/3}} \Phi_{\text{TW1}}^{-1}(1 - \alpha),$$

where $\Phi_{\text{TW1}}$ is the type 1 Tracy-Widom cumulative distribution function. For instance, $\Phi_{\text{TW1}}^{-1}(0.95) \simeq 0.98$. Main results:

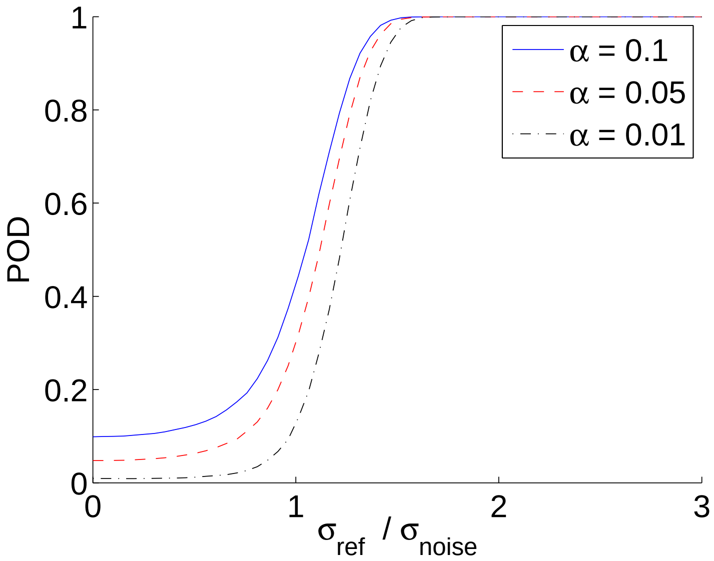
- The FAR of the test $R > r_\alpha$ is $\alpha$.
- The POD of the test $R > r_\alpha$ is

$$\text{POD} = \max\left\{\alpha, \Phi\left(\sqrt{n}\frac{\frac{\sigma_{\text{ref}}}{\sigma_{\text{noise}}} + \frac{\sigma_{\text{noise}}}{\sigma_{\text{ref}}} - r_\alpha}{\sqrt{1 - (\sigma_{\text{noise}}/\sigma_{\text{ref}})^2}}\right)\right\}$$

where $\Phi$ is the standard Gaussian cumulative distribution function.

- By Neyman-Pearson lemma the test $R > r_\alpha$ maximizes the POD for a given FAR $\alpha$.

The POD increases with the number $n$ of sensors, with $\sigma_{\mathrm{ref}}/\sigma_{\mathrm{noise}}$, and with the FAR $\alpha$.




The SV-based test becomes powerful when $\sigma_{\mathrm{ref}} > \sigma_{\mathrm{noise}}$.

# Migration-based detection test

Reverse-Time imaging functional:

$$\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x}) = \overline{\boldsymbol{g}(\boldsymbol{x})}^T \mathbf{A} \overline{\boldsymbol{g}(\boldsymbol{x})} \qquad \text{with } \boldsymbol{g}(\boldsymbol{x}) = \frac{1}{\left(\sum_{l=1}^n |\hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_l)|^2\right)^{1/2}} \left(\hat{G}_0(\boldsymbol{x}, \boldsymbol{x}_j)\right)_{j=1,\dots,n}$$

- Imaging of a point reflector without noise: $\mathcal{I}_{\mathrm{RT}}$ is a peak centered at $\boldsymbol{x}_{\mathrm{ref}}$:

$$\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x}) = \sigma_{\mathrm{ref}} h(\boldsymbol{x} - \boldsymbol{x}_{\mathrm{ref}}) = \sigma_{\mathrm{ref}} \left(\overline{\boldsymbol{g}(\boldsymbol{x})}^T \boldsymbol{g}(\boldsymbol{x}_{\mathrm{ref}})\right)^2$$

$h$ =point spread function; maximal at $\mathbf{0}$ (Cauchy-Schwartz); $h(\mathbf{0}) = 1$.

- Full aperture:

$$h(\boldsymbol{x}) = \mathrm{sinc}^2\left(\frac{\pi|\boldsymbol{x}|}{\lambda_0}\right)$$

$\hookrightarrow$ the width of the function $h(\boldsymbol{x})$ is $\lambda_0/2$ (diffraction limit).

- Imaging of noise without reflector: $\mathcal{I}_{\rm RT}$ is a speckle pattern, i.e. a stationary Gaussian random field with mean zero, variance $2\delta^2$, and covariance function:

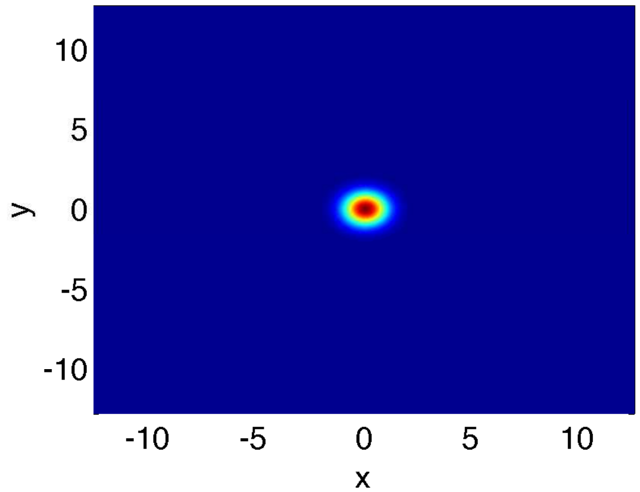
$$\mathbb{E}\big[\mathcal{I}_{\rm RT}(\boldsymbol{x})\overline{\mathcal{I}_{\rm RT}}(\boldsymbol{y})\big] = 2\delta^2 h(\boldsymbol{x}-\boldsymbol{y})$$

The hotspot volume is defined by

$$V_c = \frac{\pi^{3/2}}{(\det \mathbf{H})^{1/2}}, \qquad \mathbf{H} = \big(-\partial^2_{x_j x_l} h(\mathbf{0})\big)_{j,l=1,\ldots,3}.$$

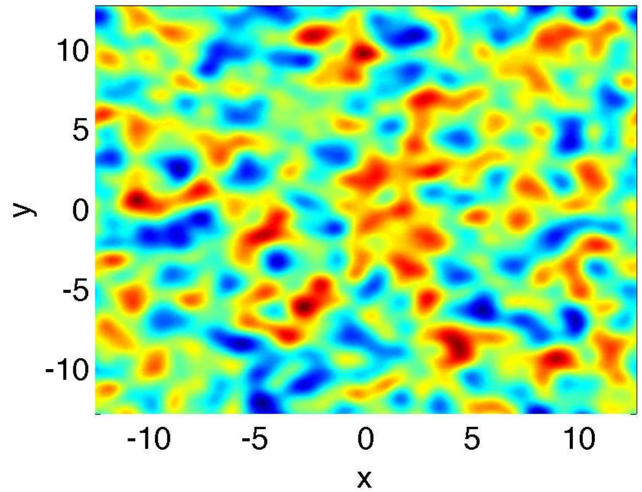
- Full aperture:

$$\mathbf{H} = \frac{8\pi^2}{3\lambda_0^2}\mathbf{I}, \qquad V_c = \frac{3^{3/2}}{(2\pi)^{3/2}}\lambda_0^3$$



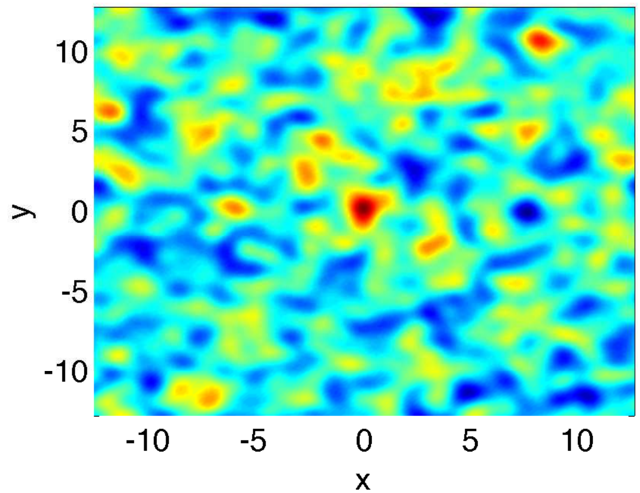

With reflector, without noise

Point spread function




Without reflector, with noise

Speckle pattern




With reflector, with noise

- Consider

$$S := \frac{\|\mathcal{I}_{\mathrm{RT}}\|_{L^\infty(\Omega)}^2 |\Omega|}{\|\mathcal{I}_{\mathrm{RT}}\|_{L^2(\Omega)}^2}.$$

In the regime $n \gg 1$ (number of sensors $\gg 1$) and $|\Omega| \gg V_c$ (search volume $\gg$ hotspot volume):

a) In the absence of a reflector, then

$$S \overset{dist.}{=} \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln\ln \frac{|\Omega|}{V_c} + Z,$$

where $Z$ follows a Gumbel distribution.

b) In the presence of a reflector, then

$$S \overset{dist.}{=} \max\left\{ \frac{\sigma_{\mathrm{ref}}^2}{2\delta^2} + \frac{\sigma_{\mathrm{ref}}}{\delta} Z_0 \,,\, \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln\ln \frac{|\Omega|}{V_c} + Z \right\},$$

where $Z_0$ follows a standard Gaussian distribution and is independent of $Z$.

Proof: extreme value theory for random fields.

• *Migration-based detection test: If the data gives $S$, sound the alarm if $S > s$.*

Fix $\alpha \in (0,1)$. Choose

$$s_\alpha = \ln \frac{|\Omega|}{V_c} + \frac{3}{2} \ln \ln \frac{|\Omega|}{V_c} + \Phi_{\rm G}^{-1}(1-\alpha),$$

where $\Phi_{\rm G}(x) = \exp(-e^{-x})$ is the Gumbel cumulative distribution function. For instance $\Phi_{\rm G}^{-1}(0.95) \simeq 2.97$.

Main results:

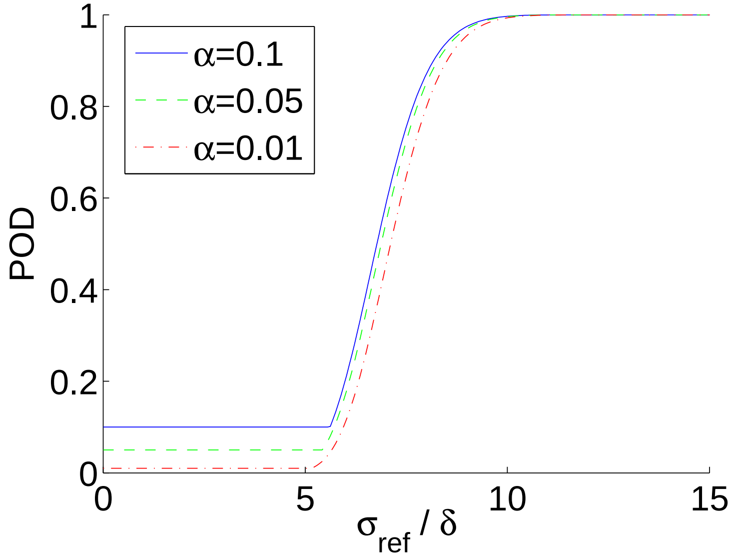
• The false alarm rate (FAR) of the test $S > s_\alpha$ is $\alpha$.

• The probability of detection (POD) of the test $S > s_\alpha$ is

$$\text{POD} = \max \left\{ \Phi\left( \frac{\frac{\sigma_{\rm ref}^2}{2\delta^2} - s_\alpha}{\frac{\sigma_{\rm ref}}{\delta}} \right), \alpha \right\}$$

where $\Phi$ is the standard Gaussian cumulative distribution function.

• By Neyman Pearson lemma, the test $S > s_\alpha$ maximizes the POD for a given FAR $\alpha$.




The migration-based test becomes powerful when $\sigma_{\mathrm{ref}} > \sqrt{2}\delta \ln^{1/2}(|\Omega|/V_c)$.

Remember: $\sigma_{\mathrm{noise}} = \sqrt{n}\delta$.

## Comparison between SV-based test and migration-based test

- The migration-based test becomes powerful when $\sigma_{\text{ref}} > \sqrt{2}\delta \ln^{1/2}(|\Omega|/V_c)$.
- The SV-based test becomes powerful when $\sigma_{\text{ref}} > \delta\sqrt{n}$.
- Therefore the migration-based test is more (resp. less) powerful than the SV-based test when $n >$ (resp. $<$) $2\ln(|\Omega|/V_c)$.
- In practice, we usually have $n > 2\ln(|\Omega|/V_c)$, and therefore the migration-based test is more efficient than the SV-based test.
- The SV-based test is simpler to implement than the migration-based test.

# Optimal migration for localization

- Given the parameters $\boldsymbol{x}$ (the reflector location), $\sigma$ (the reflector singular value), and $\delta^2$ (the noise variance), the likelihood of the observations $\mathbf{A}$ is proportional to

$$l_0\big(\mathbf{A} \mid \boldsymbol{x}, \sigma, \delta^2\big) = \frac{1}{\delta^{n^2+n}} \exp\Big( - \frac{\big\|\mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x}) \boldsymbol{g}(\boldsymbol{x})^T \big\|_F^2}{2\delta^2}\Big)$$

- Prior information on $(\boldsymbol{x}, \sigma, \delta^2)$: nothing (Jeffrey prior).

- Bayes theorem: Given the observations $\mathbf{A}$, the likelihood function of the parameters $\boldsymbol{x}$ (the reflector location), $\sigma$ (the reflector singular value), and $\delta^2$ (the noise variance) is proportional to

$$l_0\big(\boldsymbol{x}, \sigma, \delta^2 \mid \mathbf{A}\big) = \frac{1}{\delta^{n^2+n+1}} \exp\Big( - \frac{\big\|\mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x}) \boldsymbol{g}(\boldsymbol{x})^T \big\|_F^2}{2\delta^2}\Big)$$

- The maximum likelihood estimate $\hat{\boldsymbol{x}}$ of $\boldsymbol{x}$ (and the nuisance parameters $\delta^2$ and $\sigma$) is found by maximizing the likelihood function. Main result:

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\operatorname{argmax}} \, |\mathcal{I}_{\rm RT}(\boldsymbol{x})|^2$$

$\hookrightarrow$ Reverse-Time migration is the best method in the presence of additive noise.

Proof. Given the observations $\mathbf{A}$, the likelihood function of the parameters $\boldsymbol{x}$ (the reflector location), $\sigma$ (the reflector singular value), and $\delta^2$ (the noise variance) is proportional to

$$l_0\big(\boldsymbol{x}, \sigma, \delta^2 \mid \mathbf{A}\big) = \frac{1}{\delta^{n^2+n+1}} \exp\Big( - \frac{\big\|\mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})^T\big\|_F^2}{2\delta^2}\Big)$$

The maximum likelihood estimate $\hat{\boldsymbol{x}}$ of $\boldsymbol{x}$ (and the nuisance parameters $\delta^2$ and $\sigma$) is found by maximizing the likelihood function:

$$\big(\hat{\boldsymbol{x}}, \hat{\sigma}, \hat{\delta}^2\big) = \underset{\boldsymbol{x},\sigma,\delta^2}{\operatorname{argmax}}\, l_0\big(\boldsymbol{x}, \sigma, \delta^2 \mid \mathbf{A}\big).$$

We first eliminate $\delta^2$ by requiring

$$\frac{\partial l_0\big(\boldsymbol{x}, \sigma, \delta^2 \mid \mathbf{A}\big)}{\partial \delta} = 0.$$

This gives

$$\hat{\delta}^2 = \frac{\|\mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})^T\|_F^2}{n^2+n+1},$$

and the likelihood ratio is then proportional to

$$l_0\big(\boldsymbol{x}, \sigma, \hat{\delta}^2 \mid \mathbf{A}\big) \simeq \big\|\mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})^T\big\|_F^{-(n^2+n+1)/2}.$$

Since $\mathbf{A}$ is complex symmetric it admits a symmetric SVD: there exist unitary vectors $\boldsymbol{v}^{(l)}$ and nonnegative numbers $\sigma^{(l)}$ (the singular values) such that

$$\mathbf{A} = \sum_{l=1}^{n} \sigma^{(l)} \boldsymbol{v}^{(l)} \boldsymbol{v}^{(l)T}$$

We can write

$$\left\| \mathbf{A} - \sigma \boldsymbol{g}(\boldsymbol{x}) \boldsymbol{g}(\boldsymbol{x})^T \right\|_F^2 = \left\| \tilde{\boldsymbol{v}} - \sigma \tilde{\boldsymbol{g}}(\boldsymbol{x}) \right\|_2^2$$

for $\tilde{\boldsymbol{v}} = \sum_{l=1}^{n} \sigma^{(l)} \boldsymbol{v}^{(l)} \otimes \boldsymbol{v}^{(l)}$ and $\tilde{\boldsymbol{g}}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) \otimes \boldsymbol{g}(\boldsymbol{x})$. Using $\|\tilde{\boldsymbol{g}}(\boldsymbol{x})\|_2 = \|\boldsymbol{g}(\boldsymbol{x})\|^2 = 1$,

$$\hat{\sigma} = \underset{\sigma}{\operatorname{argmin}} \|\tilde{\boldsymbol{v}} - \sigma \tilde{\boldsymbol{g}}(\boldsymbol{x})\|_2^2 = \overline{\tilde{\boldsymbol{g}}(\boldsymbol{x})}^T \tilde{\boldsymbol{v}}$$

Therefore the estimate $\hat{\boldsymbol{x}}$ derives from maximizing the MUSIC-type function

$$\hat{\boldsymbol{x}} \qquad = \underset{\boldsymbol{x}}{\operatorname{argmin}} \left\| \tilde{\boldsymbol{v}} - \left( \overline{\tilde{\boldsymbol{g}}(\boldsymbol{x})}^T \tilde{\boldsymbol{v}} \right) \tilde{\boldsymbol{g}}(\boldsymbol{x}) \right\|_2^2$$

Note however that $\hat{\boldsymbol{x}}$ is not the maximizer of the MUSIC functional since all singular vectors (weighted by the singular values) contribute to $\tilde{\boldsymbol{v}}$. We have in fact

$$\begin{aligned} \left\| \tilde{\boldsymbol{v}} - \left( \overline{\tilde{\boldsymbol{g}}(\boldsymbol{x})}^T \tilde{\boldsymbol{v}} \right) \tilde{\boldsymbol{g}}(\boldsymbol{x}) \right\|_2^2 = \|\tilde{\boldsymbol{v}}\|_2^2 - \left| \overline{\tilde{\boldsymbol{g}}(\boldsymbol{x})}^T \tilde{\boldsymbol{v}} \right|^2 &= \|\tilde{\boldsymbol{v}}\|_2^2 - \left| \sum_{l=1}^{n} \sigma^{(l)} \left( \overline{\boldsymbol{g}(\boldsymbol{x})}^T \boldsymbol{v}^{(l)} \right)^2 \right|^2 \\ &= \|\mathbf{A}\|_F^2 - |\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x})|^2 \end{aligned}$$

This gives $\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\operatorname{argmin}} \left( \|\mathbf{A}\|_F^2 - |\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x})|^2 \right)$.

## Optimal migration for localization

Important remark:

The Bayesian localization scheme can be used once the detection test has passed.

Bayesian analysis is powerful but depends on the prior.

Here the prior is: there exists a reflector.

## Statistical analysis of localization error

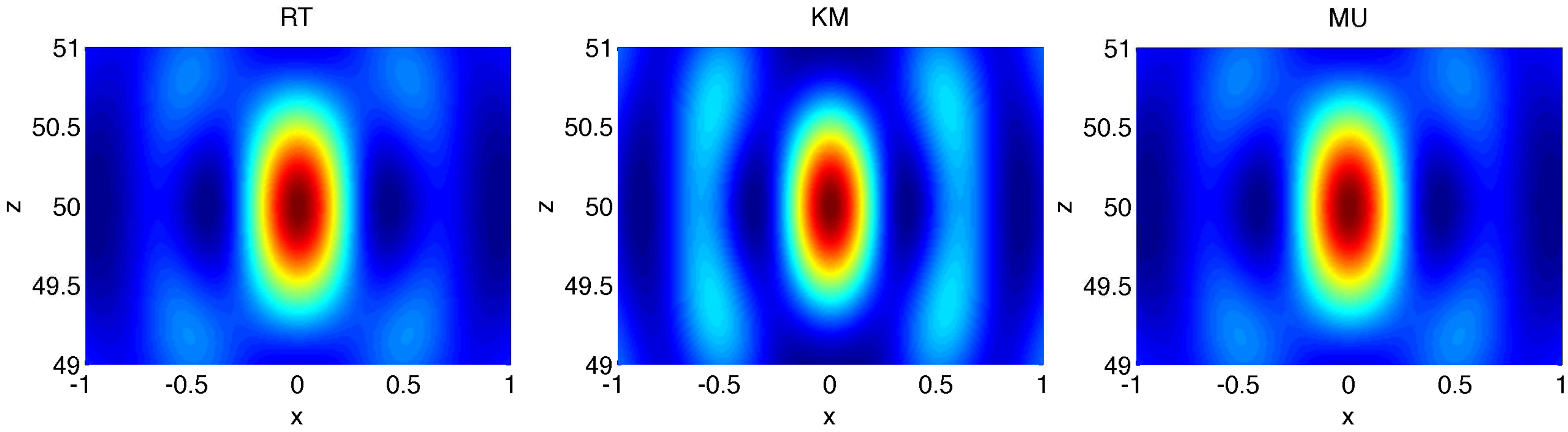
Optimal estimator:

$$\hat{\boldsymbol{x}} = \underset{\boldsymbol{x}}{\operatorname{argmax}} \left| \mathcal{I}_{\mathrm{RT}}(\boldsymbol{x}) \right|^2$$

To leading order in $\delta/\sigma_{\mathrm{ref}}$, the estimator $\hat{\boldsymbol{x}}$ is unbiased and its covariance matrix is

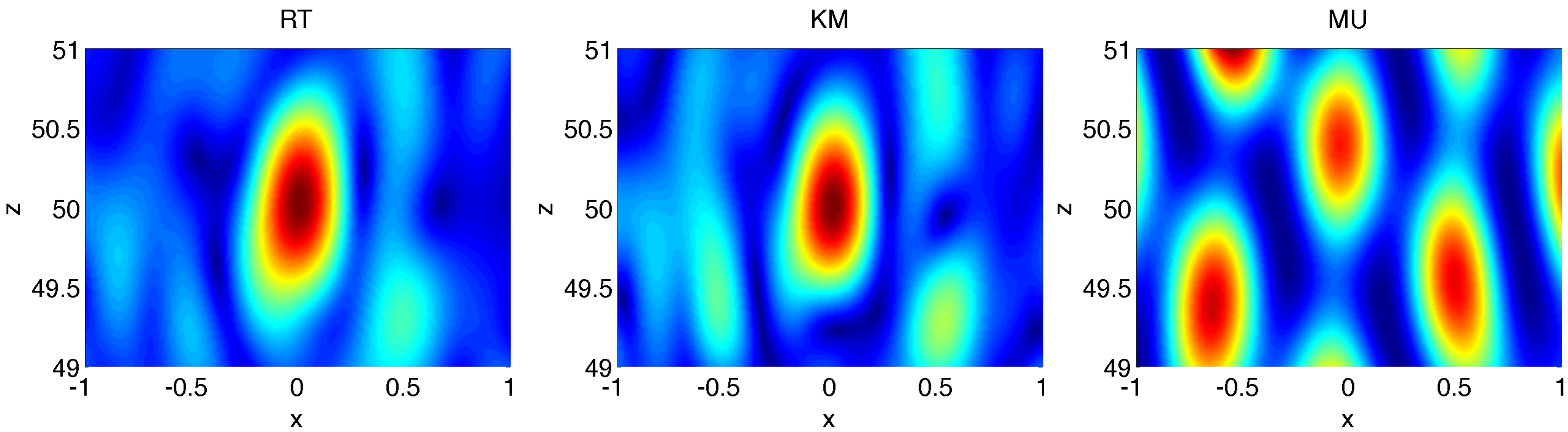
$$\mathbb{E}\big[(\hat{\boldsymbol{x}} - \boldsymbol{x}_{\mathrm{ref}})(\hat{\boldsymbol{x}} - \boldsymbol{x}_{\mathrm{ref}})^T\big] = \frac{\delta^2}{\sigma_{\mathrm{ref}}^2} \mathbf{H}^{-1}.$$

Full aperture: $\mathbb{E}\big[(\hat{x}_j - x_{\mathrm{ref},j})^2\big] = \frac{\delta^2}{\sigma_{\mathrm{ref}}^2} \frac{3}{2\pi^2} \lambda_0^2, \quad j = 1, \ldots, 3.$

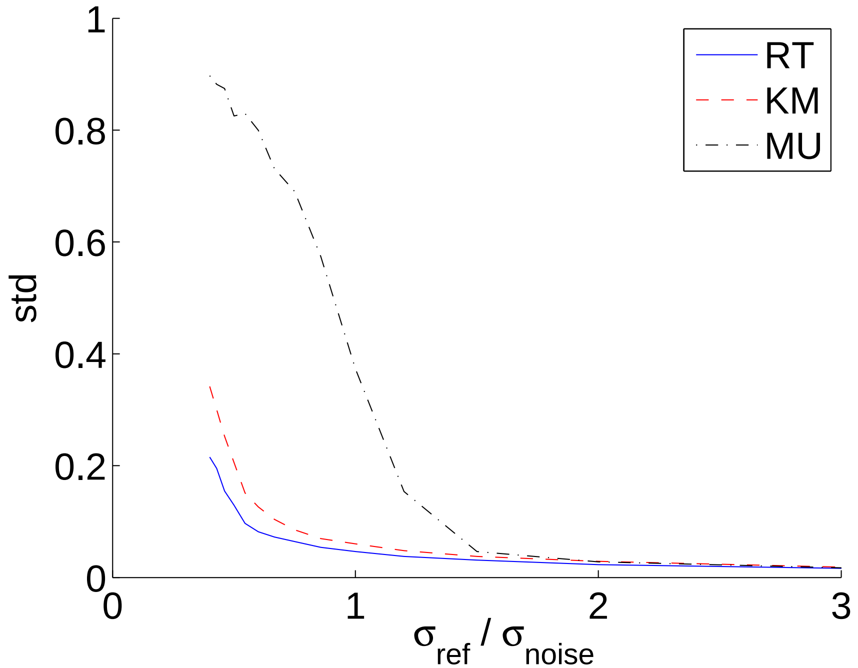



Imaging functionals in the absence of noise.

Left: reverse-time migration, center: Kirchhoff migration, right: MUSIC.




Imaging functionals in the presence of noise.




Standard deviation of the estimated reflector location obtained with three different imaging methods (here $\lambda_0 = 1$, $n = 100$).

# Conclusions

- Statistical and stochastic tools:

- Random matrix theory: description of the distribution of the singular values of the array response matrix in a noisy environment.
- Extreme value theory: description of the speckle pattern obtained by migration (backpropagation) of the array data.
- Bayesian analysis: optimal localization of a target.

- Optimal tests involve non-Gaussian distributions (Gumbel, Tracy-Widom).

- What is important is the structure of the response matrix (symmetric, Hermitian, Hankel, Toeplitz, ...), not much the marginal distribution of the entries.

- It is possible to extend the results to

- several reflectors, cracks, or inclusions,
- other noisy environments (random medium in the single-scattering regime for instance).

The main hypothesis is that the information is low-rank while the noise is high-rank.