Degenerate nonlinear parabolic-hyperbolic equations and their finite volume approximation

B. Andreianov[1]

based on joint work with
M. Bendahmane & K.H. Karlsen

[1]Laboratoire de Mathématiques CNRS UMR 6623
Université de Franche-Comté
Besançon, France

CEMRACS'11

CIRM, Luminy, July 2011

Plan of the talk

INTRODUCTION TO DEGENERATE NONLINEAR CONVECTION-DIFFUSION PROBLEMS AND THEIR FINITE VOLUME APPROXIMATION

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \operatorname{div}\,[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \quad \text{in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian, i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus '83; Otto '96; Bénilan, Wittbold '96 and '96; Carrillo '99; Ammar, Wittbold '03; Andr., Bendahmane, Karlsen, Ouaro '09

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \operatorname{div}\,[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \quad \text{in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian, i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus '83; Otto '96; Bénilan, Wittbold '96 and '96; Carrillo '99; Ammar, Wittbold '03; Andr., Bendahmane, Karlsen, Ouaro '09

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \mathrm{div}\,[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \ \text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian,
i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.

· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.

We take homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus '83; Otto '96; Bénilan, Wittbold '96 and '96; Carrillo '99; Ammar, Wittbold '03; Andr., Bendahmane, Karlsen, Ouaro '09

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \operatorname{div}[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \ \text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian,
i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus ’83; Otto ’96; Bénilan, Wittbold ’96 and ‘96; Carrillo ’99; Ammar, Wittbold ’03; Andr., Bendahmane, Karlsen, Ouaro ’09

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \operatorname{div}\, [\vec{F}(v) - \vec{a_0}(\nabla w)] = f \ \text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian,
i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T) \times \partial\Omega$.

Theory:
Alt, Luckhaus ’83; Otto ’96; Bénilan, Wittbold ’96 and ‘96; Carrillo ’99; Ammar, Wittbold ’03; Andr., Bendahmane, Karlsen, Ouaro ’09

Triply nonlinear degenerate parabolic problems...

Applications ??
Mathematical models for fluid dynamics, porous media, sedimentation, Stefan and Hele-Shaw problems involve PDEs like

$$u = b(v), w = A(v),$$
$$u_t + \operatorname{div}\,[\vec{F}(v) - \vec{a_0}(\nabla w)] = f \;\text{ in } Q = (0,T)\times\Omega$$

with $b(\cdot), A(\cdot)$ continuous nonstrictly increasing on $\mathbb{R}$,
with a continuous convection flux $\vec{F}(\cdot)$
and with $\vec{a_0} : \mathbb{R}^N \to \mathbb{R}^N$ of Leray-Lions type : the p-laplacian,
i.e., $\vec{a_0}(\vec{\xi}) = |\vec{\xi}|^{p-2}\vec{\xi}$, is a typical example.
· If $b(\cdot)$ may be constant on intervals: elliptic-parabolic
· If $A(\cdot)$ may be constant on intervals: parabolic-hyperbolic.
We take homogeneous Dirichlet boundary condition on $(0,T)\times\partial\Omega$.

Theory:
Alt, Luckhaus ’83; Otto ’96; Bénilan, Wittbold ’96 and ‘96; Carrillo ’99; Ammar, Wittbold ’03; Andr., Bendahmane, Karlsen, Ouaro ’09

Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of "approximate solutions" $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in "simpler" problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In "harder" problems, one has to treat simultaneously Steps 2+3 : "compensated compactness", entropy-process solutions...

Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of “approximate solutions” $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in “simpler” problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In “harder” problems, one has to treat simultaneously Steps 2+3 : “compensated compactness”, entropy-process solutions...

Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of “approximate solutions” $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in “simpler” problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In “harder” problems, one has to treat simultaneously Steps 2+3 : “compensated compactness”, entropy-process solutions...

Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of “approximate solutions” $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in “simpler” problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In “harder” problems, one has to treat simultaneously Steps 2+3 : “compensated compactness”, entropy-process solutions...

Convergence of approximations for degenerate parabolic problems...

Theoretical numerical analysis ?? Arguments for convergence of numerical approximations are the same as used for existence proof !

Namely:

1. Construct a sequence of “approximate solutions” $(v_h)_h$: e.g., finite volume approximation !
2. Create an accumulation point v for the sequence (compactness arguments)
3. Prove that the accumulation point is a solution of the equation $\equiv$ pass to the limit in nonlinearities: $b(v_h) \to b(v)$? $\vec{F}(v_h) \to \vec{F}(v)$? $\vec{a_0}(\nabla A(v_h)) \to \vec{a_0}(\nabla A(v))$?

NB: Steps 2 and 3 are separated in “simpler” problems :

– compactness of Step 2 uses functional analysis arguments: bounds in Sobolev spaces, compactness criteria...
– identification of nonlinear limits may use much of the PDE structure.

In “harder” problems, one has to treat simultaneously Steps 2+3 : “compensated compactness”, entropy-process solutions...

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples : Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen...
  DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov...
  gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle) + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples : Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen... DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov... gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle) + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a_0}(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit “key features” of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different “discrete duality” schemes, examples : Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen... DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov... gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle) + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples :
  Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen...
  DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov...
  gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle) + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples :
  Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen...
  DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov...
  gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle)
  + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

Hint on discretization : we often need that the discrete problem inherit "key features" of the continuous problem. Examples:

- coercivity, monotonicity of the nonlinear elliptic operator

  Preserved by different "discrete duality" schemes, examples :
  Co-Volume schemes Walkington; Afif, Amaziane; Handlovičová, Mikula et al.; Andreianov, Bendahmane, Karlsen...
  DDFV schemes Hermeline; Domelevo, Omnès; Andreianov, Boyer, Hubert; Pierre, Coudière, Bendahmane, Karlsen, Hubert, Manzini, Krell... mimetic schemes Brezzi, Lipnikov, Shashkov...
  gradient schemes : SUSHI,... Eymard, Gallouët, Herbin...

- entropy inequalities – order preservation – L^1 contraction for the convection-diffusion operator.

  Preserved by discretization of $\operatorname{div} \vec{F}(v)$ with monotone two-point finite volume schemes (e.g. Eymard, Gallouët, Herbin; Vovelle)
  + DDFV/Co-Volume/... discretization of the nonlinear elliptic operator $-\operatorname{div} \vec{a}_0(\nabla A(v))$ on orthogonal meshes .

Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

Finite volume approximation of nonlinear degenerate parabolic problems...

- L^1 contractivity of the solution semigroup
  Preserved by time-implicit Euler scheme (if previous item is OK).

NB: Structure-preservation: very nice for mathematical analysis. Efficiency ??? It depends...

When such structure-preserving schemes are used then in order to study convergence it is enough to produce "discrete" versions of "continuous" arguments for Steps 1 – 2 – 3 .

Then, the steps for construction of a convergent scheme are :

- understand the key structure properties of the continuous equ^n
- cook up meshes, discrete operators and discrete calculus tools that are "compatible" with the above structure

Let us concentrate on the following issues :

- The ideas of the arguments, at the continuous level
- A glimpse on how the ideas work, also at the discrete level
- Focus on difficulties that are proper to the discrete framework.

Theoretical foundations

Theoretical framework for elliptic-parabolic-hyperbolic problems...

Theoretical setting : entropy solutions + Leray-Lions framework. Key ideas:
Leray & Lions '65 – Alt & Luckhaus '83 ; Kruzhkov '69 – Carrillo '99

NB: Parallel theories and generalizations, not discussed here :

– semigroup solutions : Crandall, Bénilan, Carrillo & Wittbold
– kinetic solutions (quasilinear diffusion !) : Perthame, Chen & Perthame
– renormalized solutions : Murat & Lions, Carrillo & Wittbold, Ammar & Wittbold, Blanchard & Porretta, Bendahmane & Karlsen
– entropy (Bénilan et al.) solutions : Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez, Andreu-Vaillo & Igbida & Mazón & Toledo .

Nice features of the solution theory:

- well-posedness for L^∞ data u_0
- order-preservation : $u_0 \leqslant \hat{u}_0$ and $f \leqslant \hat{f}$ implies $u(t,\cdot) \leqslant \hat{u}(t,\cdot)$
- consequently, maximum principle : $\sup u(t,\cdot) \leqslant \sup u_0^+ + \int_0^t \sup f^+(\tau,\cdot)d\tau$
- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1}d\tau.$
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

Theoretical framework for elliptic-parabolic-hyperbolic problems...

Theoretical setting : entropy solutions + Leray-Lions framework. Key ideas:
Leray & Lions '65 – Alt & Luckhaus '83 ; Kruzhkov '69 – Carrillo '99

NB: Parallel theories and generalizations, not discussed here :

– semigroup solutions : Crandall, Bénilan, Carrillo & Wittbold
– kinetic solutions (quasilinear diffusion !) : Perthame, Chen & Perthame
– renormalized solutions : Murat & Lions, Carrillo & Wittbold, Ammar & Wittbold, Blanchard & Porretta, Bendahmane & Karlsen
– entropy (Bénilan et al.) solutions : Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez, Andreu-Vaillo & Igbida & Mazón & Toledo .

Nice features of the solution theory:

- well-posedness for L^∞ data u_0
- order-preservation : $u_0 \leqslant \hat{u}_0$ and $f \leqslant \hat{f}$ implies $u(t,\cdot) \leqslant \hat{u}(t,\cdot)$
- consequently, maximum principle : $\sup u(t,\cdot) \leqslant \sup u_0^+ + \int_0^t \sup f^+(\tau,\cdot)d\tau$
- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1}d\tau$.
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

Theoretical framework for elliptic-parabolic-hyperbolic problems...

Theoretical setting : entropy solutions + Leray-Lions framework. Key ideas:
Leray & Lions '65 – Alt & Luckhaus '83 ; Kruzhkov '69 – Carrillo '99

NB: Parallel theories and generalizations, not discussed here :

– semigroup solutions : Crandall, Bénilan, Carrillo & Wittbold
– kinetic solutions (quasilinear diffusion !) : Perthame, Chen & Perthame
– renormalized solutions : Murat & Lions, Carrillo & Wittbold, Ammar & Wittbold, Blanchard & Porretta, Bendahmane & Karlsen
– entropy (Bénilan et al.) solutions : Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez, Andreu-Vaillo & Igbida & Mazón & Toledo .

Nice features of the solution theory:

- well-posedness for L^∞ data u_0
- order-preservation : $u_0 \leqslant \hat{u}_0$ and $f \leqslant \hat{f}$ implies $u(t,\cdot) \leqslant \hat{u}(t,\cdot)$
- consequently, maximum principle : $\sup u(t,\cdot) \leqslant \sup u_0^+ + \int_0^t \sup f^+(\tau,\cdot)d\tau$
- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1} d\tau$.
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

Theoretical framework for elliptic-parabolic-hyperbolic problems...

Theoretical setting : entropy solutions + Leray-Lions framework. Key ideas:
Leray & Lions '65 – Alt & Luckhaus '83 ; Kruzhkov '69 – Carrillo '99

NB: Parallel theories and generalizations, not discussed here :

– semigroup solutions : Crandall, Bénilan, Carrillo & Wittbold
– kinetic solutions (quasilinear diffusion !) : Perthame, Chen & Perthame
– renormalized solutions : Murat & Lions, Carrillo & Wittbold, Ammar & Wittbold, Blanchard & Porretta, Bendahmane & Karlsen
– entropy (Bénilan et al.) solutions : Bénilan & Boccardo & Gallouët & Gariepy & Pierre & Vázquez, Andreu-Vaillo & Igbida & Mazón & Toledo .

Nice features of the solution theory:

- well-posedness for L^∞ data u_0
- order-preservation : $u_0 \leqslant \hat{u}_0$ and $f \leqslant \hat{f}$ implies $u(t,\cdot) \leqslant \hat{u}(t,\cdot)$
- consequently, maximum principle : $\sup u(t,\cdot) \leqslant \sup u_0^+ + \int_0^t \sup f^+(\tau,\cdot)d\tau$
- L^1-contraction : $\|u-\hat{u}\|_{L^1}(t) \leqslant \|u_0-\hat{u}_0\|_{L^1} + \int_0^t \|f(t,\cdot)-\hat{f}(t,\cdot)\|_{L^1} d\tau$.
- energy control : an *a priori* estimate on $\int_0^T\int_\Omega |\nabla w|^p$.

There is more: stability wrt perturbations of nonlinearities (Karlsen & Risebro; Chen & Karlsen ; Andr. & Bendahmane & Karlsen & Ouaro); some "regularity" such as existence of strong boundary traces of v (Panov),...

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v) - b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\Longrightarrow$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\Longrightarrow$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\Longrightarrow$ doubling of variables following Kruzhkov $(\mathrm{div}\,\vec{F}(v))$ and Carrillo $(\mathrm{div}\,\vec{a}_0(\nabla w))$

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v)-b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\Longrightarrow$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\Longrightarrow$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\Longrightarrow$ doubling of variables following Kruzhkov $(\mathrm{div}\,\vec{F}(v))$ and Carrillo $(\mathrm{div}\,\vec{a}_0(\nabla w))$

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v)-b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\implies$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\implies$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\implies$ doubling of variables following Kruzhkov $(\mathrm{div}\,\vec{F}(v))$ and Carrillo $(\mathrm{div}\,\vec{\mathfrak{a}}_0(\nabla w))$

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v) - b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\Longrightarrow$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\Longrightarrow$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\Longrightarrow$ doubling of variables following Kruzhkov $(\mathrm{div}\,\vec{F}(v))$ and Carrillo $(\mathrm{div}\,\vec{a}_0(\nabla w))$

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T; W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v) - b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\Longrightarrow$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\Longrightarrow$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\Longrightarrow$ doubling of variables following Kruzhkov ($\mathrm{div}\,\vec{F}(v)$) and Carrillo ($\mathrm{div}\,\vec{a}_0(\nabla w)$)

Theoretical framework for parabolic-hyperbolic problems...

Estimates easy to get (at least, formally) for approximate solutions:

- (existence) *a priori* bound on $w_h = A(v_h)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ (energy estimate) and weak compactness in L^p for $\nabla w_h = \nabla A(v_h)$
- (existence) consequently, "strong compactness in space" for $w_h = A(v_h)$ (Fréchet-Kolmogorov theorem)
- (existence) with the help of the evolution equation, "strong compactness in time" for $u_h = b(v_h)$ (Fréchet-Kolmogorov)
- (uniqueness) very formally, given two solutions $v, \hat{v}$, multiply $Eq(v) - Eq(\hat{v})$ by $\mathrm{sign}^+(v-\hat{v})$; get $\int_\Omega (b(v) - b(\hat{v}))^+(t) \leqslant 0$.
- (existence) Consequently, *a priori* L^∞ bound on $u_h = b(v_h)$ (by comparison with constant solutions)

Difficulties and hints to resolve them :

- (existence ?) No classical solutions $\Longrightarrow$ weak formulation
- (uniqueness ?) Non-uniqueness of weak solutions $\Longrightarrow$ selection by entropy inequalities (thus, entropy weak formulation)
- (uniqueness ?) Justify the formal calculation with "$\mathrm{sign}^+(v-\hat{v})$" test function $\Longrightarrow$ doubling of variables following Kruzhkov $(\mathrm{div}\, \vec{F}(v))$ and Carrillo $(\mathrm{div}\, \vec{\mathfrak{a}}_0(\nabla w))$

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont[d]):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont^d):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont[d]):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont^{d}):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont[d]):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont[d]):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Theoretical framework for parabolic-hyperbolic problems...

Difficulties and hints to resolve them (cont[d]):

- (boundary conditions ?) Many delicate issues $\Longrightarrow$ Bardos-LeRoux-Nédélec condition, Otto boundary entropies, weak or strong boundary traces, Carrillo's tricks (skipped in this talk)
- (existence ?) "Problem of time compactness" due to elliptic degeneracy $\Longrightarrow$ "structure conditions", monotone sequences of approx. solutions (skipped): Bénilan & Wittbold, Ammar & Wittbold, Andr. & Wittbold
- (existence ?) "Problem of space compactness" due to:
  - non-linearity of the diffusion operator (strong compactness of $\nabla w_h = \nabla A(v_h)$ needed) $\Longrightarrow$ Minty-Browder "compactification" trick (use of the monotonicity of the Leray-Lions operator)
  - hyperbolic degeneracy (compactness for $w_h = A(v_h)$ does not preclude oscillations "in the flat regions of $A(\cdot)$") $\Longrightarrow$ measure-valued or entropy-process solutions + compactification arguments (Tartar ; DiPerna ; Panov ; Gallouët et al.)
  - interactions ???: do Minty-Browder and entropy-process live well together ? $\Longrightarrow$ a chain rule permits to "hide" the convection term

We are specifically interested in the space discretization therefore we skip difficulties due to elliptic degeneracy: we take $b = Id$ and thus $u = b(v) \equiv v$.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.
An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0,T; W_0^{1,p}(\Omega))$;
- for all pairs $(c,\psi) \in \mathbb{R}^\pm \times \mathcal{D}([0,T)\times\overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c,\psi) \in \mathbb{R} \times \mathcal{D}([0,T)\times\Omega)$, $\psi \geqslant 0$,

$$\int_0^T\int_\Omega\Big(\eta_c^\pm(u)\partial_t\psi + \vec{q}_c^{\,\pm}(u)\cdot\nabla\psi - k(\nabla w)\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot\nabla\psi\Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{q}_c^{\,\pm})'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)
If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{q}_c^{\,\pm}(u(\cdot))$, by $\int_0^1 \vec{q}_c^{\,\pm}(\mu(\cdot;\alpha))\,d\alpha$

– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.

An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0,T;W_0^{1,p}(\Omega))$;
- for all pairs $(c,\psi) \in \mathbb{R}^\pm \times \mathcal{D}([0,T)\times\overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c,\psi) \in \mathbb{R}\times\mathcal{D}([0,T)\times\Omega)$, $\psi \geqslant 0$,

$$\int_0^T\!\!\int_\Omega \Big(\eta_c^\pm(u)\partial_t\psi + \vec{q}_c^{\,\pm}(u)\cdot\nabla\psi - k(\nabla w)\,\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot\nabla\psi\Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{q}_c^{\,\pm})'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{q}_c^{\,\pm}(u(\cdot))$, by $\int_0^1 \vec{q}_c^{\,\pm}(\mu(\cdot;\alpha))\,d\alpha$

– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an "ordinary" entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.
An entropy solution of our problem is a function $u : Q = (0, T) \times \Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0, T; W_0^{1,p}(\Omega))$;
- for all pairs $(c, \psi) \in \mathbb{R}^\pm \times \mathcal{D}([0, T) \times \overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c, \psi) \in \mathbb{R} \times \mathcal{D}([0, T) \times \Omega)$, $\psi \geqslant 0$,

$$\int_0^T \int_\Omega \Big(\eta_c^\pm(u)\partial_t\psi + \vec{q}_c^{\,\pm}(u)\cdot \nabla\psi - k(\nabla w)\,\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot \nabla\psi \Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r - c)^\pm$ (semi-Kruzhkov entropies), $(\vec{q}_c^{\,\pm})'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)
If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{q}_c^{\,\pm}(u(\cdot))$, by $\int_0^1 \vec{q}_c^{\,\pm}(\mu(\cdot;\alpha))\,d\alpha$
– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.
An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0,T; W_0^{1,p}(\Omega))$;
- for all pairs $(c,\psi) \in \mathbb{R}^\pm \times \mathcal{D}([0,T)\times\overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c,\psi) \in \mathbb{R}\times\mathcal{D}([0,T)\times\Omega)$, $\psi \geqslant 0$,

$$\int_0^T\!\!\int_\Omega \Big(\eta_c^\pm(u)\partial_t\psi + \vec{\mathfrak{q}}_c^\pm(u)\cdot\nabla\psi - k(\nabla w)\,\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot\nabla\psi\Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{\mathfrak{q}}_c^\pm)'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{\mathfrak{q}}_c^\pm(u(\cdot))$, by $\int_0^1 \vec{\mathfrak{q}}_c^\pm(\mu(\cdot;\alpha))\,d\alpha$

– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an "ordinary" entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{a}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.
An entropy solution of our problem is a function $u : Q = (0, T) \times \Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0, T; W_0^{1,p}(\Omega))$;
- for all pairs $(c, \psi) \in \mathbb{R}^\pm \times \mathcal{D}([0, T) \times \overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c, \psi) \in \mathbb{R} \times \mathcal{D}([0, T) \times \Omega)$, $\psi \geqslant 0$,

$$\int_0^T \int_\Omega \Big(\eta_c^\pm(u) \partial_t \psi + \vec{q}_c^{\,\pm}(u) \cdot \nabla\psi - k(\nabla w) \nabla \widetilde{A}_{(\eta_c^\pm)'}(w) \cdot \nabla\psi \Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0, \cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r - c)^\pm$ (semi-Kruzhkov entropies), $(\vec{q}_c^{\,\pm})'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{q}_c^{\,\pm}(u(\cdot))$, by $\int_0^1 \vec{q}_c^{\,\pm}(\mu(\cdot;\alpha))\,d\alpha$

– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0, 1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

Notion of entropy solution

Definition (entropy solution)

Assume $\vec{\mathfrak{a}}_0(\vec{\xi}) = k(\vec{\xi})\vec{\xi}$.

An entropy solution of our problem is a function $u : Q = (0,T)\times\Omega \to \mathbb{R}$,

- $u \in L^\infty(Q)$ and $w = A(u) \in L^p(0,T;W_0^{1,p}(\Omega))$;
- for all pairs $(c,\psi) \in \mathbb{R}^\pm \times \mathcal{D}([0,T)\times\overline{\Omega})$, $\psi \geqslant 0$, and also for all pairs $(c,\psi) \in \mathbb{R}\times\mathcal{D}([0,T)\times\Omega)$, $\psi \geqslant 0$,

$$\int_0^T\int_\Omega\Big(\eta_c^\pm(u)\partial_t\psi + \vec{\mathfrak{q}}_c^\pm(u)\cdot\nabla\psi - k(\nabla w)\,\nabla\widetilde{A}_{(\eta_c^\pm)'}(w)\cdot\nabla\psi\Big) + \int_\Omega \eta_c^\pm(u_0)\psi(0,\cdot) \geq 0.$$

Here $\eta_c^\pm(r) = (r-c)^\pm$ (semi-Kruzhkov entropies), $(\vec{\mathfrak{q}}_c^\pm)'(r) = (\eta_c^\pm)'(r)\vec{F}(r)$ (chain rule) and $A_\theta(r) = \theta(r)A'(r)$ and $\widetilde{A}_\theta(A(r)) = A_\theta(r)$ (another chain rule)

If we replace :

– $\eta_c^\pm(u)(\cdot)$, by $\int_0^1 \eta_c^\pm(\mu(\cdot;\alpha))\,d\alpha$ and $\vec{\mathfrak{q}}_c^\pm(u(\cdot))$, by $\int_0^1 \vec{\mathfrak{q}}_c^\pm(\mu(\cdot;\alpha))\,d\alpha$

– and if $A(\mu(\cdot;\alpha)) \equiv w$ for all $\alpha \in (0,1)$

then we get the definition of an entropy-process solution μ .

Theorem (uniqueness and reduction of an entropy-process solution)

Entropy-process solution is unique and it is an “ordinary” entropy solution.

MESHES, DISCRETE OPERATORS AND THE SCHEME

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell .

Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\operatorname{div}_c \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\operatorname{div}^{\mathfrak{T}} \vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\operatorname{div}^{\mathfrak{T}} \vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}} \cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell . Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\,\vec{a}_0(\,\nabla^{\mathfrak{T}}A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}}\,\vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}}\cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}}w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell . Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\mathfrak{a}}_0(\,\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}}\cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell . Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\mathfrak{a}}_0(\,\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}}\cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell . Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

- discrete convection operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\mathfrak{a}}_0(\,\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}}\cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$: Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

We are given a mesh $\mathfrak{T}$ of Ω and one degree of freedom per mesh cell . Finite volume methods are based upon approximation of fluxes on the interfaces between cells; we include each interface in a diamond , with diamond mesh $\mathfrak{D}$ that also forms a partition of Ω.

In our finite volume setting, the following operators are used :

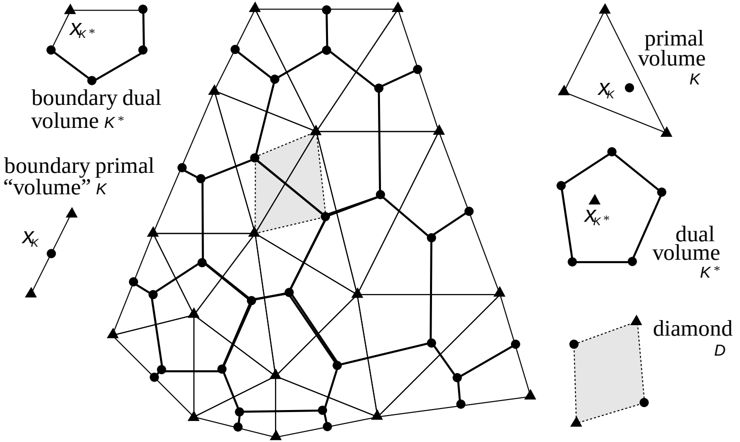
- discrete convection operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$, it applies to constant-per-cell scalar functions and gives constant-per-cell scalar functions
- discrete diffusion operator $\mathrm{div}^{\mathfrak{T}}\, \vec{\mathfrak{a}}_0(\,\nabla^{\mathfrak{T}} A(\cdot))$, where
  - the discrete divergence operator $\mathrm{div}^{\mathfrak{T}}\,\vec{\cdot}$ applies to a const.-per-diamond vector field and gives a constant-per-cell scalar
  - the discrete gradient operator $\vec{\nabla}^{\mathfrak{T}}\cdot$ applies to a constant-per-cell scalar function and gives a constant per diamond vector field

NB: Because of the nonlinearity, it is not enough to define normal components of the discrete gradient on interfaces !

There are several strategies to reconstruct the full gradient $\vec{\nabla}^{\mathfrak{T}} w^{\mathfrak{T}}$:
Co-Volume schemes, DDFV schemes, SUSHI... Recent unifying framework: Gradient Schemes (Eymard & Guichard & Herbin '11). Other approaches: e.g., Andr. & Boyer & Hubert on cartesian meshes; mimetic finite difference schemes Brezzi & Lipnikov & Shashkov ,...

Finite volume meshes and operators...

Let us describe one meshing+operators strategy: “Discrete Duality FV”. The 2D idea is due to Hermeline and to Domelevo & Omnès . One starts with a usual mesh (called “primal”) and uses both center and vertex unknowns.



Finite volume meshes and operators...

The space of discrete functions $w^{\mathfrak{T}} = \left((w_K)_K \,;\, (w_{K^*})_{K^*} \right)$ is denoted by $\mathbb{R}^{\mathfrak{T}}$, for functions zero on the boundary we use $\mathbb{R}_0^{\mathfrak{T}}$.
The set of discrete fields $(\vec{\mathcal{F}}_D)_D$ is denoted $(\mathbb{R}^d)^{\mathfrak{D}}$.

On spaces $\mathbb{R}^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$, we introduce scalar products

$$\left[\!\left[w^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] = \frac{1}{d} \sum_{K \in \mathfrak{M}} m_K \, w_K v_K + \frac{d-1}{d} \sum_{K^* \in \mathfrak{M}^*} m_{K^*} \, w_{K^*} v_{K^*}$$

and

$$\left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\!\right\} = \sum_{D \in \mathfrak{D}} m_D \, \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D;$$

The discrete divergence operator is the usual Finite Volumes' one: we apply the Green-Gauss formula in each primal cell κ and in each dual cell κ^*:

$$\operatorname{div}^{\mathfrak{T}} : (R^d)^{\mathfrak{D}} \longrightarrow R^{\mathfrak{T}}, \quad \text{with e.g. } (\operatorname{div}^{\mathfrak{T}})_K \vec{\mathcal{F}} := \sum_{D \in \mathfrak{D}} \int_{\partial K \cap D} \vec{\mathcal{F}}_D \cdot \nu_K .$$

Finite volume meshes and operators...

The space of discrete functions $w^{\mathfrak{T}} = \big((w_K)_K \, ; \, (w_{K^*})_{K^*} \big)$ is denoted by $\mathbb{R}^{\mathfrak{T}}$, for functions zero on the boundary we use $\mathbb{R}^{\mathfrak{T}}_0$.
The set of discrete fields $(\vec{\mathcal{F}}_D)_D$ is denoted $(\mathbb{R}^d)^{\mathfrak{D}}$.

On spaces $\mathbb{R}^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$, we introduce scalar products

$$\left[\!\left[w^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] = \frac{1}{d} \sum_{K \in \mathfrak{M}} m_K \, w_K v_K \; + \; \frac{d-1}{d} \sum_{K^* \in \mathfrak{M}^*} m_{K^*} \, w_{K^*} v_{K^*}$$

and

$$\left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\!\right\} = \sum_{D \in \mathfrak{D}} m_D \, \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D ;$$

The discrete divergence operator is the usual Finite Volumes' one: we apply the Green-Gauss formula in each primal cell K and in each dual cell K^*:

$$\mathrm{div}^{\mathfrak{T}} : (R^d)^{\mathfrak{D}} \longrightarrow R^{\mathfrak{T}}, \quad \text{with e.g. } (\mathrm{div}^{\mathfrak{T}})_K \vec{\mathcal{F}} := \sum_{D \in \mathfrak{D}} \int_{\partial K \cap D} \vec{\mathcal{F}}_D \cdot \nu_K .$$

Finite volume meshes and operators...

The space of discrete functions $w^{\mathfrak{T}} = \Big((w_\kappa)_\kappa \, ; \, (w_{\kappa^*})_{\kappa^*} \Big)$ is denoted by $\mathbb{R}^{\mathfrak{T}}$, for functions zero on the boundary we use $\mathbb{R}_0^{\mathfrak{T}}$.

The set of discrete fields $(\vec{\mathcal{F}}_D)_D$ is denoted $(\mathbb{R}^d)^{\mathfrak{D}}$.

On spaces $\mathbb{R}^{\mathfrak{T}}$ and $\mathbb{R}^{\mathfrak{D}}$, we introduce scalar products

$$\left[\!\left[w^{\mathfrak{T}}, v^{\mathfrak{T}} \right]\!\right] = \frac{1}{d} \sum_{\kappa \in \mathfrak{M}} m_\kappa \; w_\kappa v_\kappa \; + \; \frac{d-1}{d} \sum_{\kappa^* \in \mathfrak{M}^*} m_{\kappa^*} \; w_{\kappa^*} v_{\kappa^*}$$

and

$$\left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}}, \vec{\mathcal{G}}^{\mathfrak{T}} \right\}\!\!\right\} = \sum_{D \in \mathfrak{D}} m_D \, \vec{\mathcal{F}}_D \cdot \vec{\mathcal{G}}_D ;$$

The discrete divergence operator is the usual Finite Volumes' one: we apply the Green-Gauss formula in each primal cell κ and in each dual cell κ^*:

$$\operatorname{div}^{\mathfrak{T}} : \; (R^d)^{\mathfrak{D}} \longrightarrow R^{\mathfrak{T}}, \quad \text{with e.g. } (\operatorname{div}^{\mathfrak{T}})_\kappa \vec{\mathcal{F}} := \sum_{D \in \mathfrak{D}} \int_{\partial\kappa \cap D} \vec{\mathcal{F}}_D \cdot \nu_\kappa .$$

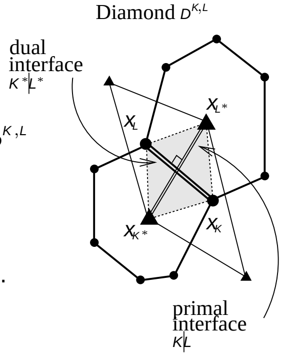
Finite volume meshes and operators...

The discrete gradient operator is of the form

$$\nabla^{\mathfrak{T}} : \; R_0^{\mathfrak{T}} \longrightarrow (R^d)^{\mathfrak{D}},$$

where the values $\nabla_D w^{\mathfrak{T}}$ are reconstructed per diamond from two projections: in $\mathbb{R}^2$,

$$\nabla_D w^{\mathfrak{T}} = \frac{w_L - w_K}{d_{KL}}\,\nu_{K,L} + \frac{w_{L^*} - w_{K^*}}{d_{K^*L^*}}\,\nu_{K^*,L^*} \quad \text{for } D = D^{K,L}$$



NB: The 3D case offers several choices: different constructions due to
Pierre & Coudière ; Coudière & Hubert ;
Hermeline, Andr. & Bendahmane & Hubert & Krell .
We follow the last one, called "3D CeVe-DDFV".
The DDFV schemes enjoy discrete duality:

Proposition (discrete duality)

For $v^{\mathfrak{T}} \in \mathbb{R}_0^{\mathfrak{T}}$ and $\vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^d)^{\mathfrak{D}}$, $\left[\!\left[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\, v^{\mathfrak{T}} \right]\!\right] = \left\{\!\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}},\; \nabla^{\mathfrak{T}} v^{\mathfrak{T}} \right\}\!\!\right\}.$

With this property, all "variational" techniques can be used at the discrete level ! The discretization of the Leray-Lions diffusion operator $-\operatorname{div} \vec{\mathfrak{a}}_0(\nabla\cdot)$ by $-\operatorname{div}^{\mathfrak{T}} \vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}}\cdot)$ preserves the key features of the continuous operator: coercivity, monotonicity, growth, existence of a potential...

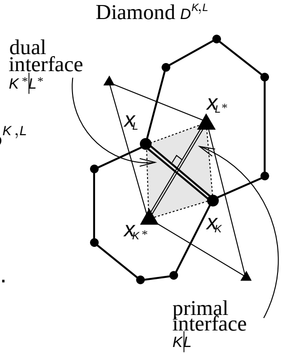
Finite volume meshes and operators...

The discrete gradient operator is of the form

$$\nabla^{\mathfrak{T}} : R_0^{\mathfrak{T}} \longrightarrow (R^d)^{\mathfrak{D}},$$

where the values $\nabla_D w^{\mathfrak{T}}$ are reconstructed per diamond from two projections: in $\mathbb{R}^2$,

$$\nabla_D w^{\mathfrak{T}} = \frac{w_L - w_K}{d_{KL}} \nu_{K,L} + \frac{w_{L^*} - w_{K^*}}{d_{K^*L^*}} \nu_{K^*,L^*} \text{ for } D = D^{K,L}$$



NB: The 3D case offers several choices: different constructions due to
Pierre & Coudière ; Coudière & Hubert ;
Hermeline, Andr. & Bendahmane & Hubert & Krell .
We follow the last one, called "3D CeVe-DDFV".
The DDFV schemes enjoy discrete duality:

Proposition (discrete duality)

For $v^{\mathfrak{T}} \in \mathbb{R}_0^{\mathfrak{T}}$ *and* $\vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^d)^{\mathfrak{D}}$, $\left[\!\left[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\, v^{\mathfrak{T}} \right]\!\right] = \left\{\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}},\, \nabla^{\mathfrak{T}} v^{\mathfrak{T}} \right\}\!\right\}.$

With this property, all "variational" techniques can be used at the discrete level ! The discretization of the Leray-Lions diffusion operator $-\operatorname{div} \vec{a}_0(\nabla\cdot)$ by $-\operatorname{div}^{\mathfrak{T}} \vec{a}_0(\nabla^{\mathfrak{T}}\cdot)$ preserves the key features of the continuous operator: coercivity, monotonicity, growth, existence of a potential...

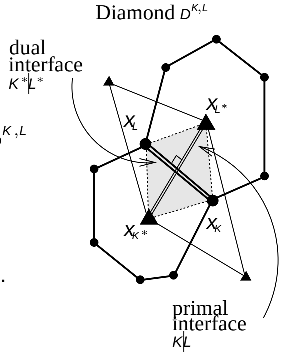
Finite volume meshes and operators...

The discrete gradient operator is of the form

$$\nabla^{\mathfrak{T}} : R_0^{\mathfrak{T}} \longrightarrow (R^d)^{\mathfrak{D}},$$

where the values $\nabla_D w^{\mathfrak{T}}$ are reconstructed per diamond from two projections: in $\mathbb{R}^2$,

$$\nabla_D w^{\mathfrak{T}} = \frac{w_L - w_K}{d_{KL}}\,\nu_{K,L} + \frac{w_{L^*} - w_{K^*}}{d_{K^*L^*}}\,\nu_{K^*,L^*} \quad \text{for } D = D^{K,L}$$



NB: The 3D case offers several choices: different constructions due to
Pierre & Coudière ; Coudière & Hubert ;
Hermeline, Andr. & Bendahmane & Hubert & Krell .
We follow the last one, called “3D CeVe-DDFV”.
The DDFV schemes enjoy discrete duality:

Proposition (discrete duality)

For $v^{\mathfrak{T}} \in \mathbb{R}_0^{\mathfrak{T}}$ *and* $\vec{\mathcal{F}}^{\mathfrak{T}} \in (\mathbb{R}^d)^{\mathfrak{D}}$, $\left[\!\left[-\operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T}}],\, v^{\mathfrak{T}} \right]\!\right] = \left\{\!\left\{ \vec{\mathcal{F}}^{\mathfrak{T}},\ \nabla^{\mathfrak{T}} v^{\mathfrak{T}} \right\}\!\right\}$.

With this property, all “variational” techniques can be used at the discrete level ! The discretization of the Leray-Lions diffusion operator $-\operatorname{div}\vec{\mathfrak{a}}_0(\nabla\cdot)$ by $-\operatorname{div}^{\mathfrak{T}}\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}}\cdot)$ preserves the key features of the continuous operator: coercivity, monotonicity, growth, existence of a potential...

Finite volume scheme for the problem...

In addition, we take a usual two-point monotone consistent flux approximation to produce a discrete operator $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$ which approximates the convection operator $\mathrm{div}\,\vec{F}(\cdot)$.

With the notation introduced above, our discretization writes:

find a discrete function $u^{\mathfrak{T},\Delta t}$ satisfying for $n = 1, \ldots, N = T/\Delta t$ the equations

$$\left|\begin{array}{l} \dfrac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} + (\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}[u^{\mathfrak{T},n}] - \mathrm{div}^{\mathfrak{T}}[\vec{a}_0(\nabla^{\mathfrak{T}} w^{\mathfrak{T},n})] = 0, \\ w^{\mathfrak{T},n} = A(u^{\mathfrak{T},n}), \end{array}\right.$$

together with the boundary and initial conditions

$$\text{for all } n = 1, \ldots, N, \qquad \begin{cases} u_K^n = 0 & \text{for all } \kappa \text{ near } \partial\Omega \\ u_{K^*}^n = 0 & \text{for all } \kappa^* \text{ near } \partial\Omega; \end{cases}$$

$$u_K^0 = \frac{1}{m_K}\int_K u_0 \quad \text{for all } \kappa, \quad u_{K^*}^0 = \frac{1}{m_{K^*}}\int_{K^*} u_0 \quad \text{for all } \kappa^*.$$

Theorem (main result of : Andr. & Bendahmane & Karlsen JHDE'11)

The discrete solutions $u^{\mathfrak{T},\Delta t}$ exist and converge to the unique entropy solution u as the discretization step (space and time) goes to zero.

Finite volume scheme for the problem...

In addition, we take a usual two-point monotone consistent flux approximation to produce a discrete operator $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$ which approximates the convection operator $\mathrm{div}\,\vec{F}(\cdot)$.

With the notation introduced above, our discretization writes:

find a discrete function $u^{\mathfrak{T},\Delta t}$ satisfying for $n = 1, \ldots, N = T/\Delta t$ the equations

$$\left|\begin{array}{l} \dfrac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} + (\mathrm{div}_c\vec{F})^{\mathfrak{T}}[u^{\overline{\mathfrak{T}},n}] - \mathrm{div}^{\mathfrak{T}}[\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}},n})] = 0, \\ w^{\overline{\mathfrak{T}},n} = A(u^{\overline{\mathfrak{T}},n}), \end{array}\right.$$

together with the boundary and initial conditions

$$\text{for all } n = 1, \ldots, N, \qquad \begin{cases} u^n_K = 0 & \text{for all } K \text{ near } \partial\Omega \\ u^n_{K^*} = 0 & \text{for all } K^* \text{ near } \partial\Omega; \end{cases}$$

$$u^0_K = \frac{1}{m_K}\int_K u_0 \quad \text{for all } K, \quad u^0_{K^*} = \frac{1}{m_{K^*}}\int_{K^*} u_0 \quad \text{for all } K^*.$$

Theorem (main result of : Andr. & Bendahmane & Karlsen JHDE'11)

The discrete solutions $u^{\mathfrak{T},\Delta t}$ exist and converge to the unique entropy solution u as the discretization step (space and time) goes to zero.

Finite volume scheme for the problem...

In addition, we take a usual two-point monotone consistent flux approximation to produce a discrete operator $(\mathrm{div}_c\, \vec{F})^{\mathfrak{T}}(\cdot)$ which approximates the convection operator $\mathrm{div}\, \vec{F}(\cdot)$.

With the notation introduced above, our discretization writes:

find a discrete function $u^{\mathfrak{T},\Delta t}$ satisfying for $n = 1, \ldots, N = T/\Delta t$ the equations

$$\left|\begin{array}{l} \dfrac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} + (\mathrm{div}_c \vec{F})^{\mathfrak{T}}[u^{\overline{\mathfrak{T}},n}] - \mathrm{div}^{\mathfrak{T}}[\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}},n})] = 0, \\ w^{\overline{\mathfrak{T}},n} = A(u^{\overline{\mathfrak{T}},n}), \end{array}\right.$$

together with the boundary and initial conditions

$$\text{for all } n = 1, \ldots, N, \qquad \begin{cases} u_\kappa^n = 0 & \text{for all } \kappa \text{ near } \partial\Omega \\ u_{\kappa^*}^n = 0 & \text{for all } \kappa^* \text{ near } \partial\Omega; \end{cases}$$

$$u_\kappa^0 = \frac{1}{m_\kappa}\int_\kappa u_0 \quad \text{for all } \kappa, \quad u_{\kappa^*}^0 = \frac{1}{m_{\kappa^*}}\int_{\kappa^*} u_0 \quad \text{for all } \kappa^*.$$

Theorem (main result of : Andr. & Bendahmane & Karlsen JHDE'11)

The discrete solutions $u^{\mathfrak{T},\Delta t}$ exist and converge to the unique entropy solution u as the discretization step (space and time) goes to zero.

Finite volume scheme for the problem...

In addition, we take a usual two-point monotone consistent flux approximation to produce a discrete operator $(\mathrm{div}_c \vec{F})^{\mathfrak{T}}(\cdot)$ which approximates the convection operator $\mathrm{div}\,\vec{F}(\cdot)$.

With the notation introduced above, our discretization writes:

find a discrete function $u^{\mathfrak{T},\Delta t}$ satisfying for $n = 1, \dots, N = T/\Delta t$ the equations

$$\left| \begin{array}{l} \dfrac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} + (\mathrm{div}_c \vec{F})^{\mathfrak{T}}[u^{\overline{\mathfrak{T}},n}] - \mathrm{div}^{\mathfrak{T}}[\vec{a}_0(\nabla^{\mathfrak{T}} w^{\overline{\mathfrak{T}},n})] = 0, \\ w^{\overline{\mathfrak{T}},n} = A(u^{\overline{\mathfrak{T}},n}), \end{array} \right.$$

together with the boundary and initial conditions

$$\text{for all } n = 1, \dots, N, \qquad \begin{cases} u_\kappa^n = 0 & \text{for all } \kappa \text{ near } \partial\Omega \\ u_{\kappa^*}^n = 0 & \text{for all } \kappa^* \text{ near } \partial\Omega; \end{cases}$$

$$u_\kappa^0 = \frac{1}{m_\kappa} \int_\kappa u_0 \quad \text{for all } \kappa, \quad u_{\kappa^*}^0 = \frac{1}{m_{\kappa^*}} \int_{\kappa^*} u_0 \quad \text{for all } \kappa^*.$$

Theorem (main result of : Andr. & Bendahmane & Karlsen JHDE'11)

The discrete solutions $u^{\mathfrak{T},\Delta t}$ exist and converge to the unique entropy solution u as the discretization step (space and time) goes to zero.

Discrete Calculus tools and Convergence Analysis

Discrete calculus tools...

Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

"Variational" arguments: take $w^{\mathfrak{T}}$ for test function, get

- Energy estimates
  ($\Longrightarrow$ Existence + Weak L^p compactness for gradients $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$
  + Estimate of space translates for $w^{\mathfrak{T},\Delta t}$)
- Estimate of time translates for $w^{\mathfrak{T},\Delta t}$.

We have to establish that $(\mathrm{div}_c \vec{F})^{\mathfrak{T}}(\cdot)$ "coexists nicely" with variational technique , i.e., $\left[\!\left[(\mathrm{div}_c \vec{F})^{\mathfrak{T}}(u^{\mathfrak{T}}) \,,\, A(u^{\mathfrak{T}}) \right]\!\right]$ behaves more or less like

$$\int_\Omega \mathrm{div}\, \vec{F}(u)\, A(u) := -\int_\Omega \vec{F}(u) \cdot \nabla A(u)$$
$$= \int_\Omega \mathrm{div}\Big(\int_0^u F(s)\, dA(s)\Big) = \int_{\partial\Omega} \Big(\int_0^u F(s)\, dA(s)\Big) \cdot \nu = 0.$$

We also have to produce discrete versions of $L^p(0,T;W^{1,p}(\Omega))$ weak compactness, of Sobolev embeddings of $W^{1,p}(\Omega)$ into $L^{sthg}(\Omega)$ (Andr. & Boyer & Hubert), and exploit discrete duality.

Discrete calculus tools...

Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

"Variational" arguments: take $w^{\mathfrak{T}}$ for test function, get

- Energy estimates
  ($\Longrightarrow$ Existence + Weak L^p compactness for gradients $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$
  + Estimate of space translates for $w^{\mathfrak{T},\Delta t}$)
- Estimate of time translates for $w^{\mathfrak{T},\Delta t}$.

We have to establish that $(\mathrm{div}_c \vec{F})^{\mathfrak{T}}(\cdot)$ "coexists nicely" with variational technique , i.e., $\left[\!\left[(\mathrm{div}_c \vec{F})^{\mathfrak{T}}(u^{\mathfrak{T}}) \, , \, A(u^{\mathfrak{T}}) \right]\!\right]$ behaves more or less like

$$\int_\Omega \mathrm{div}\, \vec{F}(u)\, A(u) := -\int_\Omega \vec{F}(u) \cdot \nabla A(u)$$
$$= \int_\Omega \mathrm{div}\Big(\int_0^u F(s)\, dA(s)\Big) = \int_{\partial\Omega}\Big(\int_0^u F(s)\, dA(s)\Big) \cdot \nu = 0.$$

We also have to produce discrete versions of $L^p(0,T;W^{1,p}(\Omega))$ weak compactness, of Sobolev embeddings of $W^{1,p}(\Omega)$ into $L^{sthg}(\Omega)$ (Andr. & Boyer & Hubert), and exploit discrete duality.

Discrete calculus tools...

Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

"Variational" arguments: take $w^{\mathfrak{T}}$ for test function, get

- Energy estimates
  ($\Longrightarrow$ Existence + Weak L^p compactness for gradients $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$
  + Estimate of space translates for $w^{\mathfrak{T},\Delta t}$)
- Estimate of time translates for $w^{\mathfrak{T},\Delta t}$.

We have to establish that $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$ "coexists nicely" with variational technique , i.e., $\left[\!\left[(\mathrm{div}_c\vec{F})^{\mathfrak{T}}(u^{\mathfrak{T}})\,,\,A(u^{\mathfrak{T}})\right]\!\right]$ behaves more or less like

$$\int_\Omega \mathrm{div}\,\vec{F}(u)\,A(u) := -\int_\Omega \vec{F}(u)\cdot\nabla A(u)$$
$$= \int_\Omega \mathrm{div}\Big(\int_0^u F(s)\,dA(s)\Big) = \int_{\partial\Omega}\Big(\int_0^u F(s)\,dA(s)\Big)\cdot\nu = 0.$$

We also have to produce discrete versions of $L^p(0,T;W^{1,p}(\Omega))$ weak compactness, of Sobolev embeddings of $W^{1,p}(\Omega)$ into $L^{sthg}(\Omega)$ (Andr. & Boyer & Hubert), and exploit discrete duality.

Discrete calculus tools...

Let's follow the steps of the "continuous" convergence proof, looking at the discrete analogues of the arguments.

"Variational" arguments: take $w^{\mathfrak{T}}$ for test function, get

- Energy estimates
  ($\Longrightarrow$ Existence + Weak L^p compactness for gradients $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$
  + Estimate of space translates for $w^{\mathfrak{T},\Delta t}$)
- Estimate of time translates for $w^{\mathfrak{T},\Delta t}$.

We have to establish that $(\mathrm{div}_c\,\vec{F})^{\mathfrak{T}}(\cdot)$ "coexists nicely" with variational technique , i.e., $\left[\!\left[(\mathrm{div}\,_c\vec{F})^{\mathfrak{T}}(u^{\mathfrak{T}})\,,\,A(u^{\mathfrak{T}})\right]\!\right]$ behaves more or less like

$$\int_\Omega \mathrm{div}\,\vec{F}(u)\,A(u) := -\int_\Omega \vec{F}(u)\cdot\nabla A(u)$$
$$= \int_\Omega \mathrm{div}\Big(\int_0^u F(s)\,dA(s)\Big) = \int_{\partial\Omega}\Big(\int_0^u F(s)\,dA(s)\Big)\cdot\nu = 0.$$

We also have to produce discrete versions of $L^p(0,T;W^{1,p}(\Omega))$ weak compactness, of Sobolev embeddings of $W^{1,p}(\Omega)$ into $L^{sthg}(\Omega)$ (Andr. & Boyer & Hubert), and exploit discrete duality.

Discrete calculus tools...

"Entropy" arguments: take $(\eta_c^{\pm})'(u^{\mathfrak{T}})$ for test function, get

- Discrete entropy inequalities
- L^∞ bound (from comparison with constant solutions)

We already know that the discrete convection operator with monotone flux leads to discrete entropy inequalities with remainder terms controlled by the "weak BV" estimate , Eymard, Gallouët, Herbin .

In addition, we have to establish that $\mathrm{div}^{\mathfrak{T}}\, \vec{\mathfrak{a}}_0(\,\nabla^{\mathfrak{T}}\cdot)$ "coexists nicely" with the entropy technique , i.e.,

$$\left[\!\left[-\,\mathrm{div}^{\mathfrak{T}} k(\,\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}))\,\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}})\,,\, \theta(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\right]$$

(with $\theta=(\eta_c^{\pm})'$, $\psi^{\mathfrak{T}}\geqslant 0$) behaves more or less like

$$-\int_\Omega \mathrm{div}\, k(\,\nabla A(u))\,\nabla A(u)\cdot(\theta(u)\psi) := \int_\Omega k(\,\nabla A(u))\,\nabla A(u)\cdot\,\nabla(\theta(u)\psi)$$
$$\geqslant \int_\Omega k(\,\nabla A(u))(\theta(u)\,\nabla A(u))\,\cdot\,\nabla\psi = \int_\Omega k(\,\nabla A(u))\,\nabla\widetilde{A}_\theta(A(u))\,\cdot\,\nabla\psi$$

Here we need to replace the chain rule by a convexity inequality and assume the orthogonality of the meshes.

Discrete calculus tools...

"Entropy" arguments: take $(\eta_c^{\pm})'(u^{\mathfrak{T}})$ for test function, get

- Discrete entropy inequalities
- L^∞ bound (from comparison with constant solutions)

We already know that the discrete convection operator with monotone flux leads to discrete entropy inequalities with remainder terms controlled by the "weak BV" estimate , Eymard, Gallouët, Herbin .

In addition, we have to establish that $\operatorname{div}^{\mathfrak{T}} \vec{a}_0(\nabla^{\mathfrak{T}} \cdot)$ "coexists nicely" with the entropy technique , i.e.,

$$\left[\!\left[-\operatorname{div}^{\mathfrak{T}} k(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}))\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}), \theta(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\right]$$

(with $\theta = (\eta_c^{\pm})'$, $\psi^{\mathfrak{T}} \geqslant 0$) behaves more or less like

$$-\int_\Omega \operatorname{div} k(\nabla A(u))\nabla A(u) \cdot (\theta(u)\psi) := \int_\Omega k(\nabla A(u))\nabla A(u) \cdot \nabla(\theta(u)\psi)$$
$$\geqslant \int_\Omega k(\nabla A(u))(\theta(u)\nabla A(u)) \cdot \nabla\psi = \int_\Omega k(\nabla A(u))\nabla \widetilde{A}_\theta(A(u)) \cdot \nabla\psi$$

Here we need to replace the chain rule by a convexity inequality and assume the orthogonality of the meshes.

Discrete calculus tools...

"Entropy" arguments: take $(\eta_c^{\pm})'(u^{\mathfrak{T}})$ for test function, get

- Discrete entropy inequalities
- L^∞ bound (from comparison with constant solutions)

We already know that the discrete convection operator with monotone flux leads to discrete entropy inequalities with remainder terms controlled by the "weak BV" estimate , Eymard, Gallouët, Herbin .

In addition, we have to establish that $\operatorname{div}^{\mathfrak{T}} \vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}}\cdot)$ "coexists nicely" with the entropy technique , i.e.,

$$\left[\!\left[-\operatorname{div}^{\mathfrak{T}} k(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}))\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}),\, \theta(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\right]$$

(with $\theta = (\eta_c^{\pm})'$, $\psi^{\mathfrak{T}} \geqslant 0$) behaves more or less like

$$\begin{aligned} -\int_\Omega \operatorname{div} k(\nabla A(u))\nabla A(u)\cdot(\theta(u)\psi) &:= \int_\Omega k(\nabla A(u))\nabla A(u)\cdot\nabla\big(\theta(u)\psi\big) \\ \geqslant \int_\Omega k(\nabla A(u))\big(\theta(u)\nabla A(u)\big)\cdot\nabla\psi &= \int_\Omega k(\nabla A(u))\nabla\widetilde{A}_\theta(A(u))\cdot\nabla\psi \end{aligned}$$

Here we need to replace the chain rule by a convexity inequality and assume the orthogonality of the meshes.

Discrete calculus tools...

"Entropy" arguments: take $(\eta_c^\pm)'(u^{\mathfrak{T}})$ for test function, get

- Discrete entropy inequalities
- L^∞ bound (from comparison with constant solutions)

We already know that the discrete convection operator with monotone flux leads to discrete entropy inequalities with remainder terms controlled by the "weak BV" estimate , Eymard, Gallouët, Herbin .

In addition, we have to establish that $\operatorname{div}^{\mathfrak{T}} \vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}}\cdot)$ "coexists nicely" with the entropy technique , i.e.,

$$\left[\!\left[-\operatorname{div}^{\mathfrak{T}} k(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}}))\,\nabla^{\mathfrak{T}} A(u^{\mathfrak{T}})\,,\ \theta(u^{\mathfrak{T}})\psi^{\mathfrak{T}} \right]\!\right]$$

(with $\theta = (\eta_c^\pm)'$, $\psi^{\mathfrak{T}} \geqslant 0$) behaves more or less like

$$-\int_\Omega \operatorname{div} k(\nabla A(u))\,\nabla A(u)\cdot(\theta(u)\psi) := \int_\Omega k(\nabla A(u))\,\nabla A(u)\cdot\nabla\big(\theta(u)\psi\big)$$
$$\geqslant \int_\Omega k(\nabla A(u))\big(\theta(u)\,\nabla A(u)\big)\cdot\nabla\psi = \int_\Omega k(\nabla A(u))\,\nabla\widetilde{A}_\theta(A(u))\cdot\nabla\psi$$

Here we need to replace the chain rule by a convexity inequality and assume the orthogonality of the meshes.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

Convergence proof...

The above points can be combined into a convergence proof:

- weak-* L^∞ compactness of $(u^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $u^{\mathfrak{T},\Delta t}(\cdot)$ "converges" to $\int_0^1 \mu(\cdot;\alpha)\,d\alpha$,
  the numerical convection flux converges to $\int_0^1 \vec{F}(\mu(\cdot;\alpha))\,d\alpha$
- strong L^1 compactness of $(w^{\mathfrak{T},\Delta t})$, weak L^p compactness of $(\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t})$
  $\Longrightarrow$ $A(u^{\mathfrak{T},\Delta t})$ tends to w strongly, $\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t})$ tends to ∇w weakly,
  and $\vec{\mathfrak{a}}_0(\nabla^{\mathfrak{T}} A(u^{\mathfrak{T},\Delta t}))$ converges to some $\vec{\chi}$ weakly.
- In addition, $A(\mu(\cdot;\alpha)) = w(\cdot)$ for all α.
- And we have discrete entropy inequalities (with vanishing remainder terms) and discrete weak formulation where we can pass to the limit, using the above convergences + consistency of $\nabla^{\mathfrak{T}}$ on test functions .
- From the weak formulation we can identify χ to $\vec{\mathfrak{a}}_0(\nabla w)$ using the Minty-Browder argument. As a byproduct, we get strong L^p convergence of $\nabla^{\mathfrak{T}} w^{\mathfrak{T},\Delta t}$ to ∇w.
- From the entropy formulation we conclude that u is an entropy-process solution , and use the reduction theorem to see that $\mu(\cdot;\alpha) \equiv u(\cdot)$.

References

R. Eymard, Th. Gallouët, R. Herbin and A. Michel

Convergence of a finite volume scheme for nonlinear degenerate parabolic equations.
Numerische Math. 92 (2002), no. 1, 41–82

B. Andreianov, M. Bendahmane and K.H. Karlsen

Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic-parabolic equations.
J. Hyperbolic Differ. Equ. 7 (2010), no. 1, 1–67.

Thank you — Merci — Danke

MERCI !