A coupling between a pointwise particle and a fluid

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Plan

We want to couple

- a pointwise particle and
- a fluid (following the Burgers equation)

in such a way that

On the point where the particle is, the particle and the fluid want to have the same velocity.

Some notation: m is the particle's mass, $h(t)$ the particle's position, $u(t,x)$ the fluid's velocity and λ is a friction parameter.

Coupled problem

$$\begin{cases} \partial_t u(t,x) + \partial_x(u^2/2)(t,x) = \lambda\,(h'(t) - u(t,x))\,\delta_{h(t)}(x) \\ mh''(t) = -\lambda(h'(t) - u(t,h(t))) \\ u(0,x) = u_0(x), \quad (h(0), h'(0)) = (0, v_0) \end{cases}$$

- Second line: If $h'(t) > u(t,h(t))$, the particule is decelerating.
- Second line: action/reaction. The force is pointwise.

Problem

The terms $u(t,x)\delta_{h(t)}(x)$ in the PDE and $u(t,h(t))$ in the ODE are to be defined since the solution can be discontinuous at the point $x = h(t)$

Aim: Define $(h'(t) - u(t,x))\,\delta_{h(t)}(x)$
Idea: Regularize the particle.

Let H^ϵ be a C^∞ function, increasing from 0 to 1 on $[-\epsilon/2, \epsilon/2]$.
The particle's trajectory h is fixed. We want to solve the previous system where we replace the Dirac δ by its regularization $(H^\epsilon)'$
We look for a special solution in the form:

- $u^\epsilon(t,x) = U^\epsilon(x - h(t))$. We denote $\xi = x - h(t)$.
- U^ϵ has bounded variations
- The jumps of U^ϵ are negative.

Therefore we have to solve

$$-h'(t)(U^\epsilon(\xi))' + \left(\frac{(U^\epsilon(\xi))^2}{2}\right)' - \lambda(h'(t) - U^\epsilon(\xi))(H^\epsilon(\xi))' = 0$$

Lemma:

If a BV function U is solution of

$$-h'(t)(U^\epsilon(\xi))' + \left(\frac{(U^\epsilon(\xi))^2}{2}\right)' - \lambda(h'(t) - U^\epsilon(\xi))(H^\epsilon(\xi))' = 0$$

then

- If ξ_0 is a discontinuity point, we have the Rankine-Hugoniot relation $U(\xi_0^+) + U(\xi_0^-) = 2h'(t)$
- On intervals where U is continuous $(U - h')(U + \lambda H)' = 0$

Corollary

If the jumps of U are negative, it exists at most one discontinuity.

Knowing the velocity on the left of the particle, we look at the possible velocities on the right of the particle.

Proposition:

If $U^\epsilon(-\epsilon/2) = u_L$ then

$$U^\epsilon(\epsilon/2) \in \begin{cases} \{u_L - \lambda\} & \text{if } u_L < h'(t) \\ [2h'(t) - u_L - \lambda, h'(t)] & \text{if } u_L \in [h'(t), h'(t) + \lambda] \\ \{u_L - \lambda\} \cup [2h'(t) - u_L - \lambda, 2h'(t) - u_L + \lambda] & \text{else} \end{cases}$$

Does not depend neither on H nor on ϵ !
We denote by $\mathcal{U}_{0,L\to R}(u_L, \lambda, v)$ this "set of admissible velocities on the right of the particle (moving at velocity v) with a given velocity u_L on the left of the particule"

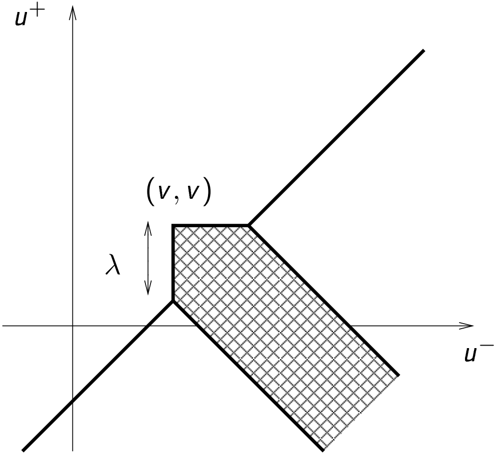


Figure: admissible couples of velocity around a particle moving at velocity v

Idea: Add some viscosity and deduce informations about the problem without viscosity.

Problem with viscosity

The problem

$$\begin{cases} \partial_t u(t,x) + \partial_x(u^2(t,x)/2) - \epsilon\partial_{xx} u = \lambda\left(h'(t) - u(t,x)\right)\delta_{h(t)}(x) \\ mh''(t) = -\lambda\left(h'(t) - u(t,h(t))\right) \\ u(0,x) = u_0(x) \\ (h(0), h'(0)) = (0, v_0) \end{cases}$$

has a unique solution, which is global and smooth.

Let G be a smooth approximation of $x \mapsto |x-\kappa|$ and ϕ be a positive test function. Following Krushkov, let us multiply the PDE by $G\phi$ and the ODE by $t \mapsto \phi(t, h(t))G'(h'(t))$, sum, integrate by parts and finally let $\epsilon \to 0$. It leads to

Entropy inequality

If $(u^\epsilon, h^\epsilon) \to (u, h)$ then $\forall \kappa \in \mathbb{R}$, $\forall \phi \geq 0$:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} |u-\kappa|\,\partial_t\phi + \operatorname{sgn}(u-\kappa)\,\frac{u^2-\kappa^2}{2}\partial_x\phi\,dx\,dt$$

$$+\int_{\mathbb{R}} \phi(0,x)\,|u_0(x)-\kappa|\,dx$$

$$+\int_{\mathbb{R}_+} m\,\partial_t\phi(h)\,|h'-\kappa|\,dt + \phi(0,h(0))\,|v_0-\kappa| \geq 0$$

With a ϕ vanishing on the particle's trajectory we recover the usual entropy inequality for the Burgers equation (without source term)
Using a sequence (ϕ_ϵ) more and more concentrated on the particle's trajectory

$$\phi(t,x) = \psi(t,x)\zeta\left(\frac{x-h(t)}{\epsilon}\right)$$

we obtain a new ODE for the particle

$$mh''(t) = \\ \left(u(t,h(t)^-) - u(t,h(t)^+)\right)\left(\frac{u(t,h(t)^-)+u(t,h(t)^+)}{2} - h'(t)\right)$$

Solutions of the coupled problem

(u, h) is a solution of the coupled problem if

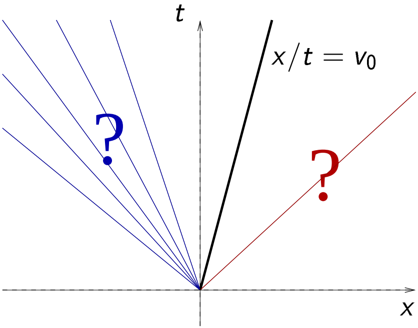
- $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$, $h \in C^1(\mathbb{R}_+)$, h' Lipschitz continuous.
- Classic entropy inequality outside of the particle trajectory
- $(u(t, h(t)^-), u(t, h(t)^+))$ is a admissible couple of velocities around the particle (moving at velocity $h'(t)$).
- $mh''(t) =$ $(u(t, h(t)^-) - u(t, h(t)^+)) \left(\frac{u(t,h(t)^-)+u(t,h(t)^+)}{2} - h'(t) \right)$

A solution can be seen as:
a Burgers solution stopped at $h(t)$ + an admissible jump across the particle + a Burgers solution starting at $h(t)$

Plan

$u_0(x) = u_L(1 - H(x)) + u_R H(x)$, H Heaviside and $h(t) = v_0 t$. We want to solve the new problem

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \lambda(u(t,x) - v_0)\delta_{v_0 t}(x)$$



Aim: Finding a Burgers wave on each side of the particle, such that the jump around the line $x = v_0 t$ is admissible.

Notation: $W(\xi, u_L, u_R)$ is the value on the straight line $x = \xi t$ of the solution of the Riemann problem (u_L, u_R) for the Burgers equation (without particule).
The fluid's velocity on the left of the particule should be the value on the line $x = v_0 t$ of a Riemann problem $(u_L, \bar{u})$, where $\bar{u} \in \mathbb{R}$:

$$u(t, h(t)^-) \in \bigcup_{\bar{u} \in \mathbb{R}} W(v_0; u_L, \bar{u}) =: \mathcal{U}^-(u_L, v_0)$$

Accessible velocities on the left of the particle

The set of accessible velocities on the left of a particle moving at velocity v_0, starting from u_L, is:

$$\mathcal{U}^-(u_L, v_0) = \begin{cases}]-\infty, v_0] & \text{if } u_L \leq v_0 \\ ]-\infty, 2v_0 - u_l[\cup \{u_L\} & \text{if } u_L > v_0 \end{cases}$$

Notation: $W(\xi, u_L, u_R)$ is the value on the straight line $x = \xi t$ of the solution of the Riemann problem (u_L, u_R) for the Burgers equation (without particule).

The fluid's velocity on the right of the particle should be the value on the line $x = v_0 t$ of a Riemann problem $(\bar{u}, u_R)$, where $\bar{u} \in \mathbb{R}$:

$$u(t, h(t)^+) \in \bigcup_{\bar{u} \in \mathbb{R}} W(v_0; \bar{u}, u_R) =: \mathcal{U}^+(u_L, v_0)$$

Accessible velocities on the right of the particle

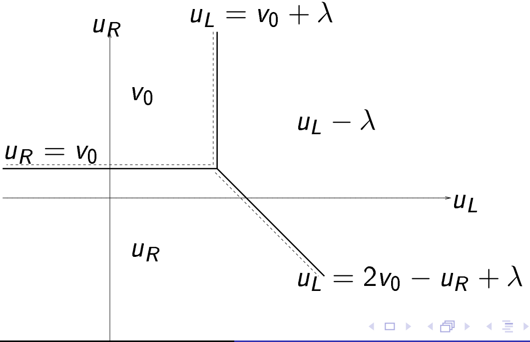
The set of accessible velocities on the right of a particle moving at velocity v_0, arriving at u_R, is:

$$\mathcal{U}^+(u_R, v_0) = \begin{cases} \{u_R\} \cup \,]\, 2v_0 - u_R, +\infty[& \text{if } u_R < v_0 \\ ]\, v_0, +\infty\,] & \text{if } u_R \geq v_0 \end{cases}$$

Determination of u^+

The following set has only one element:
$\mathcal{U}_{0,L\to R}(U^-(u_L, v_0), \lambda, v_0)) \cap U^+(u_R, v_0)$ This is the only velocity that can be put on the right of the particule which is accessible

- from the left, by a Burgers wave + an admissible jump
- from the right, by a Burgers wave



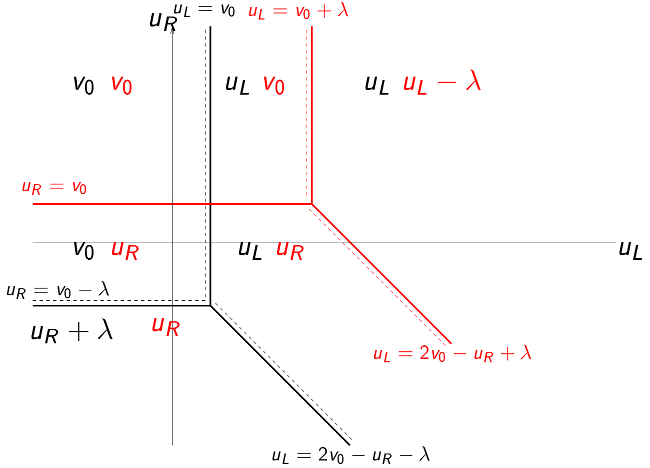
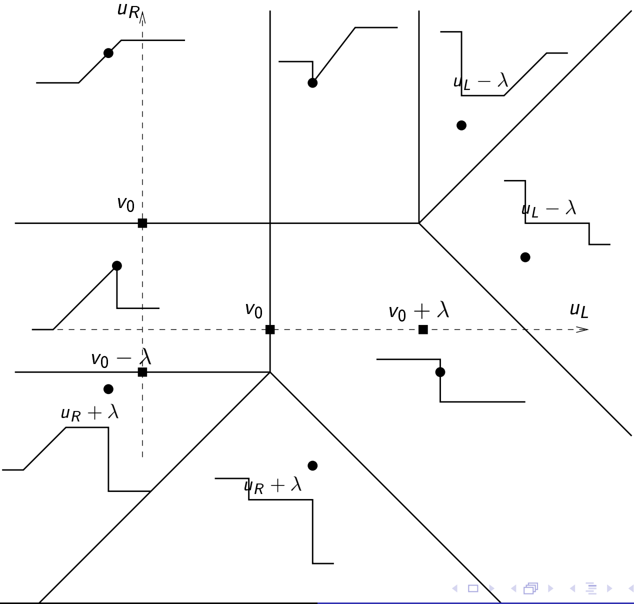


Figure: Determination of u^- and of u^+


u_R
v_0
$u_L - \lambda$
$u_L - \lambda$
v_0
$v_0 + \lambda$
u_L
$v_0 - \lambda$
$u_R + \lambda$
$u_R + \lambda$

How do we obtain a solution thanks to the problem with constant speed?
$v_0 - \lambda \leq u_R < v_0$ et $u_L \leq v_0$. We suppose $u_R < u_L$. Replacing $v_0 t$ by $h(t)$ in the solution for a fixed velocity, we find a possible solution

$$u(x,t) = \begin{cases} u_L & \text{if } x \leq u_L t \\ x/t & \text{if } u_L t < x \leq h(t) \\ u_R & \text{if } h(t) \leq x \end{cases}$$

The ODE for the particle gives

$$mh''(t) = \left(\frac{h(t)}{t} - u_R\right)\left(\frac{h(t)}{2t} + \frac{u_R}{2} - h'(t)\right)$$

Prop: $h(t)/t \to u_R$. –> The second line is not defined for all time

There exists t_1 such that

$$u(x,t_1) = \begin{cases} u_L & \text{if } x \leq h(t_1) \\ u_R & \text{if } h(t_1) \leq x \end{cases}$$

The solution after t_1 is obtain by replacing $v_0 t$ by $h(t)$ in the solution for a fixed velocity:

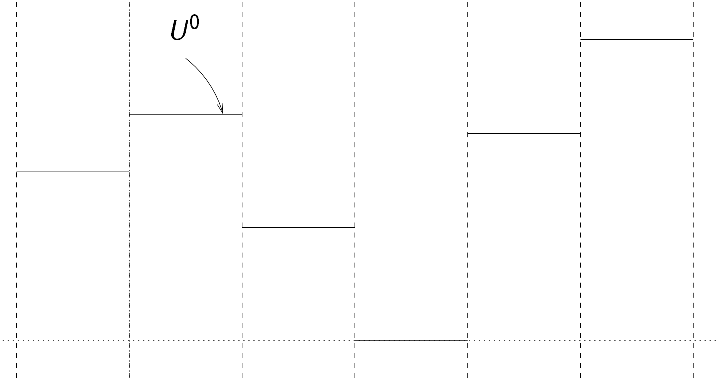
$$u(x,t) = \begin{cases} u_L & \text{if } x \leq h(t) \\ u_R & \text{if } h(t) \leq x \end{cases}$$

The ODE gives

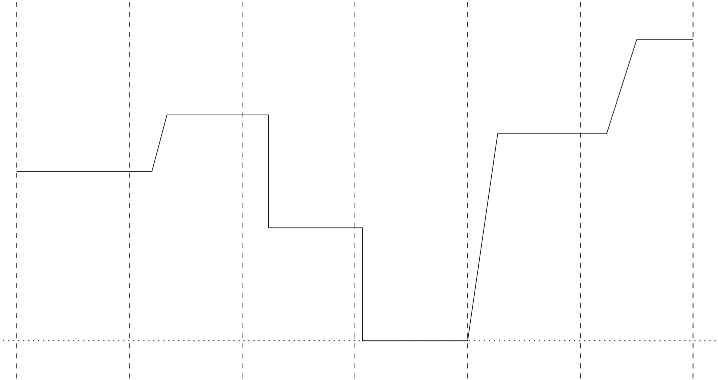
$$mh''(t) = \left(\frac{h(t)}{t} - u_R\right)\left(\frac{h(t)}{2t} + \frac{u_R}{2} - h'(t)\right)$$

Plan

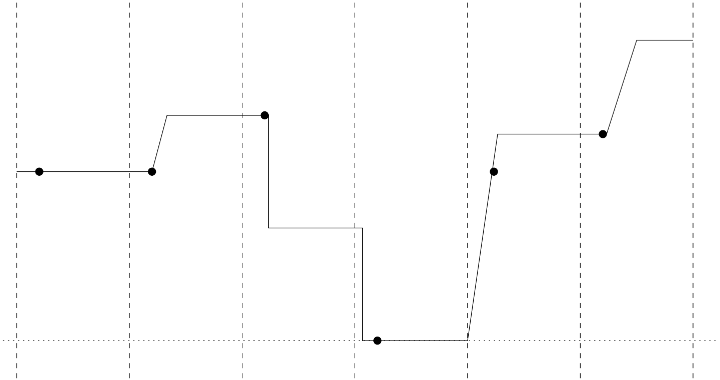
We start with a piecewise constant approximation U^0 of the initial data u^0.



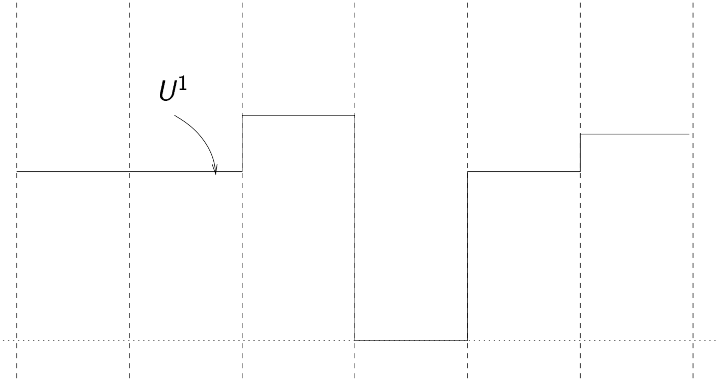
If $dt \leq \frac{dx}{2\max(|U^0|)}$, the solution of this Cauchy problem consists in gluing together the small Riemann problems.



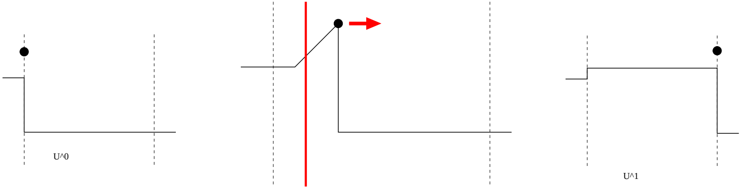
Pick uniformly at random $h \in [0, dx]$ and consider the values at points $x_i + h$.



The new value on the cell is the value of the solution at the chosen points



Adding the particle is not a problem since we have solved the Riemann problem with a particle on the interface.
New CFL condition: $t^{n+1} - t^n \leq \frac{dx}{2\max(|U^n|,|v^n|)}$.
How do we replace the particle ? Consistently with the choice of the new value.



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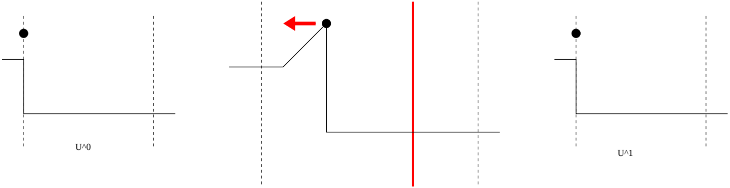


We defined the nonconservative source term thanks to a thickening of the particle. Looking closer at the proof, we better understand what happens inside the particle.

- There can be one (and only one) discontinuity verifying the Rankine Hugoniot relation

$$u(\xi^-) + u(\xi^+) = 2h'(t).$$

  This is actually a shock in the Burgers equation.
- Outside of this discontinuity, the solution continuously decreases of at most λ. This is the effect of the particle.

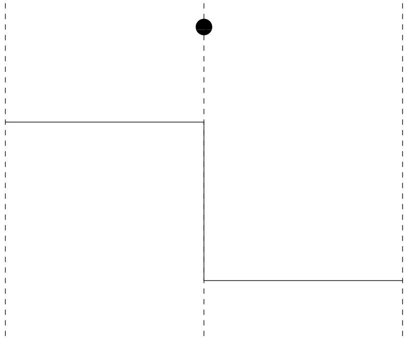
When the particle crosses a part of the fluid, it tries to change the fluid velocity to its own, but cannot change the fluid velocity of more than λ

Idea of the scheme

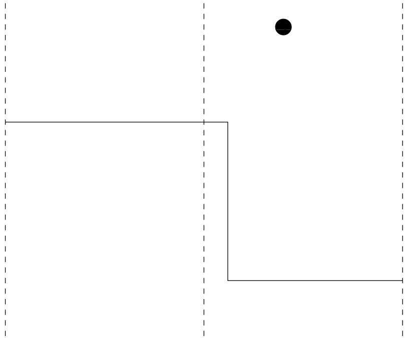
Three steps

1. Handling the particle
   When the particle crosses a part of the fluid, it tries to change the fluid velocity to its own, but cannot change the fluid velocity of more than λ –> Compute the velocity u^- and u^+ around the particle.
2. Handling the fluid
   Solve the two Dirichlet problems (now that we know the traces) on the left and right of the particle (using a Godunov scheme for example).
3. Updating the particle velocity

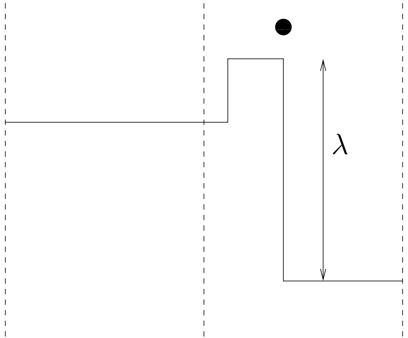
Step 1: Handling the particle
In this example, the particle is moving faster than the shock.



Step 1: Handling the particle
What happens if the particle and the shock do not see each other.

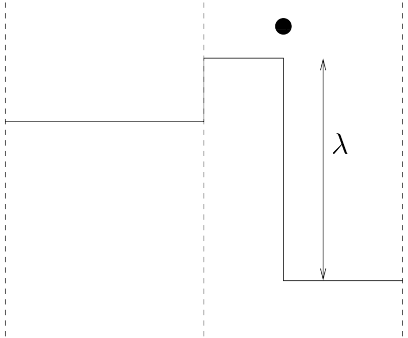


Step 1: Handling the particle
Dragging the fluid that the particle meets at velocity at most λ.

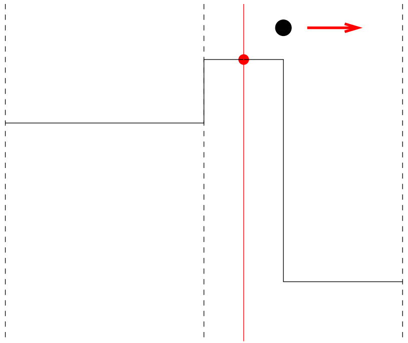


Step 1: Handling the particle

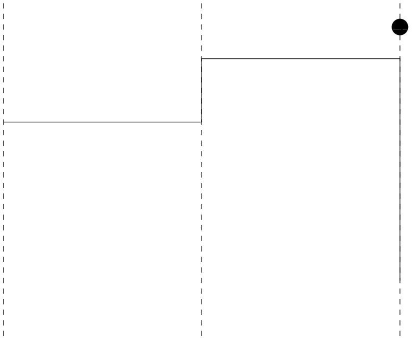
For the sake of simplicity, assume that the shock didn't move at all and only keep the dragging velocity on the right.



Step 1: Handling the particle
Pick randomly a point in the cell and replace the particle according to this choice.



Step 1: Handling the particle
Change the value of the fluid velocity.



Godunov scheme for the Burgers equation:

$$\partial_t u(t,x) + \partial_x \frac{f(u(t,x))}{2} = 0$$

$$U_k(t) = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} u(t,x)dt$$

Integrate over a cell

$$\partial_t U_k(t) + \frac{1}{x_{k+1} - x_k}(f(u(t,x_{k+1})) - f(u(t,x_{k+1}))) = 0$$

Godunov scheme:

$$u(t,x_k) = W(0; U_{k-1}(t), U_k(t))$$

Explicite Euler scheme in time:

$$U_k^{n+1} = U_k^n - \frac{t^{n+1} - t^n}{x_{k+1} - x_k}(f(W(0; U_k^n, U_{k+1}^n)) - f(W(0; U_{k-1}^n, U_k^n)))$$

Step 2: Handling the fluid
We want to solve the Burgers equation on the left of the particle.
We now have at Dirichlet condition since we know the velocity $u^- = U_{k_0-1}$ at the left of the particle.

$$U_{j_0-1}^{n+1} = U_{j_0-1}^n + \frac{t_{n+1} - t_n}{\Delta x}\left(f(W(0; U_{j_0-2}^n, U_{j_0-1}^n)) - f(W(0; U_{j_0-1}^n, u^-))\right)$$

On the right of the particule, u^+ is known.

$$U_{j_0}^{n+1} = U_{j_0}^n + \frac{t_{n+1} - t_n}{\Delta x}\left(f(W(0; u^+, U_{j_0}^n)) - f(W(0; U_{j_0}^n, U_{j_0+1}^n))\right)$$

Step 3: Updating the particle velocity
We can of course use the PDE.
BUT we have formally

$$\forall t \geq 0, \qquad \int_{\mathbb{R}} u(t,x)dx + mh'(t) = 0.$$

We easily compute

$$\Delta x \sum_j U_j^{n+1} - \Delta x \sum_j U_j^n = \Delta t \frac{(U_{k_0}^n)^2 - (U_{k_0-1}^n)^2}{2}.$$

So in order to obtain a conservative scheme we can choose:

$$v^{n+1} = v^n - \frac{(t^{n+1} - t^n)}{\Delta x} \frac{(U_{j_0}^n)^2 - (U_{j_0-1}^n)^2}{2m}.$$

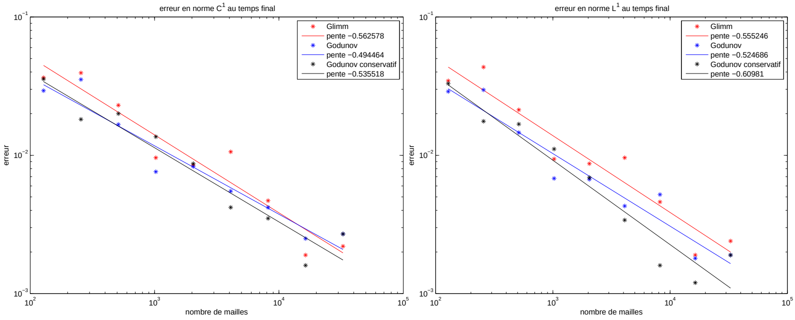
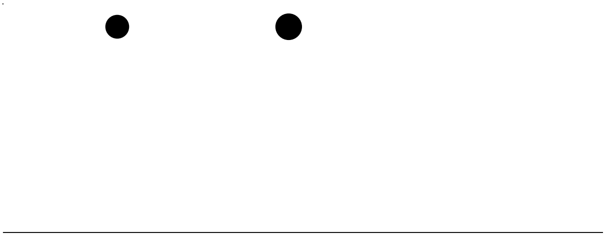
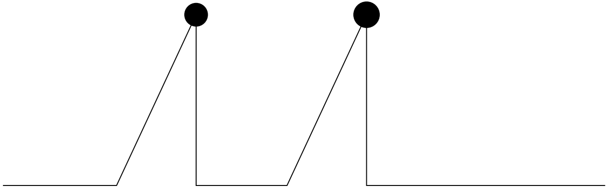


Figure: Average error in 10 simulations on the final velocity and position of the particle (left) and the L^1 norm of u (right) for different space discretizations. We find an order 1/2.

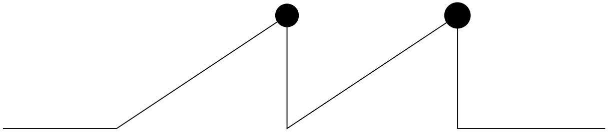
We start with a fluid with velocity 0 with two particles having the same viscosity $v_0 > 0$.



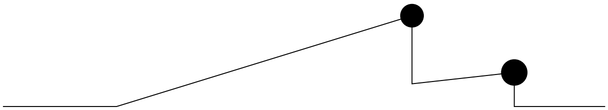
At the beginning the two particles don't see each other (here the particles manage to drag the fluid to their own velocities).



At some time t_1 the particle on the left meets the wave creating by the particle on the right.



On this moment on, this particle on the left is going faster than the other one and will eventually catch it up.



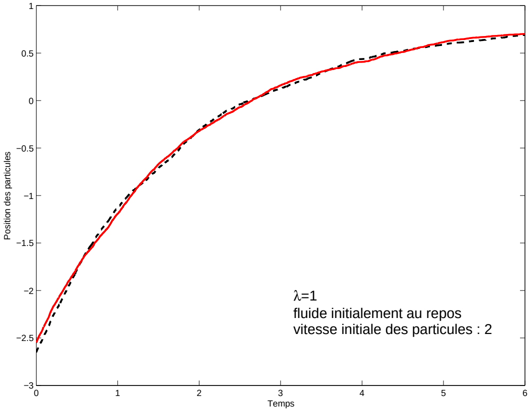


Figure: The kissing drafting tumbling phenomena

Plan

1. Uniqueness of the Riemann problem ? Existence and uniqueness for the coupled Cauchy problem ? DONE !! B. Andreianov, F. Lagoutière, N. Seguin, T. Takahashi
2. Precise modeling of the collisions between particles.
3. A more relevant equation for the fluid. The model could be, for the barotrope Euler equation:

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x \left(\rho u^2 + p(\rho)\right) = \lambda \rho (h'(t) - u(t, h(t))) \delta_{h(t)}(x) \\ m h''(t) = -\lambda \left(h'(t) - u(t, h(t))\right), \end{cases}$$