Coupling algorithms for hyperbolic systems

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

SORBONNE UNIVERSITÉS

Collaborations

with Frédéric Coquel ($\rightarrow$CMAP, Cnrs), Christophe Chalons (Cea Saclay, U. Paris Diderot), Frédéric Lagoutière ($\rightarrow$U. Paris Sud-Orsay), Pierre-Arnaud Raviart, Nicolas Seguin (LJLL-UPMC), Annalisa Ambroso (CEA Saclay$\rightarrow$Areva), Jean-Marc Hérard, Olivier Hurisse (Edf R&D)

$+$ young people (PhD): Benjamin Boutin ($\rightarrow$ Irma, U. Rennes), Filipa Caetano ($\rightarrow$U. Paris Sud-Orsay), Thomas Galie

working group LJLL-Cea-Saclay $\rightarrow$ LRC with Cea-Saclay

Content

- 0. Introduction: why algorithms to couple hyperbolic systems?
- 1.1. Systems of balance and conservation laws (some model systems, solution of the Riemann problem)
- 1.2. Finite volume methods (definition, 1d case, some usual schemes, specific properties)
- 2. Introduction to the boundary value problem, and to interface coupling
- 3.1. Interface coupling, examples, coupling algorithm
- 3.2. Interface coupling, further topics

Interface coupling: main features

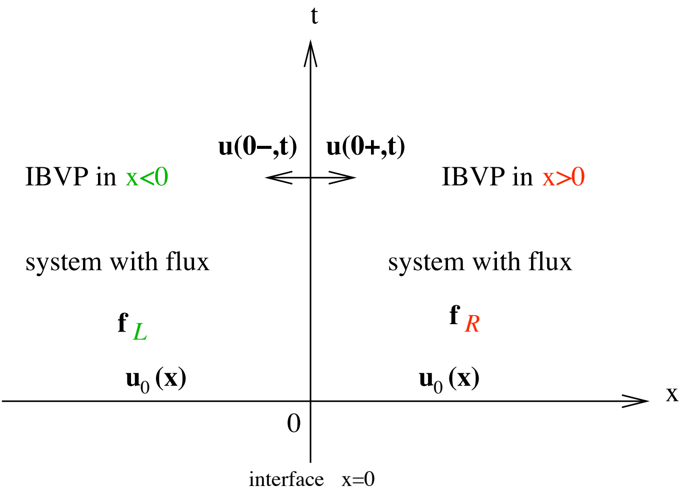
Framework: given two codes

- two (compressible) fluid codes simulating fluid flow of the same 'nature', taking into account different specificities not coupled phenomena *(monophysics)*
- fixed interface *(multidomain)*
- 'thin' interface, the codes interact
  exchange of information at the interface *(strong coupling)*
- need of a robust procedure
  understand the physics at the interface *('intelligent' coupling)*
- use existing codes
  few modifications in each domain

$\rightarrow$ give a numerical coupling procedure to 'couple' the codes.
First: what is the mathematical model?

mathematical model

- the codes simulate compressible fluid flows, modelled by systems of PDE: hyperbolic systems of balance laws
- the fixed interface is considered as a boundary (artificial, not physical)
- give a numerical coupling procedure and understand what it really computes
- understand how to model the exchange of information at the interface at both continuous and discretized levels
- $\rightarrow$ need to recall some notions about hyperbolic systems of conservation laws (HSCL) or balance laws (HSBL), finite volume schemes (FV), boundary conditions


t
u(0−,t)
u(0+,t)
IBVP in x<0
IBVP in x>0
system with flux
system with flux
f L
f R
u0 (x)
u0 (x)
x
0
interface x=0

1.1. Introduction. Conservation and balance laws

- Hyperbolic systems, entropy, characteristic fields, genuinely nonlinear / linearly degenerate
- Main example: gas dynamics. Euler system in $d = 3$, $d = 1$, barotropic model, Lagrange, $p-$system, linearized (acoustic)
- Discontinuities, weak solutions, Rankine-Hugoniot condition, entropy inequality
- Rarefactions, shocks and contact discontinuities
- Solution of the Riemann problem

Hyperbolic system of balance laws

HSBL: set of p balance laws (pde)

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \mathbf{f}_j(\mathbf{u}) = \mathbf{s}(\mathbf{u}), \ t > 0, \tag{1}$$

HSCL: hyperbolic system of conservation laws if $\mathbf{s} = 0$
$\mathbf{u} = (u_1, u_2, .., u_p)^T \in \Omega$ (in $\mathbb{R}^p$) set of *states*, $\mathbf{f}(\mathbf{u}) = (\mathbf{f}_j(\mathbf{u}))$ *flux* (each $\mathbf{f}_j(\mathbf{u})) \in \mathbb{R}^p$), $\mathbf{s}(\mathbf{u})$ source, we will also have latter $\mathbf{s}(\mathbf{u}, x)$.
we study first Cauchy problem:

$$\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ (1) + \text{initial condition } \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(x)$$

then IBVP (initial boundary value problem):

$$\mathbf{x} \in \mathcal{O} \subset \mathbb{R}^d, \ (1) + \text{initial condition } + \text{boundary condition (weak)}$$

hyperbolicity

Definition: the homogeneous system is *strongly hyperbolic* if for any direction $\mathbf{n}$ the $p \times p$ matrix

$$\mathbf{A}(\mathbf{u}, \mathbf{n}) = \sum_{j=1}^{d} \mathbf{A}_j(\mathbf{u}) n_j, \text{ where } \mathbf{A}_j(\mathbf{u}) = \mathbf{f}_j'(\mathbf{u})$$

is diagonizable with *real* eigenvalues $\lambda_k(\mathbf{u}, \mathbf{n})$,
eigenvectors $\mathbf{r}_k(\mathbf{u}, \mathbf{n}), 1 \leq k \leq p$, basis of $\mathbb{R}^p$

$$\lambda_1(\mathbf{u}, \mathbf{n}) \leq \lambda_2(\mathbf{u}, \mathbf{n}) \leq \lambda_p(\mathbf{u}, \mathbf{n}) \tag{2}$$

(ranked in increasing order), $\lambda_k(\mathbf{u}, \mathbf{n})$ is a *velocity* (wavelike solution).
In 1D, $\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}$, $\mathbf{A}(\mathbf{u}) = \mathbf{f}'(\mathbf{u})$, $\lambda_k(\mathbf{u})$, $\mathbf{r}_k(\mathbf{u})$

entropy for HSBL

Other conservation laws ?
Assume Ω convex, a convex function $\mathcal{U} : \Omega \to \mathbb{R}$ is called an *entropy* for the system (1) if $\exists d$ functions $\mathcal{F}_j : \Omega \to \mathbb{R}, 1 \leq j \leq d$, called *entropy fluxes*, such that

$$\mathcal{U}'(\mathbf{u})\mathbf{f}_j(\mathbf{u}) = \mathcal{F}_j'(\mathbf{u}), \tag{3}$$

then smooth solutions satisfy an additional (scalar) companion law

$$\frac{\partial}{\partial t}\mathcal{U}(\mathbf{u}) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\mathcal{F}_j(\mathbf{u}) = \mathcal{U}'(\mathbf{u}).\mathbf{s}(\mathbf{u}) \tag{4}$$

$\mathcal{U}'(\mathbf{u}) = (\partial_{u_1}\mathcal{U}, \partial_{u_2}\mathcal{U}, .., \partial_{u_p}\mathcal{U})$. Admissible discontinuous solutions satisfy an *inequality* ($\leq$ in (4). Existence of $\mathcal{U}$? comes from physics, not from solving (3), p equations 2 unknowns $(\mathcal{U}, \mathcal{F})$.

Example: gas dynamics

Euler system of compressible gas dynamics writes (neglecting heat conduction and viscosity)

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\varrho u_j) &= 0, \\
\frac{\partial}{\partial t}(\varrho u_i) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\varrho u_i u_j + p\delta_{i,j}) &= 0, \quad 1 \leq i \leq 3 \qquad (5) \\
\frac{\partial}{\partial t}(\varrho e) + \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\varrho e + p) u_j &= 0,
\end{aligned}
$$

ϱ density, $\mathbf{u} = (u_1, u_2, u_3)^T$ velocity, p pressure, $e = \varepsilon + |\mathbf{u}|^2/2$ specific total energy, ε internal energy, $|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2$.
The equations express the conservation of mass, momentum and energy.

Conservative variables $\mathbf{U} = (\varrho, \varrho\mathbf{u}, \varrho e)^T$.
Need of a closure law for the pressure $p = p(\mathbf{U})$. In fact, $p = p(\varrho, \varepsilon)$ for instance γ-law: $p = (\gamma - 1)\varrho\varepsilon$ for an ideal gas, $\varepsilon = e - |\mathbf{u}|^2/2, u_i = \varrho u_i/\varrho$
In (6) the fluxes $\mathbf{f}_j(\mathbf{U})$ are then easily expressed.
The set of states is $\Omega = \{\varrho > 0, \mathbf{u} \in \mathbb{R}^3, \varepsilon = e - |\mathbf{u}|^2/2 > 0\}$.
Primitive variables $\mathbf{V} = (\varrho, \mathbf{u}, p)^T$ are useful for computing the eigenvalues: $\mathbf{u} \cdot \mathbf{n} \pm c\mathbf{n}$ (c *speed of sound* , $c^2 = \gamma p/\varrho$ for a γ-law) and $\mathbf{u} \cdot \mathbf{n}$ which is an eigenvalue of multiplicity 3.
$\mathbf{A}(\mathbf{U}, \mathbf{n})$ is diagonalizable and one can compute the eigenvectors.
If $\mathbf{U} \to \mathbf{V} = \phi(\mathbf{U})$ is an admissible *change of variables*, $\mathbf{V}$ satisfies a *non conservative* system, for instance in $d = 1$

$$\partial_t \mathbf{V} + \mathbf{B}(\mathbf{V})\partial_x \mathbf{V} = \mathbf{0}$$

with matrix $\mathbf{B}(\mathbf{V})$ similar to $\mathbf{A}(\mathbf{U})$, equivalent for smooth solutions, not for discontinuous solutions.

The thermodynamic specific entropy s is defined through the fundamental relation

$$Tds = d\varepsilon + pd\tau,$$

where $T = T(\varrho, \varepsilon)$ temperature, $T = \varepsilon/C_v$ for a γ-law, $\tau = 1/\varrho$, $\mathcal{S} = -\varrho s$ is a mathematical entropy in the sense of Lax (3) with entropy flux $\mathcal{F}_i = -\varrho u_i s$ and for smooth solutions

$$\frac{\partial}{\partial t}\varrho s + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\varrho s u_j = 0. \tag{6}$$

In general, not an equality, entropy *inequality*

$$\frac{\partial}{\partial t}\varrho s + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\varrho s u_j \geq 0. \tag{7}$$

In dimension $d = 2$

$$
\begin{aligned}
\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x}(\varrho u) + \frac{\partial}{\partial y}(\varrho v) &= 0, \\
\frac{\partial}{\partial t}(\varrho u) + \frac{\partial}{\partial x}(\varrho u^2 + p) + \frac{\partial}{\partial y}(\varrho uv) &= 0, \\
\frac{\partial}{\partial t}(\varrho v) + \frac{\partial}{\partial x}(\varrho uv) + \frac{\partial}{\partial y}(\varrho v^2 + p) &= 0, \\
\frac{\partial}{\partial t}(\varrho e) + \frac{\partial}{\partial x}(\varrho e + p)u + \frac{\partial}{\partial y}((\varrho e + p)v) &= 0.
\end{aligned}
$$

The eigenvalues of $\mathbf{A}(\mathbf{u}, \mathbf{n})$ are $\mathbf{u} \cdot \mathbf{n} \pm c$ (acoustic or pressure waves) and $\mathbf{u} \cdot \mathbf{n}$ is a double eigenvalue (entropy and shear waves). Euler equations are invariant under rotation: important for the numerical approximation (project in direction $\mathbf{n} = (1, 0) \rightarrow$ study the $1d$ system)

Example: gas dynamics

Assuming a barotropic pressure law $p = p(\varrho)$ we can ignore the energy equation

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\varrho u + \frac{\partial}{\partial y}\varrho v &= 0, \\
\frac{\partial}{\partial t}\varrho u + \frac{\partial}{\partial x}(\varrho u^2 + p(\varrho)) + \frac{\partial}{\partial y}\varrho u v &= 0, \\
\frac{\partial}{\partial t}\varrho v + \frac{\partial}{\partial x}\varrho u v + \frac{\partial}{\partial y}(\varrho v^2 + p(\varrho)) &= 0.
\end{aligned}
\tag{8}
$$

Linearized acoustic: we study small perturbations $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$ of a uniform flow ϱ_0, u_0, v_0 and linearize (8)

$$
\begin{aligned}
\frac{\partial \tilde{\varrho}}{\partial t} + \frac{\partial}{\partial x}\tilde{\varrho u} + \frac{\partial}{\partial y}\tilde{\varrho v} &= 0, \\
\frac{\partial}{\partial t}\tilde{\varrho u} + \frac{\partial}{\partial x}(\widetilde{\varrho u^2} + \tilde{p}) + \frac{\partial}{\partial y}\varrho \tilde{u} v &= 0, \\
\frac{\partial}{\partial t}\tilde{\varrho v} + \frac{\partial}{\partial x}\varrho \tilde{u} v + \frac{\partial}{\partial y}(\widetilde{\varrho v^2} + \tilde{p}) &= 0.
\end{aligned}
\tag{9}
$$

In primitive variables $\tilde{\mathbf{U}} = (\tilde{p}, \tilde{u}, \tilde{v})^T$

$$p(\varrho_0 + \tilde{\varrho}) \sim p_0 + \tilde{p} \Rightarrow \tilde{p} = p'(\varrho_0)\tilde{\varrho},$$

$$\varrho_0 u_0 + \tilde{\varrho u} \sim (\varrho_0 + \tilde{\varrho})(u_0 + \tilde{u}) \Rightarrow \tilde{\varrho u} = u_0 \tilde{\varrho} + \varrho_0 \tilde{u}.$$

Simple computations lead to the linear hyperbolic system

$$\frac{\partial}{\partial t}\tilde{\mathbf{U}} + \mathbf{A}_0 \frac{\partial}{\partial x}\tilde{\mathbf{U}} + \mathbf{B}_0 \frac{\partial}{\partial y}\tilde{\mathbf{U}} = 0,$$

with constant matrices ($c_0^2 = p'(\varrho_0)$)

$$\mathbf{A}_0 = \begin{pmatrix} u_0 & \varrho_0 c_0^2 & 0 \\ 1/\varrho_0 & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix}, \ \mathbf{B}_0 = \begin{pmatrix} v_0 & 0 & \varrho_0 c_0^2 \\ 0 & v_0 & 0 \\ 1/\varrho_0 & 0 & v_0 \end{pmatrix}.$$

When $u_0 = v_0 = 0$, one can derive the *wave equation* by differentiating the first equation wrt t, the second wrt x and the third one wrt y and substracting

$$\frac{\partial^2}{\partial t^2}\tilde{\varrho} - c_0^2 \Delta \tilde{\varrho} = 0.$$

Example: gas dynamics, $d = 1$

Euler system in dimension $d = 1$ with $p = 3$ conservation laws

$$\begin{aligned}
\frac{\partial}{\partial t}\varrho + \frac{\partial}{\partial x}(\varrho u) &= 0 \\
\frac{\partial}{\partial t}(\varrho u) + \frac{\partial}{\partial x}(\varrho u^2 + p) &= 0 \\
\frac{\partial}{\partial t}(\varrho e) + \frac{\partial}{\partial x}(\varrho e + p)u &= 0
\end{aligned} \tag{10}$$

The eigenvalues of $\mathbf{A}(\mathbf{u})$ are $u \pm c$ and u, computed using the *primitive* formulation in variables $\mathbf{v} = (\varrho, u, p)$, $c^2 = \frac{\partial p}{\partial \varrho}(\varrho, s)$

$$\mathbf{B}(\mathbf{v}) = \begin{pmatrix} u & \varrho & 0 \\ 0 & u & 1/\varrho \\ 0 & \varrho c^2 & u \end{pmatrix}$$

The system is endowed with a family of entropies $\mathcal{U} = \varrho\Phi(s)$ with entropy fluxes $\mathcal{F} = \varrho\Phi(s)u$ where Φ is such that $\mathcal{U}$ is convex in the conservative variables $(\Phi' \leq 0, \Phi'' \geq 0)$.

Euler system in Lagrangian variables writes

$$\begin{aligned} \frac{\partial}{\partial t}\tau - \frac{\partial}{\partial m}u &= 0, \\ \frac{\partial}{\partial t}u + \frac{\partial}{\partial m}P &= 0, \\ \frac{\partial}{\partial t}e + \frac{\partial}{\partial m}(Pu) &= 0 \end{aligned} \tag{11}$$

where m is a mass variable, $\tau = 1/\varrho$ is the specific volume $P(\tau, \varepsilon) = p(1/\tau, \varepsilon)$. In case of a barotropic pressure law (or an isentropic flow) we get the classical $P-$system

$$\begin{aligned} \frac{\partial}{\partial t}\tau - \frac{\partial}{\partial m}u &= 0, \\ \frac{\partial}{\partial t}u + \frac{\partial}{\partial m}P &= 0, \end{aligned} \tag{12}$$

where $P = P(\tau)$ is a given function satisfying $P' < 0, P'' > 0$.

Weak solutions of balance laws

Even if $\mathbf{u}_0$ is smooth, discontinuities may develop in finite time (ex of scalar case: a smooth u is constant on characteristics)

Definition of *weak solution*: $\mathbf{u}_0(\mathbf{x})$ initial data given in $\mathrm{L}^\infty(\mathbb{R}^d)^p$ for any test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^d \times [0,\infty))^p$

$$\int_0^\infty \int_{\mathbb{R}^d} \left\{ \mathbf{u} \cdot \frac{\partial \varphi}{\partial t} + \sum_{j=1}^d \mathbf{f}_j(\mathbf{u}) \cdot \frac{\partial \varphi}{\partial x_j} \right\} d\mathbf{x} dt$$

$$+ \int_{\mathrm{R}^d} \mathbf{u}_0(\mathbf{x}) \cdot \varphi(\mathbf{x}, 0)\, d\mathbf{x} = \int_0^\infty \int_{\mathbb{R}^d} \mathbf{s}(\mathbf{u})(\mathbf{x}, t) \cdot \varphi(\mathbf{x}, t)\, d\mathbf{x} dt$$

Rankine-Hugoniot condition: the jumps $[\mathbf{u}] \equiv \mathbf{u}_+ - \mathbf{u}_-$ and $[\mathbf{f}_j(\mathbf{u})]$ are linked across a surface of discontinuity with normal $(n_t, (n_{x_j}))$

$$[\mathbf{u}] n_t + \sum_{j=1}^d [\mathbf{f}_j(\mathbf{u})]\, n_{x_j} = \mathbf{0}$$

this is a system of p equations (if the system has no differential source term $\mathbf{s}$ does not change R.H. relations)

Entropy weak solution

Non uniqueness of weak solution. Entropy criteria associated to an entropy pair $(\mathcal{U}, \mathcal{F} = (\mathcal{F}_i))$

Definition of *entropy weak solution*: $\mathbf{u}_0(\mathbf{x})$ initial data given in $\mathrm{L}^\infty(\mathbb{R}^d)^p$, for any test function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^d \times [0, \infty))^p$,$\varphi \geq 0$

$$\int_0^\infty \int_{\mathrm{R}^d} \left\{\mathcal{U}(\mathbf{u}) \cdot \frac{\partial \varphi}{\partial t} + \sum_{j=1}^d \mathcal{F}_j(\mathbf{u}) \cdot \frac{\partial \varphi}{\partial x_j}\right\} d\mathbf{x}dt$$

$$+ \int_{\mathbb{R}^d} \mathcal{U}(\mathbf{u}_0(\mathbf{x})) \cdot \varphi(\mathbf{x}, 0)\, d\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^d} \mathcal{U}'(\mathbf{u})\mathbf{s}(\mathbf{u})(\mathbf{x}, t) \cdot \varphi(\mathbf{x}, t)\, d\mathbf{x}dt \geq 0$$

$$\text{in } \mathcal{D}'(\mathbb{R}^d \times (0, \infty))^p, \ \frac{\partial}{\partial t}\mathcal{U}(\mathbf{u}) + \sum_{j=1}^d \frac{\partial}{\partial x_j}\mathcal{F}_j(\mathbf{u}) \leq \mathcal{U}'(\mathbf{u})\mathbf{s}(\mathbf{u})$$

The sign is not arbitrary: by the vanishing viscosity method

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial t} + \sum_{j=1}^d \frac{\partial}{\partial x_j}\mathbf{f}_j(\mathbf{u}_\varepsilon) - \mathbf{s}(\mathbf{u}_\varepsilon) = \varepsilon\Delta(\mathbf{u}_\varepsilon)$$

because $\mathcal{U}$ convex, $\mathcal{U}'' \geq 0$ and neglect $\varepsilon \mathcal{U}''(\mathbf{u})(\partial_{x_i}\mathbf{u}, \partial_{x_i}\mathbf{u}) \geq 0$

$$\frac{\partial}{\partial t}\mathcal{U}(\mathbf{u}_\varepsilon) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\mathcal{F}_j(\mathbf{u}_\varepsilon) - \mathcal{U}'(\mathbf{u}_\varepsilon)\mathbf{s}(\mathbf{u}_\varepsilon) \leq \varepsilon \sum_{j=1}^{d} \partial_{x_i}(\mathcal{U}'(\mathbf{u}_\varepsilon)\partial_{x_i}\mathbf{u}))$$

As for Rankine-Hugoniot condition: *entropy jump inequality*

$$[\mathcal{U}(\mathbf{u})]n_t + \sum_{j=1}^{d}[\mathcal{F}_j(\mathbf{u})]\, n_{x_j} \leq 0$$

(if source term $\mathbf{s} = \mathbf{s}(\mathbf{u})$, it does not change entropy jump inequality).

Characterization of a piecewise smooth entropy solution:

- classical solution in the domain where it is $\mathcal{C}^1$
- satisfies Rankine-Hugoniot
- and entropy inequality across a discontinuity.

In d=1, Lax-entropy condition with $k-$characteristics entering a shock.

Source term

In dimension d, $x = (x_1, ..., x_d)$, for a general scalar balance law

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i}(f_i(u, x, t)) + g(u, x, t) = 0, (t, x) \in \Pi_T$$

$\Pi_T =]0, T] \times \mathbb{R}^d$. Existence and uniqueness of Kruzkov's 'generalized solution': test function $\varphi \geq 0$,

$$\int_{\Pi_T} \Big(|u(x,t) - k| \partial_t \varphi + sgn(u(x,t) - k) \sum_{i=1}^{d} (f_i(u(x,t), x, t) - f_i(k, x, t)) \partial_{x_i} \varphi$$

$$- sgn(u(x,t) - k)(\partial_{x_i} f_i(k, x, t) + g(u, x, t))\varphi \Big) dxdt \geq 0$$

not relevant if $x \mapsto f(u, x, t)$ is discontinuous.

- production/destruction terms; external forces, body forces (gravitational); heat flux (energy equation)
- damping type: Euler with friction

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) = \varrho(g - \alpha\varphi(u)), \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) = \varrho(gu - \alpha\psi(u)) \end{cases}$$

- geometrical source term: shallow water with topography

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{1}{2}gh^2) = -ghB'(x) \end{cases}$$

- relaxation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2\partial_x u = \frac{1}{\varepsilon}(f(u) - v) \end{cases}$$

- measure (link with space-varying flux), not of the type $\mathbf{s}(\mathbf{u})$

$$\partial_t u + \partial_x f(u, x) = \mathcal{M}\delta_0, \quad \text{Dirac}$$

Linear equations with damping

$$\partial_t u + a\partial_x u = -\alpha u$$

along line $\frac{dx}{dt} = a$, the solution $\tilde{u}(t) = u(x(t), t)$ satisfies $\frac{d\tilde{u}}{dt} = -\alpha\tilde{u}$, if $\lambda > 0$, $\tilde{u}(t)$ is no longer constant, decreases if $\alpha > 0$

$$u(x,t) = e^{-\alpha t} u_0(x - at)$$

Burgers equation with damping

$$\partial_t u + \partial_x \frac{u^2}{2} = -\alpha u$$

Along a line $\frac{dx}{dt}(t) = u(x(t), t)$, $\tilde{u}(t) = u(x(t), t)$ satisfies $\frac{d\tilde{u}}{dt} = -\alpha\tilde{u}$, if $\alpha \neq 0$, $\tilde{u}$ no longer constant, the characteristics are no longer straight lines. If $x(0) = x_0$, then $\tilde{u}(t) = u_0(x_0)e^{-\alpha t}$, then from $\frac{dx}{dt}(t) = u_0(x_0)e^{-\alpha t}$, we get

$$x(t) = \frac{u_0(x_0)}{-\alpha}(e^{-\alpha t} - 1) + x_0$$

The Rankine-Hugoniot condition for $\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}$ is

$$[\mathbf{f}(\mathbf{u})] = \sigma[\mathbf{u}], \quad \text{across } x = \xi(t),$$

where $\sigma = \frac{d\xi}{dt} =$ speed of propagation of the discontinuity. For

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathcal{M}\delta_0$$

$\mathcal{M}$ weight, δ_0 Dirac measure concentrated on $x = 0$, R.H. becomes across $x = 0$

$$[\mathbf{f}(\mathbf{u})] = \mathcal{M}$$

this remark is useful in the framework of interface coupling:
$f = f(u, a) = (1 - a)f_L(u) + af_R(u)$, a Heaviside,
$[f]_{x=0} = f_R(u(0+)) - f_L(u(0-))$, $\mathcal{M} = f_R(u) - f_L(u)$, state u continuous, $\mathcal{M} = 0$, flux f continuous

Riemann problem for HSCL

From now on $d = 1$

The *Riemann problem* corresponds to a special Cauchy data

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = \mathbf{0}, \ \ t > 0, \tag{13}$$

$\mathbf{u}(x,0) = \mathbf{u}_0(x)$ where $\mathbf{u}_0(x)$ is piecewise constant

$$\mathbf{u}_0(x) = \begin{cases} \mathbf{u}_L, \ \ x < 0 \\ \mathbf{u}_R, \ \ x > 0. \end{cases} \tag{14}$$

Self similar solution $\mathbf{u}(x,t) = W_R(x/t; \mathbf{u}_L, \mathbf{u}_R)$ (if source $\mathbf{s} = 0$).
Invariance by translation: Riemann problem at any (x_0, t_0) has solution $\mathbf{u}(x,t) = W_R((x - x_0)/(t - t_0); \mathbf{u}_L, \mathbf{u}_R)$

Importance:

- gives explicit solutions (for tests)
- used in constructing numerical approximations; general $\mathbf{u}_0(x)$ approached by piecewise constant $\mathbf{u}_i^0$ on $(x_{i-1/2}, x_{i+1/2})$: juxtaposition of RP.

Linear system: $\mathbf{A}$ constant $p \times p$ matrix, diagonalizable on $\mathbb{R}$

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{0}$$

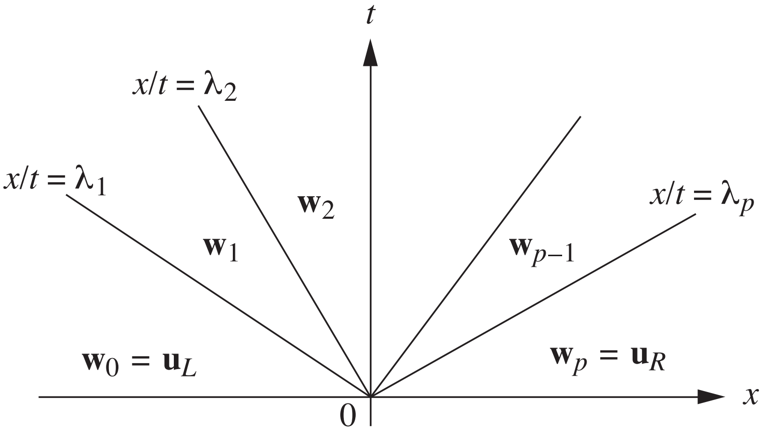
Decouples in p scalar transport equations: $\mathbf{u} = \sum_{i=1}^{p} v_i \mathbf{r}_i$

$$\partial_t v_i + \lambda_i \partial_x v_i = 0, \ 1 \leq i \leq p$$

solution: $v_i(x,t) = v_i(x - \lambda_i t, 0)$, $\mathbf{u}_{L/R} = \sum_{i=1}^{p} v_{L/R,i} \mathbf{r}_i$

$$v_i(x,0) = \begin{cases} v_{L,i}, \ x < 0 \\ v_{R,i}, \ x > 0 \end{cases}$$

Initial discontinuity at $x = 0$ gives p discontinuities propagating with characteristic speed λ_i, separating constant states (say $\mathbf{w}_i$, $i = 0, p$, and $\mathbf{w}_i - -\mathbf{w}_{i-1} = (v_{R,i} - v_{L,i})\mathbf{r}_i$.
Solution of the Riemann problem is explicit.


t
$x/t = \lambda_2$
$x/t = \lambda_1$
$x/t = \lambda_p$
$\mathbf{w}_2$
$\mathbf{w}_1$
$\mathbf{w}_{p-1}$
$\mathbf{w}_0 = \mathbf{u}_L$
$\mathbf{w}_p = \mathbf{u}_R$
0
x

-Scalar case (strictly convex) case: rarefaction or shock
-Linear system: p contact discontinuities propagating at speed λ_k
Mix both ingredients: propagation of p elementary waves (rarefaction, shock, contact) according to the nature of the characteristic field $(\lambda_k, \mathbf{r}_k)$. The analysis needs:

- Definition of Genuinely Non Linear (GNL) and Linearly Degenerate (LD) characteristic fields
- Smooth elementary $k-$waves: $k-$ rarefaction (GNL), rarefaction curve $\mathcal{R}_k(\mathbf{u}_L)$
- Discontinuous: $k-$ shock wave (k GNL) or $k-$contact discontinuity (k LD), discontinuity curve $\mathcal{C}_k(\mathbf{u}_L)$
- Lax entropy condition: admissible shocks, shock curve $\mathcal{S}_k(\mathbf{u}_L)$
- From $\mathbf{u}_L$, path in Ω following piecewise curves $\mathcal{R}_k(\mathbf{u}_k)$ or $\mathcal{S}_k(\mathbf{u}_k)$ if k GNL, $\mathcal{C}_k(\mathbf{u}_k)$ if k LD, $\mathbf{u}_L = \mathbf{u}_0$ and intermediate states $\mathbf{u}_k$, $1 \leq k \leq p$. Aim: reach $\mathbf{u}_R$.

Solution of the Riemann problem: Lax theorem for 'convex' HSCL.

Smooth self similar solutions $\mathbf{u}(x,t) = \mathbf{v}(x/t)$ satisfy

$$-\frac{x}{t^2}\mathbf{v}' + \frac{1}{t}\mathbf{A}(\mathbf{v})\mathbf{v}' = \mathbf{0}$$

Set $\xi = x/t$

$$(\mathbf{A}(\mathbf{v}) - \xi\mathbf{I})\mathbf{v}' = \mathbf{0}$$

$\exists k \in \{1, 2, .., p\}$

$$(1)\ \mathbf{v}'(\xi) = \mathbf{r}_k(\mathbf{v}(\xi)), \text{ and } (2)\ \lambda_k(\mathbf{v}(\xi)) = \xi$$

(1) will give an integral curve of $\mathbf{r}_k$, condition (2) implies k GNL field

$$D\lambda_k(\mathbf{v}(\xi))\mathbf{v}'(\xi) = D\lambda_k(\mathbf{v}(\xi))\mathbf{r}_k(\mathbf{v}(\xi)) = 1$$

whereas $D\lambda_k(\mathbf{v})\mathbf{r}_k(\mathbf{v}) \equiv 0$ for LD field. Solving (1)+ (2) with $\mathbf{v}(\xi_0) = \mathbf{u}_0$ gives the $k-$rarefaction curve from state $\mathbf{u}_0$: $\mathcal{R}_k(\mathbf{u}_0)$.

Discontinuous solutions. Write

$$\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u_0}) = \int_0^1 \frac{d}{ds}\mathbf{f}(\mathbf{u}_0 + s(\mathbf{u} - \mathbf{u}_0))ds$$

develop and use Rankine-Hugoniot condition

$$\int_0^1 \mathbf{A}(\mathbf{u}_0 + s(\mathbf{u} - \mathbf{u}_0)ds(\mathbf{u} - \mathbf{u}_0) = \sigma(\mathbf{u}_0, \mathbf{u})(\mathbf{u} - \mathbf{u}_0)$$

define the $p \times p$ matrix $\mathbf{A}(\mathbf{u}_0, \mathbf{u}) = \int_0^1 \mathbf{A}(\mathbf{u}_0 + s(\mathbf{u} - \mathbf{u}_0)ds$, it is such that $\mathbf{A}(\mathbf{u}_0, \mathbf{u}_0) = \mathbf{A}(\mathbf{u}_0)$. Then speed of discontinuity $\sigma(\mathbf{u}_0, \mathbf{u})$ is an eigenvalue, and $\mathbf{u} - \mathbf{u}_0$ an eigenvector. The eigenvalues of $\mathbf{A}(\mathbf{u})$ are known: $\lambda_k(\mathbf{u})$, by continuity, for $\mathbf{u}$ near $\mathbf{u}_0$, $\mathbf{A}(\mathbf{u}, \mathbf{u}_0)$ has p real distinct eigenvalues $\lambda_k(\mathbf{u}_0, \mathbf{u})$, eigenvectors $\mathbf{r}_k(\mathbf{u}_0, \mathbf{u})$ and 'left' eigenvectors

$$\mathbf{l}_k(\mathbf{u}_0, \mathbf{u})^T \mathbf{A}(\mathbf{u}_0, \mathbf{u}) = \lambda_k(\mathbf{u}_0, \mathbf{u})\mathbf{l}_k^T(\mathbf{u}, \mathbf{u}_0)$$

$\mathbf{u} - \mathbf{u}_0 = \mathbf{r}_k(\mathbf{u}_0, \mathbf{u})$ will give a curve (the proof needs some development). $\mathcal{S}_k(\mathbf{u}_0)$ entropy part of the curve if k GNL.

Wave curves: $k-$ field GNL

$\mathcal{R}_k(\mathbf{u}_0)$ can be parametrized: $\varepsilon \longrightarrow \Phi_k(\varepsilon) \in \Omega$

$$\Phi_k(\varepsilon) = \mathbf{u}_0 + \varepsilon \mathbf{r}_k(\mathbf{u}_0) + \frac{\varepsilon^2}{2} D\mathbf{r}_k(\mathbf{u}_0).\mathbf{r}_k(\mathbf{u}_0) + \mathcal{O}(\varepsilon^3),\ 0 \le \varepsilon \le \varepsilon_0$$

$\mathcal{S}_k(\mathbf{u}_0)$ can be parametrized: $\varepsilon \longrightarrow \Psi_k(\varepsilon) \in \Omega$

$$\Psi_k(\varepsilon) = \mathbf{u}_0 + \varepsilon \mathbf{r}_k(\mathbf{u}_0) + \frac{\varepsilon^2}{2} D\mathbf{r}_k(\mathbf{u}_0).\mathbf{r}_k(\mathbf{u}_0) + \mathcal{O}(\varepsilon^3),\ -\varepsilon_0 < \varepsilon \le 0$$

$$\sigma_k(\varepsilon) = \lambda_k(\mathbf{u}_0) + \frac{\varepsilon}{2} D\lambda_k(\mathbf{u}_0).\mathbf{r}_k(\mathbf{u}_0) + \mathcal{O}(\varepsilon^2),\ -\varepsilon_0 < \varepsilon \le 0$$

The k wave curve $\mathcal{C}_k(\mathbf{u}_0)$ is thus a $\mathcal{C}^2$ curve $\varepsilon \mapsto \chi_k(\varepsilon)$

$$\chi_k(\varepsilon) = \mathbf{u}_0 + \varepsilon \mathbf{r}_k(\mathbf{u}_0) + \frac{\varepsilon^2}{2} D\mathbf{r}_k(\mathbf{u}_0).\mathbf{r}(\mathbf{u}_0) + \mathcal{O}(\varepsilon^3),\ -\varepsilon_0 < \varepsilon \le \varepsilon_0$$

Same equation valid for $k-$ field LD, only $\sigma_k(\varepsilon) \equiv \lambda_k(\mathbf{u}_0)$

Assume all fields are either GNL or LD. Find p elementary waves, and intermediate states: $\mathbf{u}_L = \mathbf{u}_0 \to \mathbf{u}_1 \to \ldots \to \mathbf{u}_p$

$$\mathbf{u}_1 = \chi_1(\varepsilon_1; \mathbf{u}_0), \mathbf{u}_2 = \chi_2(\varepsilon_2; \mathbf{u}_1), ...\mathbf{u}_p = \chi_p(\varepsilon_p; \mathbf{u}_{p-1})$$

Solve the RP means reach $\mathbf{u}_R = \mathbf{u}_p$

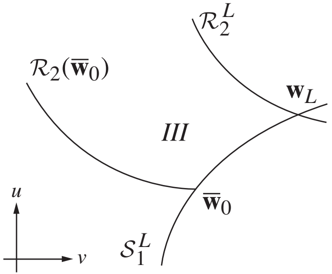
$$\mathbf{u}_p = \chi_p(\varepsilon_p; \chi_{p-1}(\varepsilon_{p-1}; \chi_{p-2}(\varepsilon_{p-2}; ...\chi_1(\varepsilon_1; \mathbf{u}_0))...) \equiv \chi(\varepsilon)$$

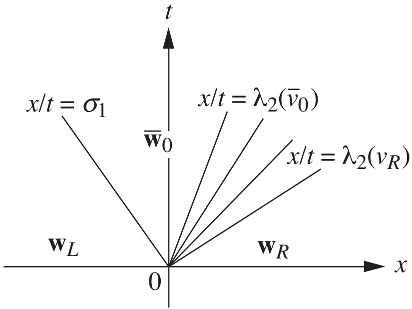
$\varepsilon = (\varepsilon_1, ..., \varepsilon_p) \in \mathbb{R}^p$. Find ε such that $\chi(\varepsilon) = \mathbf{u}_R$.

$$\chi_k(\varepsilon) = \mathbf{u}_L + \varepsilon \mathbf{r}_k(\mathbf{u}_L) + \mathcal{O}(\varepsilon^2),\ |\varepsilon| \leq \varepsilon_0,\ \chi_k'(\mathbf{0}) \sim \mathbf{r}_k(\mathbf{u}_L)$$

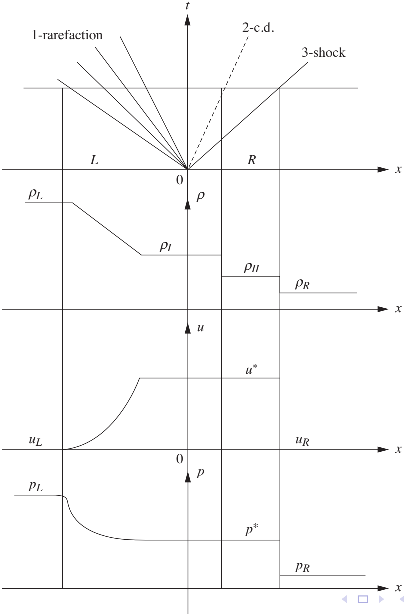
Local inversion theorem: $\chi : \mathbb{R}^p \to \mathcal{V}$, $\mathcal{V}$ neighborhood of $\mathbf{u}_L$ in $\mathbb{R}^p$, $\chi'(\mathbf{0}) \sim (\mathbf{r}_1(\mathbf{u}_L), .., \mathbf{r}_p(\mathbf{u}_L))$ is invertible (basis of $\mathbb{R}^p$).
Lax theorem: $\exists \mathcal{V}(\mathbf{u}_L)$, $\forall \mathbf{u}_R \in \mathcal{V}$, the Riemann problem has a *unique* solution built with elementary $k-$ waves.

$p-$system has 2 GNL fields, only shocks or rarefactions in the solution of the Riemann problem. Example: 1-shock and 2-rarefaction



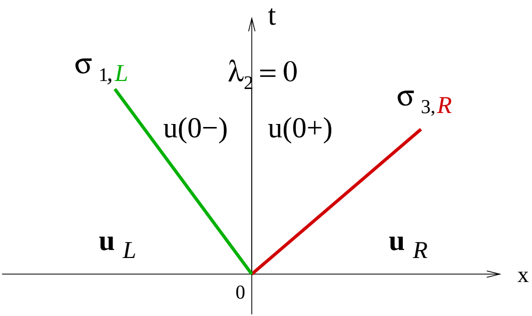


Example of RP for Euler system: shock tube



Coupled Riemann problem

coupled Riemann problem (CRP) $=$ Cauchy problem for coupled systems with Riemann data $\mathbf{u}_L, \mathbf{u}_R$
Take CRP for two Euler systems in Lagrangian frame



Example: solution of a CRP with two shocks $1 - L$ and $3 - R$, one stationary wave $\mathbf{u}(0-)$, $\mathbf{u}(0+)$
(easy because $\lambda_{1,L}(\mathbf{u}) < 0 < \lambda_{3,R}(\mathbf{u})$, no change of sign)

Coupling algorithms for hyperbolic systems

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

1.2. Introduction to finite volume methods

- Finite volume methods: definition, conservation, consistency
- 1d conservative schemes; monotone schemes
- simple schemes: Lax-Friedrichs, Lax-Wendroff
- Godunov scheme
- Roe scheme
- Approximate Riemann solver
- Some required properties for applications

Definition

3 principles in the derivation of a FV method for approximating the solution of a pde (space derivative)

- partition of Ω by cells Ω_i or 'finite volumes'
- 1 unknown per cell

$$u_i \sim \frac{1}{|\Omega_i|} \int_{\Omega_i} u(x)dx$$

  $|\Omega_i|$ measure of Ω_i (length if $d = 1$, surface if $d = 2$, volume if $d = 3$)

- integrate the pde on Ω_i to derive the scheme for computing u_i

Here pde in time too: $u(x, t)$. Above lines give an ODE in $u_i(t)$, *method of line*; add a scheme for advancing in time.
Same formalism for a system: $\mathbf{u}_i$ has p components.

Definition

$$\text{Start from } \; u_i^0 = \frac{1}{|\Omega_i|}\int_{\Omega_i} u(x,0)dx$$

$$\text{define exact } \tilde{u}_i(t) = \frac{1}{|\Omega_i|}\int_{\Omega_i} u(x,t)dx, \;\; u_i(t) \sim \tilde{u}_i(t)$$

$$\text{integrate the equation } \; \int_{\Omega_i} (\frac{\partial u}{\partial t} + \operatorname{div} f(u(x,t)))dx = 0$$

$$\frac{\partial}{\partial t}\int_{\Omega_i} u(x,t)dx + \int_{\partial\Omega_i} f(u(x,t)).n\, dx = 0$$

$$\int_{\partial\Omega_i} f(u(x,t)).n\, dx = \sum_{j\in\mathcal{N}(i)} \int_{\partial\Omega_i\cap\partial\Omega_j} f(u(x,t)).n_{i,j}\, dx$$

($\mathcal{N}(i)$ neighbors of Ω_i). Replace $\frac{\partial}{\partial t}\int_{\Omega_i} u(x,t)dx = |\Omega_i|\partial_t \tilde{u}_i(t) \sim |\Omega_i|\partial_t u_i(t)$, replace the exact normal flux through an edge $e_{i,j} = \partial\Omega_i \cap \partial\Omega_j$ by a numerical one.

Definition

On $e_{i,j}$ approach the exact normal flux using only the unknown values $(u_k(t))$; simplest: use only $u_i(t), u_j(t)$

$$\int_{e_{i,j}} f(u(x,t)).n_{i,j}\,dx \sim |e_{i,j}|\Phi(u_i(t), u_j(t), n_{i,j})$$

method of line $|\Omega_i| \dfrac{\partial}{\partial t} u_i(t) + \sum_{j\in\mathcal{N}(i)} |e_{i,j}|\Phi(u_i(t), u_j(t), n_{i,j}) = 0$

ODE; for a first order scheme, use Euler method. Fully discretized

scheme $\dfrac{|\Omega_i|}{\Delta t}(u_i^{n+1} - u_i^n) + \sum_{j\in\mathcal{N}(i)} |e_{i,j}|\Phi(u_i^n, u_j^n, n_{i,j}) = 0$

$\Phi(u,v,n)$ is a numerical flux
Consistency with exact flux: $\Phi(u,u,n) = f(u).n$
Conservation: $\Phi(u_i, u_j, n_{i,j}) = -\Phi(u_j, u_i, -n_{i,j})$
Same formalism for a system: $\Phi(\mathbf{u}, \mathbf{u}, n)$ has p components.

In dimension $d = 2$: along an 'infinite' edge $e = e_{i,j}$, axis $(n, n^{\perp})$, the independent variables are noted (ζ, τ), given a constant on each side of the edge, the solution does not depend on the tangent variable τ, only on the variable ζ along n, the normal axis.
Solve a 1d Riemann problem for $f.n$ with Riemann data (u_i, u_j).
In dimension $d = 3$: the same with an 'infinite' face, depends only on the variable on the normal axis.
Example: vertical edge, $n = (1, 0), n^{\perp} = (0, 1)$, $(\zeta, \tau) = (x, y)$: use a one-dimensional numerical flux $g(u_i, u_j)$ which approaches the exact flux.
In general, use this one-dimensional flux for the projected equation with continuous flux $f.n$ ('simple' if the equations are rotational invariant).
Conclusion: derive good numerical fluxes for 1d problems!
Test ideas on scalar conservation laws. Try to extend to systems.

1d numerical schemes

$\Omega_i = (x_{i-1/2}, x_{i+1/2})$, $x_i = (x_{i-1/2} + x_{i+1/2})/2$, $|\Omega_i| = \Delta x_i$. Finite volume schemes, numerical flux $g : \mathbb{R}^2 \to \mathbb{R}$

$$\Delta x_i \frac{u_i^{n+1} - u_i^n}{\Delta t} + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n) = 0$$

sign $-$ expresses conservation; g *consistent* with f: $g(u,u) = f(u)$

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i}(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n))$$

Theorem (Lax-Wendroff): IF converges (in a reasonable way), the limit is a weak solution.

Finite difference form: $\Delta x_i = \Delta x$, $\lambda = \Delta t / \Delta x$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)}{\Delta x} \sim \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\partial}{\partial x} f(u(x_i, t_n))$$

$$\text{set} \quad g_{i+1/2} = g(u_i, u_{i+1})$$

3-point scheme. More generally, $g_{i+1/2} = g(u_{i-1}, u_i, u_{i+1}, u_{i+2})$
5-point scheme (necessary for higher order methods) ...

Different types

1. finite difference type schemes
2. using the properties of the HSCL: exact or approximate Riemann solver, FVS (flux vector splitting)
3. using other approaches: Lagrange-projection, relaxation, kinetic

Examples

1. Lax-Friedrichs, Lax-Wendroff
2. Godunov, Roe, HLLE, Osher
3. relaxation schemes, kinetic schemes

Links between 1, 2 and 3, for instance: Lax-Friedrichs and Rusanov, Rusanov and relaxation, kinetic and flux vector splitting, relaxation and Godunov-type schemes...

upwind, Godunov and Roe scheme

- linear case: in the scalar case $\partial_t u + a\partial_x u = 0$, upwind scheme $g(u,v) = a_+ u + a_- v$; for a system, same in characteristic var.
- nonlinear case if $f'(u)$ $(p = 1)$ or $\lambda_i(\mathbf{u})$ (system) changes sign: Godunov's scheme. Discretize $\mathbf{u}_0(x)$ by averaging on each cell: $\mathbf{u}_0(x) \mapsto \mathbf{u}_i^0 = \frac{1}{\Delta x}\int_{\Omega_i} \mathbf{u}_0(x)dx$
  1. $(\mathbf{u}_i^0) \mapsto \mathbf{u}_\Delta(x,0)$ piecewise constant
  2. solve exactly the HSCL with i.c. $\mathbf{u}_\Delta(x,0)$: $t \in]0,\Delta t] +$ CFL, is a juxtapositon of Riemann problems, gives $\rightarrow \mathbf{u}(x,\Delta t)$
  3. project back $\mathbf{u}(x,\Delta t)$ on piecewise constant functions $\rightarrow \mathbf{u}_i^1$

  solve a Riemann problem at each interface $x_{i+1/2}$, gives: $g(\mathbf{u}_i,\mathbf{u}_{i+1}) = f(w_R(0\pm;\mathbf{u}_i,\mathbf{u}_{i+1}))=$ Godunov's flux
- if too complex! solve a linear Riemann problem at each interface $x_{i+1/2}$ with some matrix $A_{i+1/2}$ (Roe's matrix)
- more generally, use an approximate Riemann solver (HLL)

Usual properties

Order of accuracy. Taylor expansion: 3-point schemes are first order (if monotone) or second order (Lax-Wendroff)

Stability.

- $\mathbb{L}^2$ linear stability: use Fourier transform or normal modes
- $\mathbb{L}^\infty$ stability (convex combination)
- monotone schemes: *scalar* property $u^0 \leq v^0 \Rightarrow u^n \leq v^n$
- monotonicity preserving, TVD (*scalar*)

CFL condition for explicit schemes: $\frac{\Delta t}{\Delta x} \max |f'(u)| \leq cfl \leq 1$ for system becomes $\frac{\Delta t}{\Delta x} \max |\lambda_i(\mathbf{u})| \leq cfl \leq 1$

Entropy : discrete entropy inequality with entropy $\mathcal{U}$ and consistent numerical entropy flux $\mathcal{G}$

$$\mathcal{U}(u_i^{n+1}) \leq \mathcal{U}(u_i^n) + \lambda_i \big(\mathcal{G}(u_i^n, u_{i+1}^n) - \mathcal{G}(u_{i-1}^n, u_i^n)\big)$$

Monotone schemes are $\mathbb{L}^\infty$ stable, TVD, Entropy satisfying... but only first order accurate. Example: Godunov, Lax-Friedrichs, Osher...

stability:

-preservation of invariant domains, keep the approximate solution in the physical set of states, for instance preserve the positivity of ϱ, p, $\alpha \in [0,1]$ for a volume fraction...

-discrete entropy inequalities, maximum principle for the specific entropy satisfied by Godunov, HLLE

accuracy:

-exactly resolve stationary contact discontinuity (satisfied by Godunov, Roe, VFRoe, not by HLLE)

-capture stationary discrete shocks with at most two intermediate states (satisfied by Godunov, Roe)

some specific problems:

- well-balanced: preserve equilibria at the discrete level
- asymptotic preserving: when the continuous equation has some asymptotic behavior, mimic that at the discrete level
- compute low-mach, slowly moving shocks ($\sigma / \max_i |\lambda_i| << 1$)

Some tools

- entropy fix
- choice of variables: non conservative VFRoenc
- entropy variables
- add diffusion/antidiffusion
-
-

Positivity of ϱ and p satisfied by Godunov (not by Roe) at least for a $\gamma-$law $p = (\gamma - 1)\varrho\varepsilon$:

$$\text{if } \varrho_i^0 \geq 0, p_i^0 \geq 0, \text{ then } \forall n > 0, \varrho_i^n \geq 0, p_i^n \geq 0.$$

Proof: $\mathbf{u}_i^n = (\varrho_i^n, (\varrho u)_i^n, (\varrho e)_i^n)$ is the mean value of the solution of an exact evolution step: $\mathbf{u}_i^n = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x, \Delta t)dx$.

It yields $\varrho_i^n \geq 0, \forall i \in \mathbb{Z}$, because $\mathbf{u}(x,t) = (\varrho, \varrho u, \varrho e)(x,t)$ is an admissible physical state.

The expression for p_i^n is less straightforward. We have

$$p_i^n = (\gamma - 1)\Big((\varrho e)_i^n - \frac{1}{2}\varrho_i^n (u_i^n)^2\Big), \; u_i^n = \frac{(\varrho u)_i^n}{\varrho_i^n}.$$

The energy component is given by

$$\Delta x_i (\varrho e)_i^n = \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho\varepsilon(x, \Delta t)dx + \frac{1}{2}\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u^2(x, \Delta t)dx,$$

then by Cauchy-Schwarz inequality

$$\Big(\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u \, dx \Big)^2 \leq \Big(\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho \, dx \Big) \Big(\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u^2 \, dx \Big)$$

gives by definition of u_i^n

$$\varrho_i^n (u_i^n)^2 \leq \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u^2 \, dx$$

thus since

$$\frac{\Delta x_i p_i^n}{(\gamma - 1)} = \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho \varepsilon \, dx + \frac{1}{2} \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u^2 \, dx - \frac{1}{2} \varrho_i^n (u_i^n)^2$$

we get

$$p_i^n \geq (\gamma - 1) \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho \varepsilon \, dx \geq 0$$

this last integral is positive since $\varrho \varepsilon \geq 0$ again because $\mathbf{u}(x, t)$ is an admissible physical state.

Roe scheme

Roe-type linearization: $A(\mathbf{u},\mathbf{v})$ Roe matrix if $A(\mathbf{u},\mathbf{v})$ is a $p\times p$ matrix satisfying

- $\mathbf{f}(\mathbf{v})-\mathbf{f}(\mathbf{u})=A(\mathbf{u},\mathbf{v})(\mathbf{v}-\mathbf{u})$
- $A(\mathbf{u},\mathbf{v})$ has real eigenvalues $a_k(\mathbf{u},\mathbf{v})$
- and a corresponding set of eigenvectors, basis of $\mathbb{R}^p$: $r_k(\mathbf{u},\mathbf{v})$.

Theoretical result: if the system has a strictly convex entropy $\mathcal{U}$, $A(\mathbf{u},\mathbf{v})$ exists.
in practice: $A(\mathbf{u},v)=A(m(\mathbf{u},v))$, find an averaging operator, exists for Euler.
The scheme is given by

$$\Delta x\mathbf{u}_j^{n+1}=\int_0^{\Delta x/2} w_R^\ell(\frac{x}{\Delta t};\mathbf{u}_{j-1}^n,\mathbf{u}_j^n)dx+\int_{-\Delta x/2}^0 w_R^\ell(\frac{x}{\Delta t};\mathbf{u}_j^n,\mathbf{u}_{j+1}^n)dx$$

w_R^ℓ exact solution of a *linear* Riemann problem associated resp. to $A_{j-1/2}^n=A(\mathbf{u}_{j-1}^n,\mathbf{u}_j^n)$ on (x_{i-1},x_i) and $A_{j+1/2}^n=A(\mathbf{u}_j^n,\mathbf{u}_{j+1}^n)$ on (x_i,x_{i+1}).

Roe scheme

The scheme is

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\lambda}{2}\left((A_{j+1/2}^n - |A_{j+1/2}^n|)(\mathbf{u}_{j+1}^n - \mathbf{u}_j^n) + (A_{j-1/2}^n + |A_{j-1/2}^n|)(\mathbf{u}_j^n - \mathbf{u}_{j-1}^n\right.$$

'matrix upwind form'

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\lambda}{2}\left((A_{j+1/2}^n)^-(\mathbf{u}_{j+1}^n - \mathbf{u}_j^n) + (A_{j-1/2}^n)^+(\mathbf{u}_j^n - \mathbf{u}_{j-1}^n)\right)$$

eigenvector decomposition: α_k coefficient of $\Delta\mathbf{u}$ on r_k, $a_k^\pm$ eigenvalues of $A^\pm$

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \lambda \sum_{k=1}^{p} \left((\alpha_k a_k^- r_k)_{j+1/2}^n + (\alpha_k a_k^+ r_k)_{j-1/2}^n\right)$$

conservative form with numerical flux

$$g(\mathbf{u},\mathbf{v}) = \frac{1}{2}(\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})) - \frac{1}{2}|A(\mathbf{u},\mathbf{v})|(\mathbf{v} - \mathbf{u})$$

the viscosity matrix is $|A(\mathbf{u},\mathbf{v})|$.

$A(\mathbf{u}_L, \mathbf{u}_R) = A(m(\mathbf{u}_L, \mathbf{u}_R))$, m mean operator computed by *parameter vector* = change of variables $\mathbf{u} \longrightarrow \mathbf{w}(\mathbf{u})$ such that we get homogeneous quadratic functions of $\mathbf{w}$: $\mathbf{u}(\mathbf{w})$ and $g(\mathbf{w}) = f(\mathbf{u}(\mathbf{w}))$

$$\Delta g = g'((\mathbf{w}_- + \mathbf{w}_+)/2)\Delta\mathbf{w}, \;\; \Delta\mathbf{u} = \mathbf{u}'((\mathbf{w}_- + \mathbf{w}_+)/2)\Delta\mathbf{w}$$

then $m(\mathbf{u}_-, \mathbf{u}_+) = \mathbf{u}((\mathbf{w}_- + \mathbf{w}_+)/2)$. $H = e + p/\varrho$ total specific enthalpy $(\varrho e + p)u = \varrho H u$

$$\mathbf{w} = (\sqrt{\varrho}, \sqrt{\varrho}u, \sqrt{\varrho}H)^T, \mathbf{u} = (w_1^2, w_1 w_2, w_1 w_3 - p,)^T$$

$$g(\mathbf{w}) = (w_1 w_2, w_2^2 + p, w_2 w_3)^T$$

for an ideal gas $p = -(\gamma - 1)/2\gamma w_2^2 + (\gamma - 1)/\gamma w_1 w_3$, also OK for Gruneisen law, not possible for any real gas equation of state.
Note $\overline{\mathbf{u}}$ the Roe average state of $\mathbf{u}_L, \mathbf{u}_R$

$$\overline{u} = \frac{\varrho_L u_L + \varrho_R u_R}{\varrho_L + \varrho_R}, \overline{H} = \frac{\varrho_L H_L + \varrho_R H_R}{\varrho_L + \varrho_R}$$

comes from $u = w_2/w_1$, $H = w_3/w_1$.

Coefficients α_k of $\Delta\mathbf{u}$ on $\mathbf{r}_k$ given by nice formulas
For a more general equation of state $p = p(\varrho, \varrho\varepsilon)$, possible to define $A(\overline{\mathbf{u}})$ if one can find mean values of κ, χ such that

$$\Delta p = \overline{\chi}\Delta\varrho + \overline{\kappa}\Delta(\varrho)$$

Properties of Roe's scheme. Accuracy: solves exactly pure discontinuities (shocks and contacts).
Drawbacks: ϱ, p are not necessarily positive (in case of the interaction of strong shocks) and no entropy inequality, needs an entropy correction near sonic points, for instance diagonalize $Q(u,v) = \lambda|A(u,v)|$ in the basis $\mathbf{r}_k(u,v)$ $\lambda diag(|a_k|)$ and add some entropy by a smooth quadratic regularization of $|x|$ near 0

$$Q_\delta(x) = \begin{cases} \lambda|x|, \ |x| \geq \delta \\ \lambda(x^2/2\delta + \delta/2), \ |x| \leq \delta \end{cases} \tag{1}$$

δ chosen in function of spectral radius of $\overline{A}$, $\delta = \alpha(|\overline{u}| + \overline{c})$, α constant depends on the applications.

Extensions

Replace $Q(u,v)$ by a diagonal matrix $\frac{\alpha(u,v)}{\lambda}\mathbf{I}$ gives a Lax-Friedrichs type scheme

$$g^{Roe}(\mathbf{u},\mathbf{v}) = \frac{1}{2}(\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})) - \frac{1}{2}|A^{Roe}(\mathbf{u},\mathbf{v})|(\mathbf{v}-\mathbf{u})$$

$$g(\mathbf{u},\mathbf{v}) = \frac{1}{2}(\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})) - \frac{1}{2\lambda}\alpha(\mathbf{u},\mathbf{v})(\mathbf{v}-\mathbf{u})$$

Stability: $\alpha \leq 1$, LF for $\alpha = 1$. Rusanov:
$\alpha(\mathbf{u},\mathbf{v}) = \lambda max(max_i(|\lambda_i(\mathbf{u})|, max_i(|\lambda_i(\mathbf{v})|)$ and CFL 1/2
Extensions of Godunov's scheme:

- using shock curve decomposition, associated to a path

$$\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{u}) = A(\mathbf{u},\mathbf{v})(\mathbf{v}-\mathbf{u})$$

$$A(\mathbf{u},\mathbf{v}) = \int_0^1 A(\mathbf{u} + s(\mathbf{v}-\mathbf{u}))ds$$

extends to nonconservative systems

- linearization at another state: VFRoe scheme
$A^{VFR}(\mathbf{u},\mathbf{v}) = A((\mathbf{u}+\mathbf{v})/2)$.

VFRoe scheme in nonconservative variables: $\mathbf{w} = \mathbf{w}(\mathbf{u})$

$$\partial_t \mathbf{w} + B(\mathbf{w})\partial_x \mathbf{w} = 0$$

linearization

$$\partial_t \mathbf{w} + B(\hat{\mathbf{w}})\partial_x \mathbf{w} = 0$$

$\hat{\mathbf{w}} = (\mathbf{w}(\mathbf{u}_L) + \mathbf{w}(\mathbf{u}_R)/2)$, $B(\mathbf{w}_L, \mathbf{w}_R) = B((\mathbf{w}_L + \mathbf{w}_R)/2)$ then $A^{VF}(\mathbf{u}_L, \mathbf{u}_R) = A(\mathbf{u}(\hat{\mathbf{w}}))$.
Simple, no theoretical good properties (can produce negative ϱ), but gives practical good results with a good choice of $\mathbf{w}$.
Example: for isentropic gas dynamics,

$$\begin{aligned} \partial_t \varrho + \partial_x \varrho u &= 0 \\ \partial_t \varrho u + \partial_x(\varrho u^2 + p) &= 0 \end{aligned}$$

$p = p(\varrho) = \kappa \varrho^\gamma$

Choose $\mathbf{w} = (\varphi(\varrho), u)$, where φ is involved in Riemann invariant w, $w_\pm = u \pm \varphi(\varrho), \varphi'(\varrho) = \sqrt{p'(\varrho)}/\varrho$, quasilinear formulation

$$\begin{aligned} \partial_t\varphi + u\partial_x\varphi + \sqrt{p'(\varrho)}\partial_x\varrho u &= 0, \\ \partial_t\ u + u\partial_x u + \sqrt{p'(\varrho)}\partial_x\varrho u &= 0 \end{aligned}$$

diagonalizable with $w_\pm$.
Vacuum appears if $u_R - u_L \geq \varphi_R + \varphi_L$
Linearize with $\hat{u} = (u_L + u_R)/2, \hat{\varphi} = (\varphi(\varrho_L) + \varphi(\varrho_L))/2, \sqrt{p'(\hat{\varrho})}$
Linear Riemann problem has an intermediate state for
$\lambda_1 = \hat{u} - \sqrt{p'(\hat{\varrho})} < x/t < \lambda_2 = \hat{u} + \sqrt{p'(\hat{\varrho})}$
$u^* = (u_L + \varphi_L + u_R - \varphi_R)/2, \varphi^* = (u_L + \varphi_L - u_R + \varphi_R)/2,$
φ^* defines $\varrho \geq 0$ only if $u_L + \varphi_L - u_R + \varphi_R \geq 0$, take φ^*_+ might 'ensure' $\varrho > 0$ (no vacuum).

Second order extension

Use the same numerical flux on more 'accurate' values (MUSCL approach), in time use RK or some 2nd order scheme
-piecewise constant reconstruction, one value per mesh is first order

$$u_i = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x) dx$$

-define two values per mesh $u_{i+1/2-}$, $u_{i+1/2+}$ (using $u_{i\pm 1}, u_i$) such that both $u_{i+1/2\pm} = u(x_{i+1/2}) + \mathcal{O}(\Delta x)^2$, and define the new numerical flux: $g_{i+1/2} = g(u_i, u_{i+1})$ replaced by
$g(u_{i+1/2-}, u_{i+1/2+})$
-second order accurate reconstruction operator: scalar piecewise linear $u_\delta(x) = u_i + \delta_i(x - x_i)$ in Ω_i with slope δ_i computed from nearby values and limited,

$$u_{i-1/2+} = u_\delta(x_{i-1/2}) = u_i - \Delta x_i \delta_i / 2$$

$$u_{i+1/2-} = u_\delta(x_{i+1/2}) = u_i + \Delta x_i \delta_i / 2$$

- TVD property $\Rightarrow$ need of limiter, for example

$$\delta_i = \text{ minmod}\left(2\frac{u_i - u_{i-1}}{\Delta x_{i-1} + \Delta x_i}, 2\frac{u_{i+1} - u_i}{\Delta x_{i+1} + \Delta x_i}\right)$$

- for systems, which variables are piecewise linear (+limited): conservative / primitive?
- second order needs 'slopes', in 2d, approximating the gradient is less obvious than in 1d.

Introduction to the treatment of source terms

Treatment depends on the nature of the ‘source’.

- External force, gravity: explicit treatment
- need of upwinding in some cases
- stiff source terms (reacting flow, relaxation): implicit treatment or
- frequent tool: operator splitting
- geometric source terms: well-balanced schemes = preserve some discrete steady states
- friction like source terms: asymptotic preserving schemes = preserve asymptotic behavior
- higher order terms: diffusion, dispersion...

Introduction to operator splitting

A simple example

$$\partial_t u + a\partial_x u = -\alpha u$$

$u(x,t) = e^{-\alpha t} u_0(x - at)$, if $a > 0$, an upwind method and explicit treatment of source term give

$$u_j^{n+1} = u_j^n - \lambda a(u_j^n - u_{j-1}^n) - \alpha \Delta t u_j^n$$

An operator splitting consists in solving in two steps (in time)

- 1. $\partial_t u + a\partial_x u = 0$ with upwind: $u_j^{n+1-} = u_j^n - \lambda a(u_j^n - u_{j-1}^n)$
- 2. $\partial_t u = -\alpha u$ gives with Euler: $u_j^{n+1} = u_j^{n+1-} - \alpha \Delta t u_j^{n+1-}$

results in a first order accurate (setting $\nu = \lambda a$)

$$u_j^{n+1} = u_j^n - \lambda a(u_j^n - u_{j-1}^n) - \alpha \Delta t u_j^n + a\alpha\lambda\Delta t(u_j^n - u_{j-1}^n)$$

$$u_j^{n+1} = u_j^n - \nu(u_j^n - u_{j-1}^n) - \alpha \Delta t(u_j^n(1-\nu) + \nu u_{j-1}^n)$$

$1. \partial_t u + a\partial_x u = 0 \Rightarrow u(x,t) = u_0(x - at)$
$2. \partial_t u = -\alpha u \Rightarrow u(x,t) = e^{-\alpha t} u_0(x)$ each solved on a time step gives $u(x, t + \Delta t) = e^{-\alpha \Delta t} u(x - a\Delta t, t)$= exact solution

In general

$$\partial_t u + (A + B)u = 0$$

where A, B are operators (differential or not, previous example: $Au = \partial_x u, Bu = \alpha u$); in general advection (differential) and source (may be stiff). Solve

- $\partial_t u + Au = 0$ on one time step, from i.c. $u(x, t)$ gives $\tilde{u}(x, t + \Delta t) = e^{-\Delta t A} u(x, t)$
- $\partial_t u + Bu = 0$ on one time step, from i.c. $\tilde{u}(x, +\Delta t)$ gives $\check{\tilde{u}}(x, t + \Delta t) = e^{-\Delta t B} \tilde{u}(x, t + \Delta t) = e^{-\Delta t B} e^{-\Delta t A} u(x, t)$
- exact solution would be $u(x, t + \Delta t) = e^{-\Delta t (A+B)} u(x, t)$

If A and B do not commute, there is a splitting error, it results in a first order method. Can be improved by Strang's splitting.

Strategy

Both schemes (splitting, upwinding) converge towards the same solution, as the mesh size vanishes, and with the same rate. When the mesh is given, which is best? Answer: problem dependent

- relaxation scheme: splitting with instantaneous relaxation
- preserve equilibria $(A + B)u = 0$ (steady solutions) at the discrete level, in general, by balancing exactly flux gradient and source term: well balanced scheme (or interface Riemann solver). If source terms exactly balance convective effects, source terms have to be *upwinded* in accordance with upwinded convective fluxes. Splitting gives poor accuracy on coarse meshes. The first step may introduce non equilibrium states (ex. simulating atmosphere at rest may create 'catastrophic' behavior).
- compute unsteady flows, with some external time scale; splitting behaves better, well balanced scheme should be improved.

Introduction to relaxation schemes

Consider a simple example

$$\begin{aligned} \partial_t u + \partial_x v &= 0, \\ \partial_t v + \partial_x p(u) &= \lambda(f(u) - v), \end{aligned} \tag{2}$$

with $p(u) = au$, $a > 0$ constant, satisfying Whitham

$$-\sqrt{a} < f'(u) < \sqrt{a}.$$

Appropriate discretization of (2) will approximate the solution u of the conservation law $\partial_t u + \partial_x f(u) = 0$ for λ large enough.
Diagonalize (2), with 'Riemann invariants'
$w = v - \sqrt{a}u, z = v + \sqrt{a}u$ propagating with speed $\pm\sqrt{a}$, and use upwind scheme.
Inverse relations $u = (z - w)/2\sqrt{a}, v = (w + z)/2$

Flux of (2) is (v, au), numerical flux $g_{i+1/2} = (v_{i+1/2}, au_{i+1/2})$. Upwind scheme in (w, z) gives fluxes $(w_{i+1/2} = w_{i+1}, z_{i+1/2} = z_i)$, hence

$$w_{i+1/2} = (v - \sqrt{a}u)_{j+1/2} = v_{j+1} - \sqrt{a}u_{j+1}$$
$$z_{i+1/2} = (v + \sqrt{a}u)_{j+1/2} = v_j + \sqrt{a}u_j$$

and

$$u_{j+1/2} = \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2\sqrt{a}}(v_{j+1} - v_j)$$
$$v_{j+1/2} = \frac{1}{2}(v_j + v_{j+1}) - \frac{1}{2}\sqrt{a}(u_{j+1} - u_j).$$

For the fully discrete first order scheme, this gives with $\nu = \Delta t/\Delta x$

$$u_j^{n+1} - u_j^n + \tfrac{\nu}{2}(v_{j+1}^n - v_{j-1}^n) - \tfrac{\nu}{2}\sqrt{a}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0$$
$$v_j^{n+1} - v_j^n + \tfrac{\nu\sqrt{a}}{2}(u_{j+1}^n - u_{j-1}^n) - \tfrac{\nu}{2\sqrt{a}}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) =$$
$$\lambda\Delta t(f(u_j^n) - v_j^n).$$

Since the system is linear, the upwind scheme for the first order system is the exact Godunov solver in variables (w, z)

$$W_R(x/t; (w_l, z_l), (w_r, z_r)) = \begin{cases} (w_l, z_l) & \frac{x}{t} < -\sqrt{a} \\ (w_r, z_l) & -\sqrt{c} < \frac{x}{t} < \sqrt{a} \\ (w_r, z_r) & \frac{x}{t} > \sqrt{a} \end{cases}$$

The relaxed spatial discretization with $v = f(u)$ gives a Lax-Friedrichs type scheme

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\nu}{2}\sqrt{a}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

associated with the approximate Riemann solver and projection on the equilibrium variety $(u = (w+z)/2, v = f(u))$

$$w_R(\xi; u_l, u_r) = \begin{cases} u_l & \xi < -\sqrt{a} \\ \dfrac{u_l + u_r}{2} - \dfrac{f(u_r) - f(u_l}{2\sqrt{a}}, & -\sqrt{a} < \xi < \sqrt{a} \\ u_r & \xi > \sqrt{a} \end{cases}$$

Jin-Xin relaxation scheme

and thus the numerical flux

$$g(u_l, u_r) = \frac{1}{2}(f(u_l) + f(u_r)) - \frac{\sqrt{a}}{2}(u_r - u_l)$$

Remark : the Rusanov scheme is obtained by optimizing the choice of a, under the subcharacteristic constraint

$$\sqrt{a} = \sup_{u_l, u_r} |f'(u)|.$$

Generalization to a system of p equations gives $2p$ equations

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u} + \frac{\partial}{\partial x}\mathbf{v} &= \mathbf{0}, \\ \frac{\partial}{\partial t}\mathbf{v} + \mathbf{A}\frac{\partial}{\partial x}\mathbf{u} &= \lambda(\mathbf{f}(\mathbf{u}) - \mathbf{v}), \end{aligned} \tag{3}$$

where $\mathbf{A}$ is now a constant diagonal matrix with positive entries. The choice of a in Rusanov scheme is now

$$\sqrt{a} = \sup_{\mathbf{u}_l, \mathbf{u}_r} \sup_j |\lambda_j(\mathbf{u})|.$$

2d FV schemes

some remarks

- grid effects possible on a cartesian grid
- for a given mesh (T_i), choice of 'cells' Ω_i: cell-center $\Omega_i = T_i$ or cell-vertex scheme $\Omega_i = T_i^*$ dual mesh
- some difficulties to obtain bounds (TVD?), consistency
- attempts to construct truly 2d FV schemes
- second order needs 'slopes', approximating the gradient is less obvious than in 1d
- implementation...

Coupling algorithms for hyperbolic systems

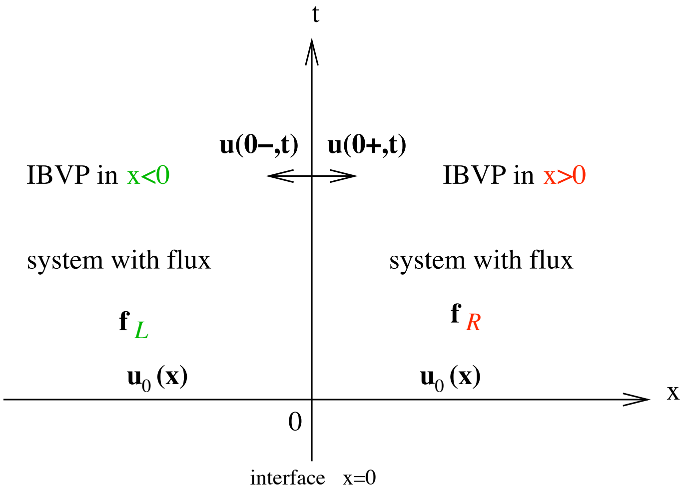
Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

UPMC
SORBONNE UNIVERSITÉS

Our coupling model

the interface is a *boundary* for both (left and right) systems: two IBVP



2. Introduction to the boundary value problem, introduction to interface coupling

- introduction to the IBVP
- simple examples: linear 1d and 2d (scalar, system)
- scalar nonlinear
- nonlinear system
- modeling
- numerical approach
- introduction to interface coupling

Introduction to the boundary value problem

HSCL: set of p conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \mathbf{f}_j(\mathbf{u}) = \mathbf{0}, \ t > 0, \tag{1}$$

$\mathbf{u} = (u_1, u_2, .., u_p)^T \in \Omega$ in $\mathbb{R}^p$ set of *states*, $\mathbf{f}(\mathbf{u}) = (\mathbf{f}_j(\mathbf{u}))$ flux (each $\mathbf{f}_j(\mathbf{u})) \in \mathbb{R}^p$). Initial condition, $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$ on the 'boundary' $t = 0$, and for an IBVP (initial boundary value problem)

$$\mathbf{x} \in \mathcal{O}, \ + \text{ boundary condition } \mathbf{g} \ \ \textit{on } \partial\mathcal{O} \times (0, T)$$

Problems at all levels

- theoretical
- modeling
- numerical approach

Introduction: linear case, advection equation

$$\begin{aligned}\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0, \quad x \in (0,1),\ t > 0, &(2)\\ u(x,0) &= u_0(x), \quad x \in (0,1), &(3)\end{aligned}$$

solutions are constant along *characteristic* lines $x - at =$const. If $a > 0$, they enter the domain from $x = 0$, leave it from $x = 1$. One needs to prescribe the solution on the boundary $x = 0$,

$$u(0,t) = g(t), \quad t > 0 \tag{4}$$

where g is some given function. If $M = (x,t)$ is any point in the domain $(0,1) \times \mathbb{R}_+^\star$, the value of u at M is then uniquely determined. One cannot prescribe the solution on the boundary $x = 1$. The solution u of (2), (3) (4) is then given for $t > 0$ by

$$u(x,t) = u_0(x - at) \quad \text{if } at < x < 1,$$
$$u(x,t) = g\left(t - \frac{x}{a}\right) \quad \text{if } 0 < x < \min(at, 1)$$

The solution of the pure Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y} = 0, & (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^\star, \\ u(x, y, 0) = u_0(x, y), & x, y \in \mathbb{R} \times \mathbb{R} \end{cases} \tag{5}$$

is

$$u(x, y, t) = u_0(x - at, y - bt),$$

and is constant on the characteristic lines $x - at = cst$, $y - bt = cst$, advection direction $\mathbf{C} = (\mathbf{c}, 1)$, $\mathbf{c} = (a, b)^T$.
For an I.B.V.P., $(x, y, t) \in Q = \mathcal{O} \times \mathbb{R}_+^\star$ of $\mathbb{R}^2 \times \mathbb{R}_+$, with boundary Σ; Q is a cylinder and two different kinds of data are given on the surface Σ:
(i) initial data on the set $\mathcal{O}$ of the plane $t = 0$,
(ii) boundary data on the remaining part Γ of Σ (Γ is the side of the cylinder): $\Gamma = \partial\mathcal{O} \times \mathbb{R}_+^\star$. On this surface Γ, $n_t = 0$
Let $\mathbf{n} = (n_x, n_y)^T$ be the outward normal to $\partial\mathcal{O}$ in the plane $t = 0$.

One says that the boundary of $\mathcal{O}$ is *characteristic* at a point if $an_x + bn_y = \mathbf{c} \cdot \mathbf{n} = 0$ at this point.
Boundary data have to be prescribed on the part Γ_- of the boundary that corresponds to incoming characteristics

$$\partial\mathcal{O}_- = \{(x, y) \in \partial\mathcal{O}; \mathbf{c} \cdot \mathbf{n}(x, y) < 0\}, \tag{6}$$

$$u(\cdot, t) = g(\cdot, t) \text{ on } \partial\mathcal{O}_- \iff u = g \text{ on } \Gamma_- = \partial\mathcal{O}_- \times \mathbb{R}_+, \tag{7}$$

and not on the part $\partial\mathcal{O}_+ = \{(x, y) \times \partial\mathcal{O}; \mathbf{c} \cdot \mathbf{n}(x, y) \geq 0\}$ where they are outgoing. Note that if $\mathcal{O}$ is characteristic at $m_0 = (x_0, y_0)$, u cannot be specified on the corresponding line of Γ.

Linear system in 1d

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} = \mathbf{0},\ x \in (0,1), t > 0$$

First if $\mathbf{A} = \mathbf{\Lambda} = diag(a_i)$ constant diagonal $p \times p$ matrix and $a_i \neq 0$ assume q positive eigenvalues $a_i > 0$
$\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-$, $\mathbf{A}^+ = diag(a_i+) \equiv \Lambda^I$, $\mathbf{A}^- = diag(a_i-) \equiv \Lambda^{II}$, $\mathbf{u} = (\mathbf{u}^I, \mathbf{u}^{II})$. Boundary conditions are

$$\mathbf{u}^I(0,t) = \mathbf{g}^I(t), \mathbf{u}^{II}(1,t) = \mathbf{g}^{II}(t)$$

i such that $a_i = 0$ is in II for $x = 0$, in I for $x = 1$. More generally

$$\mathbf{u}^I(0,t) = \mathbf{g}^I(t) + S^I\mathbf{u}^{II}(0,t), \mathbf{u}^{II}(1,t) = \mathbf{g}^{II}(t) + S^{II}\mathbf{u}^I(1,t)$$

S^I, S^{II} rectangular matrices (allow reflection of the outgoing wave) if $\mathbf{A}$ diagonalizable $\mathbf{A} = T\mathbf{\Lambda}T^{-1}$, characteristic variables $\mathbf{w} = T^{-1}\mathbf{u}$, $\mathbf{w} = (\mathbf{w}^I, \mathbf{w}^{II}) \in \mathbb{R}^q \times \mathbb{R}^{p-q}$. Boundary conditions are

$$\mathbf{w}^I(0,t) = \mathbf{g}^I(t) + S^I\mathbf{w}^{II}(0,t), \mathbf{w}^{II}(1,t) = \mathbf{g}^{II}(t) + S^{II}\mathbf{w}^I(1,t)$$

In conservative variables $\mathbf{u}$?
Can we prescribe $E\mathbf{u}(0,t) = \mathbf{g}(t)$, E is a $N \times p$ matrix, $\mathbf{g}(t) \in \mathbb{R}^N$ given (prescribe N linear combinations of the conservative variables).
It requires $N = q$ (number of > 0 eigenvalues) and if $T = (T^I, T^{II})$ (T^I matrix of eigenvectors assoc. to $a_i > 0$, and T^{II} to $a_i < 0$), ET^I must be a $q \times q$ invertible matrix

Example: linearized acoustic in $1d$ given by the linear system

$$\partial t\mathbf{U} + \mathbf{A}_0 \partial x\mathbf{U} = 0,\ x \in [a, b]$$

$\mathbf{U} = (p, u)$ with constant matrix (assume $u_0 = 0$)

$$\mathbf{A}_0 = \begin{pmatrix} 0 & \varrho_0 c_0^2 \\ 1/\varrho_0 & 0 \end{pmatrix}.$$

Characteristic variables are $w_1 = (-p + \varrho_0 c_0 u)/2\varrho_0 c_0$,
$w_2 = (p + \varrho_0 c_0 u)/2\varrho_0 c_0$, resp. propagate at $-c_0$ and c_0,
$p = \varrho_0 c_0 (w_2 - w_1), u = w_1 + w_2$.
Can we prescribe $u(a, t) = u(b, t) = 0$?
$E = (0, 1),\ T^I = \mathbf{r}_1 = (-\varrho_0 c_0, 1)^T,\ T^{II} = \mathbf{r}_2 = (\varrho_0 c_0, 1)^T$,
$ET^I = ET^{II} = 1$ invertible.
At $x = a$ means $w_2(a, t) = -w_1(a, t)$ (known from initial data), at $x = b$ means $w_1(b, t) = -w_2(a, t)$.

Linear system in 2d

$$\partial_t \mathbf{u} + \mathbf{A}\partial_x \mathbf{u} + \mathbf{B}\partial_y \mathbf{u} = \mathbf{0},\ x > 0, y \in \mathbb{R}, t > 0$$

$\mathbf{u}(x, y, 0) = \mathbf{u}_0(x, y), E\mathbf{u}(0, y, t) = \mathbf{g}(y, t)$.
Much more difficult ! assume $\mathbf{A}$ invertible ($x = 0$, $\mathbf{n} = (-1, 0)$ non characteristic boundary), even assume $\mathbf{A}$ diagonal with $q > 0$ eigenvalues.
Necessary condition: q boundary conditions prescribed on $x = 0$ $\mathbf{u}^I(0, y, t) = \mathbf{g}^I(y, t)$.
Not sufficient ! other necessary condition: uniform Kreiss condition (Kreiss-Lopatinski) says

$$\det E\mathbf{N}(\eta, s) \neq 0, \forall \eta \in \mathbb{R}, \mathrm{Re}(s) > 0$$

Use Laplace transform, $\mathbf{D}(\eta, s) = \mathbf{A}^{-1}(s\mathbf{I} - i\eta\mathbf{B})$ has q eigenvalues ξ_i with $Re < 0$, normal modes $\mathbf{u}(x, y, t) = \varphi(x)e^{i\eta y - st}$, N matrix of eigenvectors of $\mathbf{D}$ (cor. to ξ_i), the condition excludes the modes that yield an ill-posed problem.

Nonlinear equation (scalar)

$$\partial_t u + \sum \partial_{x_i} f_i(u) = 0,\ x \in \mathcal{O}, t > 0$$

$d = 1$ already difficult ! Theoretical result (vanishing viscosity method) Bardos-LeRoux-Nédelec (1979): there exists a unique entropy (weak) solution u in $BV(\mathcal{O} \times (0,T))$ in a sense well-defined with Kruzkov's entropy (formulation with test functions) $\varphi \in \mathcal{C}_0^2(\overline{\mathcal{O}} \times [0,T[)$, $\varphi \geq 0$ and any $k \in \mathbb{R}$

$$\int_0^T \int_{\mathcal{O}} \Big\{ |u-k| \frac{\partial \varphi}{\partial t} + \mathrm{sgn}(u-k) \sum_{i=1}^d (f_i(u) - f_i(k)) \frac{\partial \varphi}{\partial x_i} \Big\} d\mathbf{x}\, dt$$

$$+ \int_0^T \int_{\partial \mathcal{O}} \mathrm{sgn}(b-k) (\sum_{i=1}^d (f_i(k) - f_i(\gamma u)) \nu_i) \varphi(\mathbf{s}, t) d\mathbf{s}\, dt,$$

$$+ \int_{\mathcal{O}} \varphi(\mathbf{x}, 0) |u_0(\mathbf{x}) - k| d\mathbf{x} \geq 0, \tag{8}$$

where ν is the unit outward normal to $\partial\mathcal{O}$, γu is the trace of u and $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ a.e. in $\mathcal{O}$, b boundary data

In $1d$, domain $x > 0$ boundary $x = 0$, easy characterization: given a 'boundary value' $b(t)$, the solution is such that $u(0,t)$ satisfies

$$\frac{f(u) - f(k)}{u - k} \leq 0, \forall k \text{ between } u = u(0,t) \text{ and } b = b(t)$$

slope of the chord $[(u, f(u)), (b, f(b))]$ negative.
If $f' > 0$, forces $u(0,t) = b(t)$, if $f' < 0$, no condition. If f' may vanish, nonlinear effects are possible.
Example with Burgers:
• $u_0 = 1$, $f'(u_0) > 0$, $b = -2$, $f'(b) < 0$, rarefaction, $u(0,t) = 0$.
• $u_0 = -1$, $f'(u_0) < 0$, $b = 2$ $f'(b) > 0$, shock entering, with speed $\sigma = 1/2$, $u(0,t) = b = 2$
• $u_0 = -1$, $f'(u_0) < 0$, $b = 1/2$ $f'(b) > 0$, shock leaving, with speed $\sigma = -1/4$, $u(0,t) = -1$.

Nonlinear system

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0},\ x > 0, t > 0$$

Theoretical results: given boundary data $\mathbf{g}$, necessary condition in the form $\mathbf{u}(0,t) \in \mathcal{E}(\mathbf{g}(\mathbf{t}))$ (residual boundary condition, result of Gisclon-Serre)
Easy 'characterization' with Riemann problem: $\mathbf{u}(0,t) \in \mathcal{V}(\mathbf{g}(\mathbf{t}))$, where $\mathcal{V}$ = set of traces at 0 of all possible Riemann problems with given left data $\mathbf{g}$ (Dubois-LeFloch)

$$\mathbf{u}(0,t) = \mathbf{W}_R(0; \mathbf{g}, \mathbf{v}) \text{ for some } \mathbf{v} \in \Omega$$

$\mathcal{E} = \mathcal{V}$ for scalar (nonlinear) equations and also for linear systems: if $\mathbf{g} = cst$, $a_1 \leq a_2 \leq .. \leq a_r \leq 0$, $r = p - q$ non positive eigenvalues

$$\mathcal{V}(\mathbf{g}) = \{\mathbf{u}, \exists \alpha_i \in \mathbb{R}^r, \mathbf{u} = \mathbf{g} + \sum_{i=1}^{r} \alpha_i \mathbf{r}_i\}$$

Modeling and numerics

Different types of boundaries: physical / artificial boundary

Solid boundary: rigid wall. Boundary condition is $\mathbf{u}.\mathbf{n} = 0$: the fluid cannot cross the wall ($u = 0$ in d=1, slip boundary conditions in $d = 2$, the flow moves tangentially to the boundary)

Fluid boundary: linearization. Example in $1d$:

- supersonic inflow: $u_0 > c_0$, $q = 3$, 3 boundary conditions (the whole state must be prescribed),
- subsonic inflow: $0 < u_0 < c_0$, $q = 2$, 2 conditions. Prescribe any linear combination in conservative variables, in primitive variables, (ϱ, u), (ϱ, p), not (u, p)
- subsonic inflow: $q = 1$, $-c_0 < u_0 < 0$, 1 condition, ϱ or u or p
- supersonic inflow: $q = 0$ no condition

choice of the prescribed condition given by modeling

Artificial boundary

For computation, need of a bounded domain, if part of an infinite domain (example: exterior flow)

- absorbing boundary or nonreflecting conditions: 'easy' in 1d, less in 2d (unless flow normal to the boundary)
- other approach: PML (perfectly matched layer)

• For a finite difference scheme, you need the whole boundary state (u^I, u^{II}) even if only u^I is prescribed. Use interpolation techniques, or an upwind scheme to compute these values from the interior known values. For a solid wall, use 'mirror state'.
• In $1d$, for a finite volume monotone scheme, the boundary is an interface, say $x = 0$, you need a flux at the interface. You may use the whole boundary state b even if only part of it is 'used':
$g(b, u_{1/2})$ where is g is a monotone numerical flux, it picks up the relevant data.
• same idea in 2d in the normal direction
• Scalar case: theorem of convergence to the entropy solution of the IBVP (convergence of the traces in some cases)
Some result also exists for Godunov scheme for a convex system (non characteristic boundary).

For a system, usual treatment:
–if you have a known state U_{ext} satisfying the linearized condition, use it in the flux $\Phi(U_{int}, U_{ext})$, where U_{int} is the known state in the interior cell adjacent to the boundary.
–If nonlinear effects are possible, solve partial Riemann problems. What is required is that U_{ext} belongs to a manifold with codimension $q =$ number of specified conditions $=$ number of positive eigenvalues.
Example: supersonic outflow $\mathbf{V}_{ext}$ with subsonic internal computed state $\mathbf{V}_i$. Look for one ($q = 0$) supersonic (or sonic) state $\mathbf{V}_0$ that can be connected to $\mathbf{V}_i$ by 4-wave in $d = 2$ (a 3-wave in $d = 1$). Since $a_4(\mathbf{V}_0, -\mathbf{n}) = -u_{n0} + c_0 \leq 0 \leq a_4(\mathbf{V}_i, -\mathbf{n}) = -u_{-ni} + c_i$, this wave is a 4-rarefaction (3-rarefaction in $d = 1$).

Interface coupling: (theoretical) introduction

2 hyperbolic systems of conservation laws : $\mathbf{u} \in \Omega \subset \mathbb{R}^p$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_L(\mathbf{u}) = \mathbf{0}, \quad x < 0, \ t > 0 \tag{9}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_R(\mathbf{u}) = \mathbf{0}, \quad x > 0, \ t > 0 \tag{10}$$

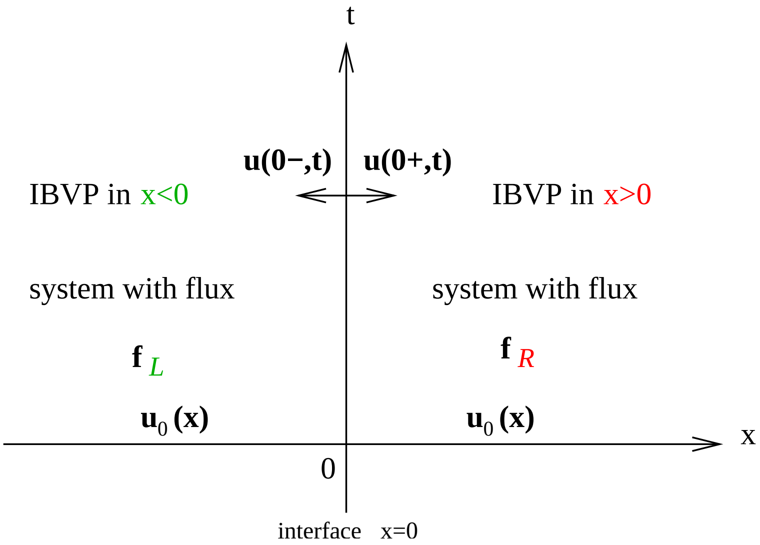
$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}$$

and a coupling condition at $x = 0$

$$\mathbf{u}(0-,t) \in \mathcal{V}_L(\mathbf{u}(0+,t)), \ \mathbf{u}(0+,t) \in \mathcal{V}_R(\mathbf{u}(0-,t)) \tag{11}$$

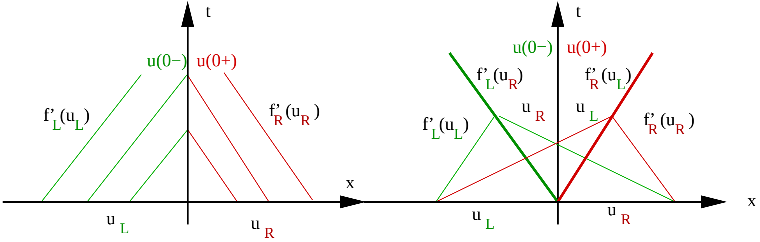
which says that the 2 IBVP are well posed. $\mathbf{u}(0+,t) \in \mathcal{V}_R(\mathbf{b})$ means for some $\mathbf{u} \in \mathbb{R}^p$, $\mathbf{u}(0+,t) = \mathbf{W}_R(0+;\mathbf{b},\mathbf{u})$ $\mathbf{W}_R$ solution of the Riemann problem for $\mathbf{f}_R$ (sim. for $\mathcal{V}_L$ and $\mathbf{f}_L$)

Interface coupling



Interface coupling: scalar case

'characteristic' interface: difficulties are possible !



transmission

The coupling condition

$$\mathbf{u}(0-,t) \in \mathcal{V}_L(\mathbf{u}(0+,t)),\ \mathbf{u}(0+,t) \in \mathcal{V}_R(\mathbf{u}(0-,t))$$

'often' leads to the continuity $\mathbf{u}(0+,t) = \mathbf{u}(0-,t)$:
the conservative variables are *transmitted*
Is it possible to *transmit* other variables (*primitive*)?
Change of dependent variables : $\mathbf{u} \in \Omega \rightarrow \mathbf{v} \in \Omega_{\mathbf{v}}$
$\mathbf{v} \rightarrow \mathbf{u} = \varphi_\alpha(\mathbf{v}); \alpha = L, R$ admissible i.e. $\varphi'_\alpha(\mathbf{v})$ isomorphism of $\mathbb{R}^p$
$\mathbf{c}$ a given boundary *physical* data, set $\mathbf{b}_\alpha = \varphi_\alpha(\mathbf{c})$, define
$\mathcal{V}_L(\mathbf{b}_L) = \{\mathbf{w} = \mathbf{W}_L(0-; \mathbf{u}_-, \mathbf{b}_L); \mathbf{u}_- \in \Omega\}$
$\mathcal{V}_R(\mathbf{b}_R) = \{\mathbf{w} = \mathbf{W}_R(0+; \mathbf{b}_R), \mathbf{u}_+); \mathbf{u}_+ \in \Omega\}$
are admissible boundary sets for L, R
transmission of variable $\mathbf{v}$ obtained by

$$\mathbf{u}(0-,t) \in \mathcal{V}_L(\varphi_L(\mathbf{v}(0+,t)))$$
$$\mathbf{u}(0+,t) \in \mathcal{V}_R(\varphi_R(\mathbf{v}(0-,t)))$$

'often' yields continuity: $\mathbf{v}(0-,t) = \mathbf{v}(0+,t)$

Numerical interface coupling

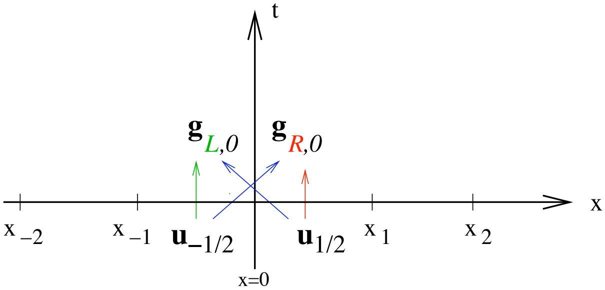
Finite volume method: Δx, Δt, $\mu = \frac{\Delta t}{\Delta x}$, $t_n = n\,\Delta t$, $n \in \mathbb{N}$
cell (x_j, x_{j+1}), center $x_{j+1/2} = \left(j + \frac{1}{2}\right)\Delta x$, $j \in \mathbb{Z}$,
$\mathbf{u}^0_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \mathbf{u}_0(x)dx$, $j \in \mathbb{Z}$.
2 numerical fluxes $\mathbf{g}_L$, $\mathbf{g}_R$, $\mathbf{g}_\alpha$ consistent with $\mathbf{f}_\alpha$
3-point monotone scheme (under CFL condition):

$$\mathbf{g}^n_{\alpha,j} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}\right)$$

- $\mathbf{u}^{n+1}_{j-1/2} = \mathbf{u}^n_{j-1/2} - \mu\left(\mathbf{g}^n_{L,j} - \mathbf{g}^n_{L,j-1}\right), \quad j \leq 0$
- $\mathbf{u}^{n+1}_{j+1/2} = \mathbf{u}^n_{j+1/2} - \mu\left(\mathbf{g}^n_{R,j+1} - \mathbf{g}^n_{R,j}\right), \quad j \geq 0$

2 fluxes at $x = 0$: $\mathbf{g}^n_{L,0}$, $\mathbf{g}^n_{R,0}$

Numerical interface coupling



2 (numerical) fluxes at the interface: $g_{L,0} = g_L(u_{-1/2}, u_{1/2})$, $g_{R,0} = g_R(u_{-1/2}, u_{1/2})$ need of a state $u_{1/2}$ for $g_{L,0}$, $u_{-1/2}$ for $g_{R,0}$

Coupling algorithms for hyperbolic systems

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

Interface coupling, coupling algorithm, examples

- Interface coupling: main features, examples
- Mathematical model: coupling condition
- State coupling / flux coupling
- Numerical interface coupling
- Father model, interface model

Interface coupling: main features

Recall the framework: given two codes

- two (compressible) fluid codes simulating fluid flow of the same 'nature', taking into account different specificities not coupled phenomena *(monophysics)*
- fixed interface *(multidomain)*
- 'thin' interface, the codes interact exchange of information at the interface *(strong coupling)*
- need of a robust procedure understand the physics at the interface *('intelligent' coupling)*
- use existing codes few modifications in each domain

→ give a numerical coupling procedure to 'couple' the codes. The real problem is difficult; try some simpler situations, identifying some specificities on simpler cases.

Examples

- 'Real' examples of coupling codes in thermohydraulics
  - homogeneous models: HEM-HRM (assuming thermodynamic equilibrium or not)
  - 1D - 2D, 1D - 3D models (taking into account symmetry or keeping multidimensional effects)
  - bifluid - drift flux models (1 velocity per fluid or algebraic closure for the drift)
- Some (theoretical) mathematical models for coupling
  - (scalar) conservation laws
  - linear systems (of the same dimension)
  - relaxation (2x2) system / relaxed (scalar) conservation law
  - Euler systems in Lagrangian coordinates
  - " " systems: barotropic (2x2)/ with energy (3x3)
  - coupled Riemann problem for two Euler systems
  - linearly degenerate systems (relaxing to Euler)

Mathematical model for interface coupling

- Two hyperbolic systems of conservation laws (possibly nonconservative)

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0}, \mathbf{u} \in \mathbb{R}^\alpha,\ \alpha = L, x < 0, R, x > 0,\ t > 0 \quad (1)$$

  possibly $\mathbf{u} \in \mathbb{R}^p$ left, $\mathbf{U} \in \mathbb{R}^q$ right
- 'compatibility' between systems (or not): either $p = q$ or $p \neq q$ but (if say $p < q$) $\exists \mathcal{L}$ (lift), $\exists \mathcal{P}$ (projection), $\mathbf{u} \rightarrow \mathbf{U} = \mathcal{L}\mathbf{u}$ and $\mathbf{U} \rightarrow \mathbf{u} = \mathcal{P}\mathbf{U}$
  1. plasma models: same equations, only one flux component is discontinuous
  2. models 1D-2D: 2D system reduces to the 1D system
  3. $p-$system coupled with Euler (in Lagrangian coord.) are compatible
  4. multiphase models: 7 equations (2 velocities) and drift - flux
- two boundary value problems, one on each side of the interface $x = 0$ (*thin* interface, no 'interface model')
  coupling model through the 'choice' of transmitted variables

Coupling Condition

- Given $\mathbf{b}$, IBVP in $x > 0$, one cannot impose $\mathbf{u}(0+,t) = \mathbf{b}$
  $\longrightarrow$ *weak formulation* of the boundary condition:
  $\mathbf{u}(0+,t) \in \mathcal{O}_R(\mathbf{b})$ means $\mathbf{u}(0+,t) = \mathbf{W}_R(0+;\mathbf{b},\mathbf{u})$ for some $\mathbf{u} \in \mathbb{R}^p$
  $\mathbf{W}_R(0+;\mathbf{u}_\ell,\mathbf{u}_r)$ solution of the Riemann problem(RP) with $\mathbf{f}_R$
  $\mathcal{O}_R(\mathbf{b})$ = traces at $x = 0$ of all possible RP between $\mathbf{b}$ and a right state
  (sets $\mathcal{O}$ previouly noted $\mathcal{V}$)
- define the sets $\mathcal{O}_L(\mathcal{P}\mathbf{U}(0+,t))$, $\mathcal{O}_R(\mathcal{L}\mathbf{u}(0-,t))$
- coupling condition (CC):
  $$\boxed{\mathbf{u}(0-,t) \in \mathcal{O}_L(\mathcal{P}\mathbf{U}(0+,t))), \quad \mathbf{U}(0+,t) \in \mathcal{O}_R(\mathcal{L}\mathbf{u}(0-,t))}$$
  state coupling
- transmission possible with other (primitive) variables $\mathbf{u} \mapsto \mathbf{v}$ and $\mathbf{U} \mapsto \mathbf{V}$

Why a thin interface ? why this mathematical model ?
several levels of answer

- codes should not be modified: only the (boundary) data
- need to understand what a 'natural' scheme computes
- in case of non uniqueness, instability linked to resonance is avoided (ex. plasma)
- if one 'regularizes', for large time, behaves like a *coupled problem* (CRP)
- thickening requires more physics

When $p = q$, there is another 'natural' conservative approach.

State coupling / Flux coupling

1. A natural link exists with equations with discontinuous coefficients ($p = q$): conservative approach

$$\partial_t \mathbf{u} + \partial_x \big((1 - H(x)) \mathbf{f}_L(\mathbf{u}) + H(x) \mathbf{f}_R(\mathbf{u}) \big) = \mathbf{0}$$

   yields $\boxed{\mathbf{f}_L(\mathbf{u}(0-,t)) = \mathbf{f}_R(\mathbf{u}(0+,t))}$ *flux* coupling as opposed to *state* coupling
   A conservative form (given by physics) involves some *natural* entropy condition

2. Even for 'identical' systems ($\mathbf{f}_L = \mathbf{f}_R$), the conservative formulation is a choice for transmission: one *decides* to 'transmit' the flux. In some cases, it is not physical (ex. nozzles with discontinuous but constant section, the rate of flow is not conserved)
   $\rightarrow$ we choose to study all possibilities: state and flux coupling

3. One can model the *transmission* of other variables

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_\alpha(\mathbf{u}) = \mathbf{0},\ \alpha = L, x < 0, R, x > 0,\ t > 0 \quad (1)$$

- Flux coupling = conservative approach

$$\partial_t \mathbf{u} + \partial_x \big((1 - H(x))\mathbf{f}_L(\mathbf{u}) + H(x)\mathbf{f}_R(\mathbf{u})\big) = \mathbf{0}, x \in \mathbb{R}$$

yields $\boxed{\mathbf{f}_L(\mathbf{u}(0-, t)) = \mathbf{f}_R(\mathbf{u}(0+, t))}$ the flux is *transmitted*

- $\neq$ State coupling

$$\partial_t \mathbf{u} + \partial_x \big((1 - H(x))\mathbf{f}_L(\mathbf{u}) + H(x)\mathbf{f}_R(\mathbf{u})\big) = \mathcal{M},\ x \in \mathbb{R}$$

  • when $x = 0$ is *non characteristic* the coupling condition CC 'often' yields continuity $\boxed{\mathbf{u}(0+, t) = \mathbf{u}(0-, t)}$ conservative variables are *transmitted*, NOT the flux

  • when $x = 0$ is *characteristic* not all, only part of the conservative variables can be *transmitted*

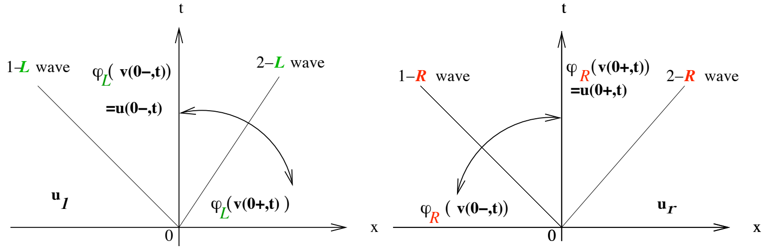
- In some particular case, with transmission of primitive variables state coupling= flux coupling !

- change of variables : $\mathbf{u} \in \Omega \to \mathbf{v} \in \Omega_{\mathbf{v}}$ (conservative/primitive)
- $\mathbf{v} \to \mathbf{u} = \varphi_\alpha(\mathbf{v}); \alpha = L, R$ admissible i.e. $\varphi'_\alpha(\mathbf{v})$ isomorphism of $\mathbb{R}^p$
- $\mathbf{c}$ given by *physics* (pressure), $\mathbf{b}_L = \varphi_L(\mathbf{c})$, $\mathbf{b}_R = \varphi_R(\mathbf{c})$, set
  $\mathcal{O}_L(\mathbf{b}_L) = \{\mathbf{w} = \mathbf{W}_L(0-; \mathbf{u}_-, \mathbf{b}_L); \mathbf{u}_- \in \Omega\}$
  $\mathcal{O}_R(\mathbf{b}_R) = \{\mathbf{w} = \mathbf{W}_R(0+; \mathbf{b}_R, \mathbf{u}_+); \mathbf{u}_+ \in \Omega\}$
  sets of admissible boundary values for L, R
- *transmission of variables* $\mathbf{v}$ obtained by

$$\mathbf{u}(0-, t) \in \mathcal{O}_L(\varphi_L(\mathbf{v}(0+, t)))$$
$$\mathbf{u}(0+, t) \in \mathcal{O}_R(\varphi_R(\mathbf{v}(0-, t)))$$

(note that $\varphi_L(\mathbf{v}(0+, t)) \neq \mathbf{u}(0+, t) = \varphi_R(\mathbf{v}(0+, t))$)
It yields 'continuity' of (or part of) $\mathbf{v}$: $\boxed{\mathbf{v}(0-, t) = \mathbf{v}(0+, t)}$

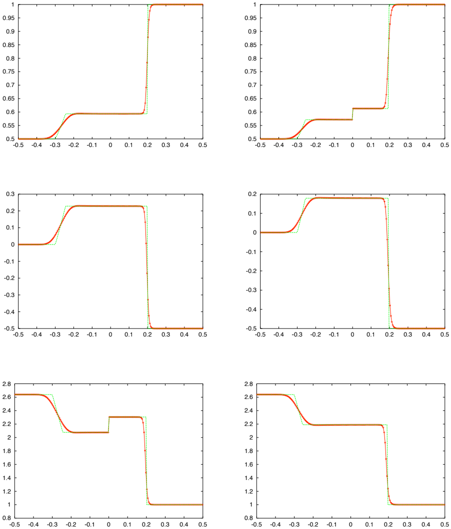
Barotropic Euler system in Lagrangian coordinates $\mathbf{u} = (\tau, v)^T$,
$\mathbf{f}(\mathbf{u}) = (-v, p)^T, \lambda_1 = -C < 0 < \lambda_2 = +C \ \ (C = \sqrt{-p'(\tau)})$
two systems with $p = p_\alpha(\tau), \ \alpha = L, R$
interface $x = 0$, *non characteristic* separates the $1-$ and $2-$waves
CC by transmission of $\mathbf{v} = (v, p)$ yields continuity of $\mathbf{v} = (v, p)$



left RP : $\mathbf{v}(0-) \to \mathbf{v}(0+)$
$2L-$wave

right RP : $\mathbf{v}(0-) \to \mathbf{v}(0+))$
by a $1R-$ wave

τ, v, p in transmission of $\mathbf{u} = (\tau, v)$ left vs $\mathbf{v} = (v, p)$ right

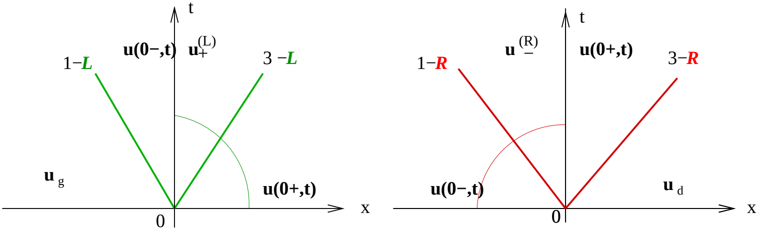


There is a simple explanation:
the flux is an admissible change of variables so the flux can be *transmitted*:

$$(\tau, v) \mapsto (v, p)$$

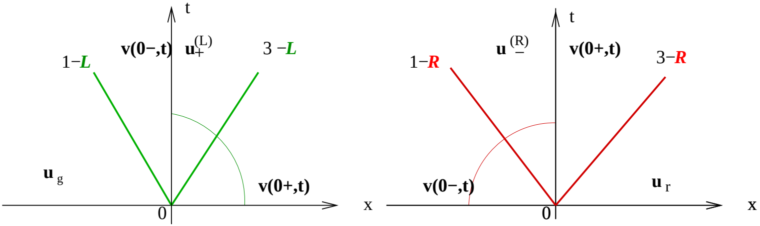
because $\tau \mapsto p(\tau)$ satisfies $p'(\tau) < 0$

Two (full) Euler systems in Lagrangian coordinates
$\mathbf{u} = (\tau, v, e), \mathbf{f}_\alpha(\mathbf{u}) = (-v, p, pv)$, $p = p_\alpha(\tau, \varepsilon)$, $\lambda_2 = 0$ eigenvalue



Coupling condition CC
$\mathbf{u}(0-, t) \in \mathcal{O}_L(\mathbf{u}(0+, t))$ (left), $\mathbf{u}(0+, t) \in \mathcal{O}_R(\mathbf{u}(0-, t))$(right)
$\mathbf{u}(0-) = \mathbf{W}_L(0-; \mathbf{u}_g, \mathbf{u}(0+)), \quad \mathbf{u}(0+) = \mathbf{W}_R(0+; \mathbf{u}(0-), \mathbf{u}_d)$

two Euler systems in Lagrangian coordinates with gamma law:
$\mathbf{u} = (\tau, v, e), \mathbf{f}_\alpha(\mathbf{u}) = (-v, p, pv),\ p = (\gamma_\alpha - 1)\varepsilon/\tau$



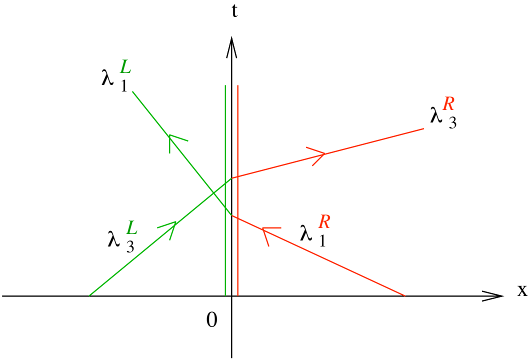
CC in primitive variable $\mathbf{v} = (\tau, v, p)$ yields continuous flux

$$\boxed{p(0-,t) = p(0+,t), v(0-,t) = v(0+,t)}$$

intersection of two wave curves in $(v,p)-$plane:
$\mathbf{v}(0+) \in \tilde{\mathcal{C}}^3_L(\mathbf{v}(0-)) \cap \tilde{\mathcal{C}}^1_R(\mathbf{v}(0-)) = \{\mathbf{v}(0-)\}$
CC in conservative variable $\mathbf{u} = (\tau, v, e)$ yields

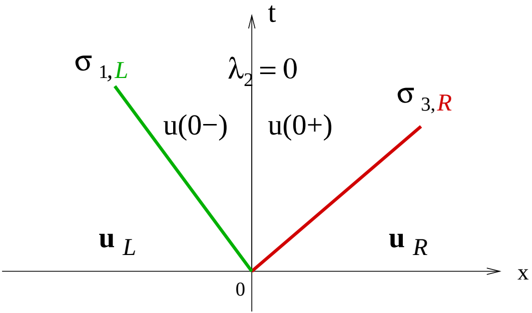
$$\boxed{\varepsilon/\tau(0-,t) = \varepsilon/\tau(0+,t), v(0-,t) = v(0+,t)}$$

heuristic



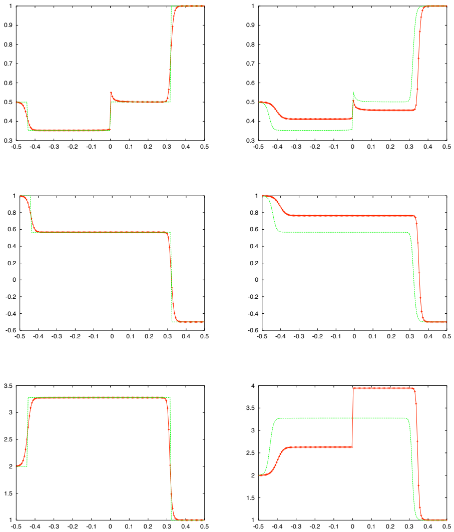
$\lambda_{L,1} < 0 < \lambda_{L,3}$, $\lambda_{R,1} < 0 < \lambda_{R,3}$ $\lambda_{L,2} = \lambda_{R,2} = 0$ *characteristic* case: the flux is NOT an admissible change of variables
heuristic: transmission of 2 quantities (justified by a linearized analysis)
coupled RP (CRP): only $1L-$ waves, $0-$wave and $3R-$waves

coupled Riemann problem (CRP) = Cauchy problem for (1) with Riemann data $\mathbf{u}_L, \mathbf{u}_R$ for two (full) Euler systems (Lagrangian)



Example: solution of a CRP with two shocks $1 - L$ and $3 - R$, one stationary wave $\mathbf{u}(0-)$, $\mathbf{u}(0+)$
(easy because $\lambda_{1,L}(\mathbf{u}) < 0 < \lambda_{3,R}(\mathbf{u})$, no change of sign)

τ, v, p for Euler with CC $\mathbf{v} = (\tau, v, p)$ Left, vs $\mathbf{u} = (\tau, v, e)$ Right



Comments

Nonuniqueness of CC; different CC give different solutions
$\longrightarrow$ need of a *physical* criteria for choosing the transmitted variables (besides conservation of mass), conservation of some stationary solutions (material wave)?

Some difficulties linked to the state coupling approach:

- non conservative systems
- singular source terms
- possible resonance: the eigenvalues may change sign (ex. Euler system in Eulerian coordinates) at $x = 0$
- non uniqueness of the solution

A natural answer: add viscosity

- Dafermos regularization (does not bring uniqueness)
- numerical (uniqueness, but which solution is computed?)

Numerical coupling by a two-flux FV method

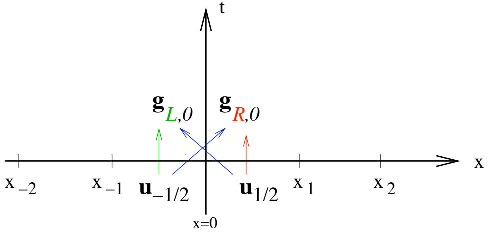
Finite volume method: Δx, Δt, $\mu = \frac{\Delta t}{\Delta x}$, $t_n = n\,\Delta t$, $n \in \mathbb{N}$
cell (x_j, x_{j+1}), center $x_{j+1/2} = \left(j + \frac{1}{2}\right) \Delta x$, $j \in \mathbb{Z}$,
$\mathbf{u}^0_{j+1/2} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \mathbf{u}_0(x) dx$, $j \in \mathbb{Z}$
two numerical fluxes $\mathbf{g}_L$, $\mathbf{g}_R$, $\mathbf{g}_\alpha$ consistent with $\mathbf{f}_\alpha$
3-point monotone scheme (under CFL condition):

$$\mathbf{g}^n_{\alpha,j} = \mathbf{g}_\alpha \left(\mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}\right)$$

- $\mathbf{u}^{n+1}_{j-1/2} = \mathbf{u}^n_{j-1/2} - \mu \left(\mathbf{g}^n_{L,j} - \mathbf{g}^n_{L,j-1}\right), \quad j \leq 0$
- $\mathbf{u}^{n+1}_{j+1/2} = \mathbf{u}^n_{j+1/2} - \mu \left(\mathbf{g}^n_{R,j+1} - \mathbf{g}^n_{R,j}\right), \quad j \geq 0$
- $x_0 = 0$ is a boundary between two cells: two fluxes for $j = 0$
  $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha \left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right), \alpha = L, R$

two numerical fluxes at $x = 0$,

- $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right), \alpha = L, R$ ensures **u**-state coupling



- $\mathbf{g}^n_{L,0} = \mathbf{g}_L(\mathbf{u}^n_{-1/2}, \varphi_L(\mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{0,R} = \mathbf{g}_R(\varphi_R(\mathbf{v}^n_{-1/2}), \mathbf{u}^n_{+1/2})$
ensures **v**-state coupling
- other approach (JM. Hérard): interface model to compute the two numerical fluxes at $x = 0$

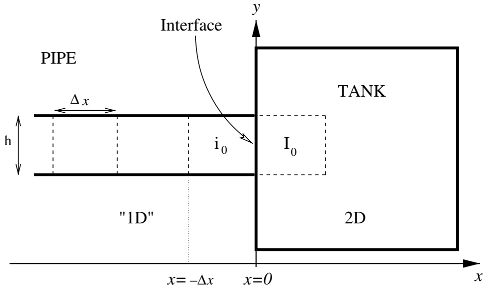
Other approaches

Coupling *consistent* systems with relaxation term:
$(R)\ \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \lambda \mathbf{S}(\mathbf{U})$ as $\lambda \to \infty$ gives
$(E)\ \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}$ (equilibrium)
Example: coupling HEM/HRM (homogeneous models for two-phase flow): $\mathbf{U} = (\varrho, \varrho u, \varrho e, \varrho_1 z)$ (ϱ =mixture density, and ϱ_1 =phase one),$\mathbf{F}(\mathbf{U}) = (\varrho u, \varrho u^2 + p^R, (\varrho e + p^R)u, \varrho_1 z u)$,
$\mathbf{S}(\mathbf{U}) = (0, 0, 0, \varrho_1^* z^*(\varrho) - \varrho_1 z)$,
$\mathbf{u} = (\varrho, \varrho u, \varrho e)$, $\mathbf{f}(\mathbf{u}) = \varrho u, \varrho u^2 + p^E, (\varrho e + p^E)u)$,
HEM is obtained from HRM through relaxation (thermodynamical equilibrium) $p^R(\varrho_1 z, \varrho, \varepsilon) \equiv p^E(\varrho, \epsilon)$ if $\varrho_1 z = \varrho_1^* z^*(\varrho)$

- conservative state coupling: transmission of $(\varrho, \varrho u, \varrho e)$
- state coupling: transmission of *primitive* variables (ϱ, u, p)
- keep the larger system everywhere and $\lambda_L = \infty / \lambda_R$ finite allows *flux coupling*: conservative system with discontinuous flux

Introduce a larger system = father model

Coupling *compatible* models through an interface model used to define interface fluxes (J.-M. Hérard-O. Hurisse).
Example 1D/2D: $\mathbf{u} = (\varrho, \varrho u, \varrho e)$ and $\mathbf{U} = (\varrho, \varrho u, \varrho v, \varrho e)$



transverse velocity v; need to compute a flux on the ϱv component use 'well balanced' approach of LeRoux et al: add a color function z, $\partial_t z = 0$, thus a standing wave at interface, nonconservative pde $\partial_t v + zu\partial_x v = 0$, solve Riemann problem $w_R(0\pm)$ for the numerical interface fluxes
Remark: gives same discrete fluxes as state coupling

Developments: theoretical results

1. Scalar case
   - Existence theorem in some generic situations (and uniqueness in some cases)
   - convergence of the two-flux scheme (monotone, E-scheme)
   - Coupled Riemann problem
   - coupling of the 2x2 relaxation system with the relaxed equation with F. Caetano
   - Dafermos regularization with Benjamin Boutin
2. The case of systems
   - coupling of linear systems
   - multiple choice of transmitted variables
   - coupling of Lagrange-type systems (characteristic interface)
   - coupling Euler system (3x3) and p-system (2x2)
   - coupling two Euler systems (Eulerian coordinates)
     - coupled Riemann problem (state coupling, not easy)
     - relaxation model: explicit solution of CRP for a relaxation system with LD fields; flux coupling for Euler

1. Plasma model: *same model* *(same pde)*, one neglects the current density. Case of non uniqueness
2. Coupling two Euler systems: *same* *model*, different closure laws
   - choice of transmitted variables
   - example u, p for a material wave
   - choice of scheme (relaxation, Lagrange+projection)
   - examples of coupled Riemann problem
3. Coupling multi-phase models: HRM-HEM, two different but *consistent* models, 4 equations / 3 equations
4. Work in progress: 4 equations (mixture model with drift) / 7 equations (bifluid model) *compatibility is not obvious*
5. Hérard-Hurisse: 2D/1D
   bifluid (6 eqns)/HRM
   transition free/porous media

- coupling bifluid and drift flux models:
  - asymptotic expansion of a bifluid model ($\longrightarrow$ drift flux model)
  - relaxation approximation of a bifluid ($\longrightarrow$ a drift flux) model
- control of the transmission procedure, optimization
- asymptotic preserving schemes

Coupling algorithms for hyperbolic systems

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

Interface coupling, coupling algorithm, further topics

- Regularization for state coupling
  - Dafermos regularization
  - finite volume scheme
- Relaxation state coupling solver for flux coupling
- Conclusion and future directions

Interface coupling

CC = coupling conditions
Nonuniqueness of CC; different CC give different solutions
Some difficulties linked to the present approach:

- non conservative systems
- singular source terms
- possible resonance: the eigenvalues may change sign (ex. Euler system in Eulerian coordinates) at $x = 0$
- non uniqueness of the solution

A natural answer: add viscosity
idea: Dafermos regularization (B. Boutin) but does not bring uniqueness

State coupling = non-conservative approach

$$\partial_t u + \partial_x f(u, a) = \mathcal{M}, \quad x \in \mathbb{R},\ t > 0,\ \partial_t a = 0,$$

$f(u, a) = af_L(u) + (1 - a)f_R(u)$, $\mathcal{M}$ measure (Dirac), weight jump

$$[f(u, a)] = f_R(u(0+, t)) - f_L(u(0-, t))$$

Riemann data for a: $a_L = 1, a_R = 0$ and $\partial_t a = 0$
$\Rightarrow a(x)$ is a Heaviside function, $\partial_x a = -\delta_0$,

$$\begin{cases} \partial_t u + \partial_x \Big(af_L(u) + (1 - a)f_R(u)\Big) + \Big(f_R(u) - f_L(u)\Big)\partial_x a = 0 \\ \partial_t a = 0 \end{cases}$$

If u continuous, $(f_R(u) - f_L(u))\partial_x a$ (non conservative product) is well defined. Write the 1st equation

$$\partial_t u + (af_L{}'(u) + (1 - a)f_R{}'(u))\partial_x u = 0$$

System with eigenvalues 0 and $\lambda(u, a) = af_L{}'(u) + (1 - a)f_R{}'(u)$.
Extends to v-coupling

Non conservative system

$$\begin{cases} \partial_t u + \lambda(u,a)\partial_x u = 0 \\ \partial_t a = 0 \end{cases}$$

add a regularization term

$$\begin{cases} \partial_t u_\varepsilon + \lambda(u_\varepsilon, a_\varepsilon))\partial_x u_\varepsilon = t\varepsilon\partial_{xx} u_\varepsilon \\ \partial_t a_\varepsilon = t\varepsilon^2 \partial_{xx} a_\varepsilon \end{cases}$$

initial data $u_\varepsilon(x,0) = u_0(x), a_\varepsilon(x,0) = a_0(x)$

$$u_0(x) = \begin{cases} u_L, \ x < 0 \\ u_R, \ x > 0 \end{cases} \quad a_0(x) = \begin{cases} 1, \ x < 0 \\ 0, \ x > 0. \end{cases}$$

Regularization with t in the RHS was proposed by Dafermos. It corresponds to a classical viscous regularization in variable $\xi = x/t$, $T = \ln t$ and allows to study the approximation of self-similar solutions.

Look for self similar solutions: $\xi = x/t$, $u_\varepsilon(\xi), a_\varepsilon(\xi)$
$u_\varepsilon, a_\varepsilon$ exist, $\exists u$, '$u_{\varepsilon_k} \to u$' as $\varepsilon \to 0$,

- u solution of the CRP, entropy solution in $x < 0$, $x > 0$
- at interface possible boundary layer $\to$ *zoom*: *fast* variable

$y = \xi/\varepsilon$ $\mathcal{U}_\varepsilon(y) = u_\varepsilon(\varepsilon y)$, $\mathcal{A}_\varepsilon(y) = a_\varepsilon(\varepsilon y)$.

- $\mathcal{A}_\varepsilon(y)$ converges to $\mathcal{A}(y) = (1 - \mathrm{erf}(y/\sqrt{2}))/2$, $\mathcal{A}(-\infty) = 1, \mathcal{A}(+\infty) = 0$, non trivial profile connecting 1 to 0 thanks to ε^2 (if ε, $\mathcal{A}(y) = 1/2$)
- possible non trivial profiles for $\mathcal{U}$. If f_α strictly convex:

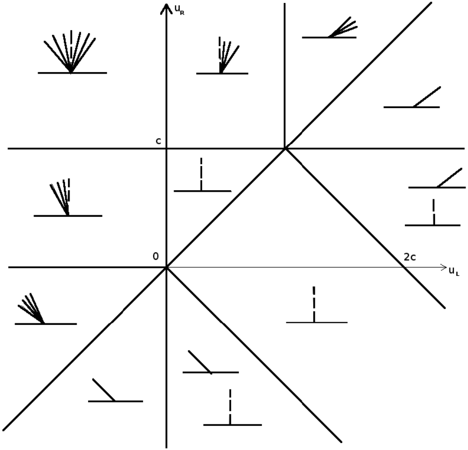
-left: $\mathcal{U}(-\infty) = u(0-)$ or $\mathcal{U}(-\infty) < u(0-)$

$$f'_L(\mathcal{U}(-\infty)) < 0 < f'_L(u(0-))$$

-right: $\mathcal{U}(+\infty) = u(0+)$ or $\mathcal{U}(+\infty) > u(0+)$

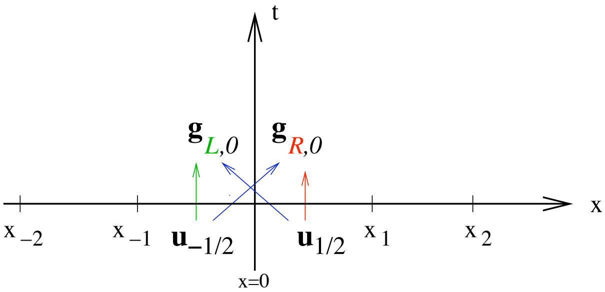
$$f'_R(\mathcal{U}(+\infty)) > 0 > f'_R(u(0+))$$

Structure of the discontinuity $u(0-), u(0+)$: $u(0-), \mathcal{U}(-\infty)$ L-stationary shock, $\mathcal{U}(-\infty), \mathcal{U}(+\infty)$, $\mathcal{U}(+\infty), u(0+)$ R- stationary shock. Rules out some *unstable* solutions, possible nonuniqueness solution of the CRP, in the plane (u_L, u_R)



$f_L(u) = u^2/2, f_R(u) = (u-c)^2/2, c > 0$ (B. Boutin) numerical coupling with FV methods and 2 fluxes at $x = 0$: one can always compute a numerical solution (which?)

- $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right), \alpha = L, R$ ensures **u**-state coupling

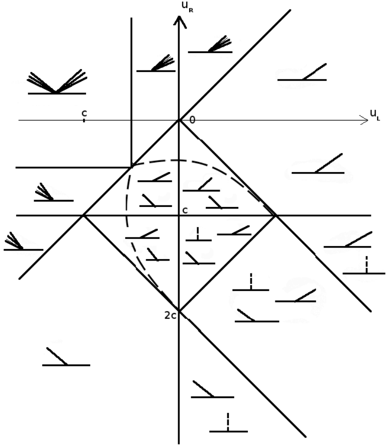


- $\mathbf{g}^n_{L,0} = \mathbf{g}_L(\mathbf{u}^n_{-1/2}, \varphi_L(\mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{0,R} = \mathbf{g}_R(\varphi_R(\mathbf{v}^n_{-1/2}), \mathbf{u}^n_{+1/2})$ ensures **v**-state coupling

If* the two-flux FV scheme converges ($u_\Delta \to u$) in some 'sensible way', (*proven in the scalar case, with rather general assumptions) then u is solution of the coupled problem with our CC.
In case of

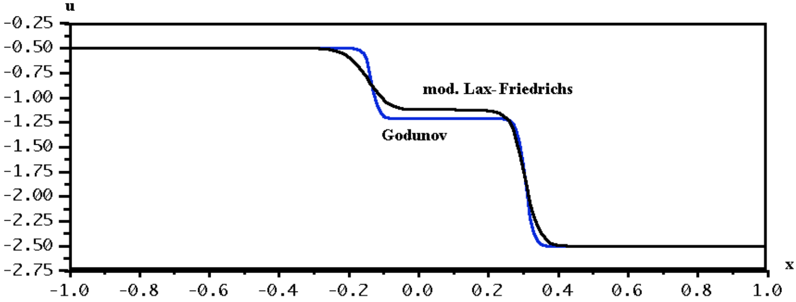
- uniqueness, u is the unique solution
- non-uniqueness, u is a solution, which solution?

scalar quadratic case: $f_L(u) = u^2/2, f_R(u) = (u-c)^2/2, c < 0$:
possible solutions obtained by Dafermos regularization



double shock missing in central area

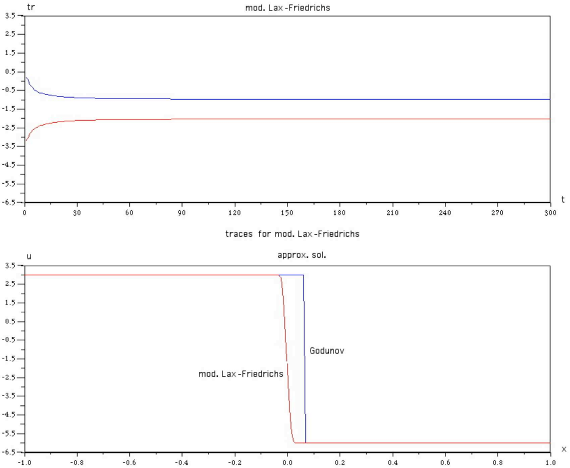
scalar quadratic case: $f_L(u) = u^2/2, f_R(u) = (u+4)^2/2$
CRP with $u_L = -0.5, u_R = -2.5,\ f'_L(u_L) < 0, f'_R(u_R) > 0$



exact solution: 2 shocks computed with Godunov's scheme and Lax-Friedrichs modified: $u_G^m = -1,21, u_{LF}^m = -1,12$

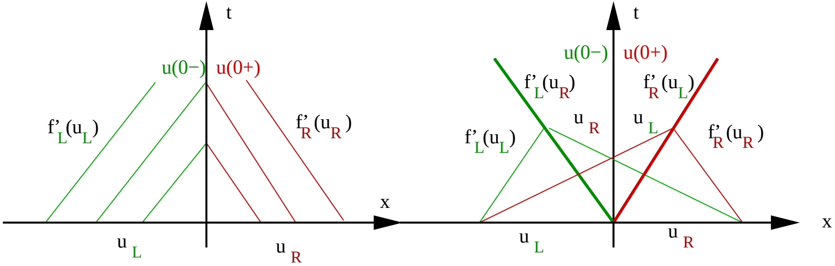
which solution is computed?

$f_L(u) = u^2/2$, $f_R(u) = (u+3)^2/2$, CRP with $u_L = 3$, $u_R = -6$



computed solution: a R−shock with Godunov's scheme,
L−shock + stationary discontinuity + R−shock with mod. L.-F.

$f_L(u) = u^2/2, f_R(u) = (u+3)^2/2$, same CRP with $u_L = 3, u_R = -6$
data such that $f_L'(u_L) > 0, f_L'(u_R) < 0, f_R'(u_L) > 0, f_R'(u_R) < 0,$



mod. LF computes a compound discontinuity with boundary layer: L−shock $u_L \to u(0-)$, discontinuity $u(0-) \to u(0+)$, R−shock $u(0+) \to u_R$

Numerical flux coupling for Euler

For Euler, $\mathbf{u} = (\varrho, \varrho u, \varrho e)$, flux $\mathbf{f}(\mathbf{u}) = (\varrho u, \varrho u^2 + p, (\varrho e + p)u)$, two gamma laws γ_L, γ_R. The eigenvalues may change sign, the flux is not an admissible change of variables.
Numerical flux coupling via a *global relaxation coupling solver*

- a larger relaxation system relaxing towards Euler as $\epsilon \to 0$
- a numerical coupling of the convective part of the relaxation systems with judicious choice of CC
- a splitting method: convection + instantaneous relaxation $\epsilon = 0$

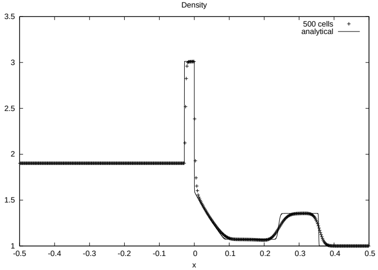
Results in a standard finite volume method: if it ‘converges’ to $\mathbf{u}$, $\mathbf{u}$ is solution of a *coupled problem* with *continuous flux*, and entropy solution in $x < 0$ and $x > 0$
Extends to general fluid models

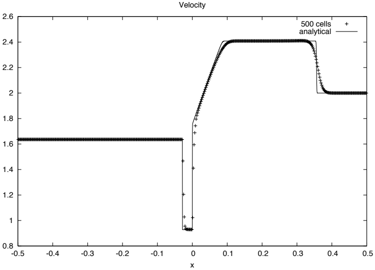
Euler (barotropic case): $\mathbf{u} = (\varrho, \varrho u)$, flux $\mathbf{f}(\mathbf{u}) = (\varrho u, \varrho u^2 + p)$
Relaxation system (Suliciu):

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) = 0 \\ \partial_t(\varrho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) = \lambda \varrho(\tau - \mathcal{T}) \end{cases}$$

with $\Pi_\alpha = \tilde{\Pi}(\tau, \mathcal{T}) \equiv \tilde{p}_\alpha(\mathcal{T}) + a^2(\mathcal{T} - \tau)$, $\tau = 1/\varrho, \tilde{p}(\tau) = p(\varrho)$.
Formally $\mathcal{T} \to \tau$, $\Pi \to p$ as $\lambda \to \infty$.
3 LD fields, RP are easily computed $\to$ Godunov's scheme:
$j \neq 0$ (left and right) $\mathbf{g}^n_{\alpha,j} = \mathbf{f}_\alpha(\mathbf{W}_\alpha(0; \mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}))$
$j = 0$, solve a CRP with transmission of $\mathbf{v} = (\tau, u, \Pi)$ then
$\mathbf{g}^n_{\alpha,0} = \mathbf{f}_\alpha(\mathbf{W}_c(0; \mathbf{v}^n_{-1/2}, \mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{L,0} = \mathbf{g}^n_{R,0}$
$\to$ results in a *conservative consistent scheme* for Euler, entropy scheme in $x < 0, x > 0$

Example of a (numerical) flux coupling:





CRP for Euler: data $\varrho_L = 1.902, u_L = 1.6361$, $p_L = 2.4598$; $\varrho_R = 1, u_R = 2, p_R = 1$ *(computation by Thomas Galié)*
exact solution: $1L-$shock, stationary (coupling) wave, $1R-$ sonic rarefaction, $2R-$CD and $3R-$shock

Conclusion

- This analysis was necessary: it gives in many cases
  - a theoretical model for interface coupling
  - a better understanding of what can be transmitted
  - a robust coupling scheme

  and useful tools (even for other approaches)
- Some questions left
- It is not the ultimate approach
  - thickened interface
- Related topics of interest are
  - interface coupling with small scale phenomena
  - coupling more complex fluid systems (multiphysics)
  - model adaptation

Coupling algorithms for hyperbolic systems

Edwige Godlewski
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris)

Cemracs, July 2011

Interface coupling, coupling algorithm, further topics

- Regularization for state coupling
  - Dafermos regularization
  - finite volume scheme
- Relaxation state coupling solver for flux coupling
- Conclusion and future directions

Interface coupling

CC = coupling conditions
Nonuniqueness of CC; different CC give different solutions
Some difficulties linked to the present approach:

- non conservative systems
- singular source terms
- possible resonance: the eigenvalues may change sign (ex. Euler system in Eulerian coordinates) at $x = 0$
- non uniqueness of the solution

A natural answer: add viscosity
idea: Dafermos regularization (B. Boutin) but does not bring uniqueness

State coupling = non-conservative approach

$$\partial_t u + \partial_x f(u,a) = \mathcal{M}, \quad x \in \mathbb{R},\ t > 0,\ \partial_t a = 0,$$

$f(u,a) = af_L(u) + (1-a)f_R(u)$, $\mathcal{M}$ measure (Dirac), weight jump

$$[f(u,a)] = f_R(u(0+,t)) - f_L(u(0-,t))$$

Riemann data for a: $a_L = 1, a_R = 0$ and $\partial_t a = 0$
$\Rightarrow a(x)$ is a Heaviside function, $\partial_x a = -\delta_0$,

$$\begin{cases} \partial_t u + \partial_x\Big(af_L(u) + (1-a)f_R(u)\Big) + \Big(f_R(u) - f_L(u)\Big)\partial_x a = 0 \\ \partial_t a = 0 \end{cases}$$

If u continuous, $(f_R(u) - f_L(u))\partial_x a$ (non conservative product) is well defined. Write the 1st equation

$$\partial_t u + (af_L{}'(u) + (1-a)f_R{}'(u))\partial_x u = 0$$

System with eigenvalues 0 and $\lambda(u,a) = af_L{}'(u) + (1-a)f_R{}'(u)$.
Extends to v-coupling

Non conservative system

$$\begin{cases} \partial_t u + \lambda(u,a)\partial_x u = 0 \\ \partial_t a = 0 \end{cases}$$

add a regularization term

$$\begin{cases} \partial_t u_\varepsilon + \lambda(u_\varepsilon, a_\varepsilon))\partial_x u_\varepsilon = t\varepsilon \partial_{xx} u_\varepsilon \\ \partial_t a_\varepsilon = t\varepsilon^2 \partial_{xx} a_\varepsilon \end{cases}$$

initial data $u_\varepsilon(x,0) = u_0(x), a_\varepsilon(x,0) = a_0(x)$

$$u_0(x) = \begin{cases} u_L, \ x < 0 \\ u_R, \ x > 0 \end{cases} \quad a_0(x) = \begin{cases} 1, \ x < 0 \\ 0, \ x > 0. \end{cases}$$

Regularization with t in the RHS was proposed by Dafermos. It corresponds to a classical viscous regularization in variable $\xi = x/t$, $T = \ln t$ and allows to study the approximation of self-similar solutions.

Look for self similar solutions: $\xi = x/t$, $u_\varepsilon(\xi), a_\varepsilon(\xi)$
$u_\varepsilon, a_\varepsilon$ exist, $\exists u$, '$u_{\varepsilon_k} \to u$' as $\varepsilon \to 0$,

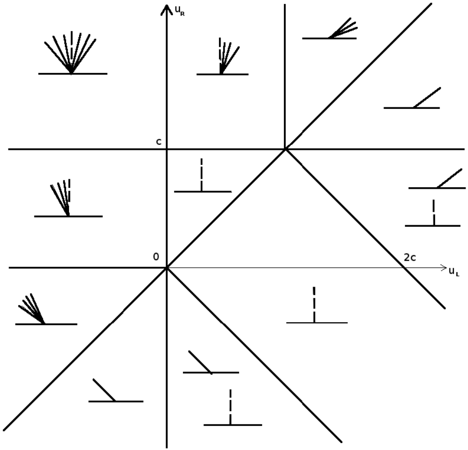
- u solution of the CRP, entropy solution in $x < 0$, $x > 0$
- at interface possible boundary layer $\to$ *zoom*: *fast* variable

$y = \xi/\varepsilon$ $\mathcal{U}_\varepsilon(y) = u_\varepsilon(\varepsilon y)$, $\mathcal{A}_\varepsilon(y) = a_\varepsilon(\varepsilon y)$.

- $\mathcal{A}_\varepsilon(y)$ converges to $\mathcal{A}(y) = (1 - \mathrm{erf}(y/\sqrt{2}))/2$, $\mathcal{A}(-\infty) = 1, \mathcal{A}(+\infty) = 0$, non trivial profile connecting 1 to 0 thanks to ε^2 (if ε, $\mathcal{A}(y) = 1/2$)
- possible non trivial profiles for $\mathcal{U}$. If f_α strictly convex:

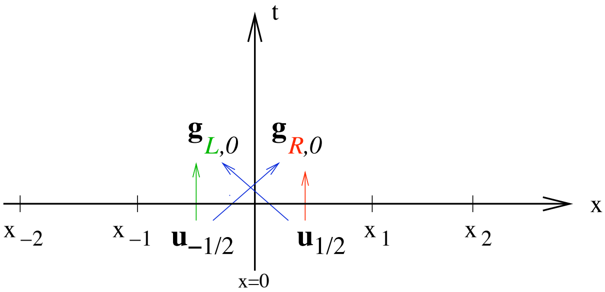
-left: $\mathcal{U}(-\infty) = u(0-)$ or $\mathcal{U}(-\infty) < u(0-)$
$$f_L'(\mathcal{U}(-\infty)) < 0 < f_L'(u(0-))$$
-right: $\mathcal{U}(+\infty) = u(0+)$ or $\mathcal{U}(+\infty) > u(0+)$
$$f_R'(\mathcal{U}(+\infty)) > 0 > f_R'(u(0+))$$
Structure of the discontinuity $u(0-), u(0+)$: $u(0-), \mathcal{U}(-\infty)$ L-stationary shock, $\mathcal{U}(-\infty), \mathcal{U}(+\infty)$, $\mathcal{U}(+\infty), u(0+)$ R- stationary shock. Rules out some *unstable* solutions, possible nonuniqueness

solution of the CRP, in the plane (u_L, u_R)



$f_L(u) = u^2/2, f_R(u) = (u-c)^2/2, c > 0$ (B. Boutin) numerical coupling with FV methods and 2 fluxes at $x = 0$: one can always compute a numerical solution (which?)

- $\mathbf{g}^n_{\alpha,0} = \mathbf{g}_\alpha\left(\mathbf{u}^n_{-1/2}, \mathbf{u}^n_{+1/2}\right), \alpha = L, R$ ensures **u**-state coupling

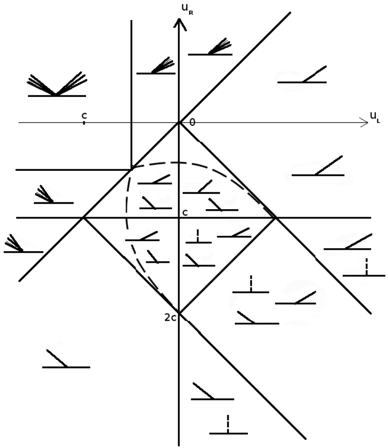


- $\mathbf{g}^n_{L,0} = \mathbf{g}_L(\mathbf{u}^n_{-1/2}, \varphi_L(\mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{0,R} = \mathbf{g}_R(\varphi_R(\mathbf{v}^n_{-1/2}), \mathbf{u}^n_{+1/2})$ ensures **v**-state coupling

If* the two-flux FV scheme converges ($u_\Delta \to u$) in some 'sensible way', (*proven in the scalar case, with rather general assumptions) then u is solution of the coupled problem with our CC.
In case of

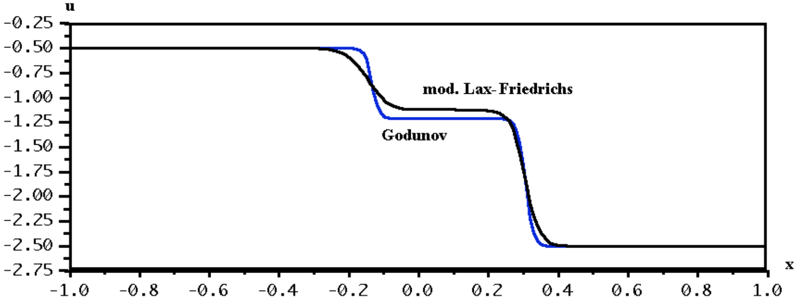
- uniqueness, u is the unique solution
- non-uniqueness, u is a solution, which solution?

scalar quadratic case: $f_L(u) = u^2/2$, $f_R(u) = (u-c)^2/2$, $c < 0$: possible solutions obtained by Dafermos regularization



double shock missing in central area

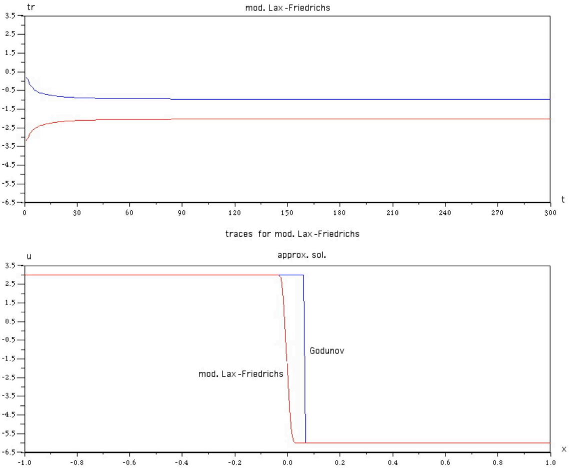
scalar quadratic case: $f_L(u) = u^2/2, f_R(u) = (u+4)^2/2$
CRP with $u_L = -0.5, u_R = -2.5,\ f_L'(u_L) < 0, f_R'(u_R) > 0$



exact solution: 2 shocks computed with Godunov's scheme and Lax-Friedrichs modified: $u_G^m = -1,21, u_{LF}^m = -1,12$

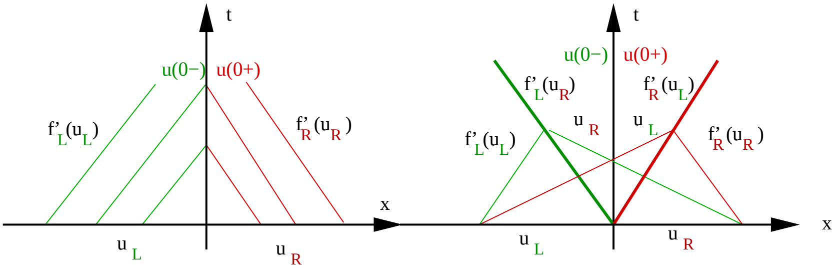
which solution is computed?

$f_L(u) = u^2/2, f_R(u) = (u+3)^2/2$, CRP with $u_L = 3, u_R = -6$



computed solution: a R−shock with Godunov's scheme,
L−shock + stationary discontinuity + R−shock with mod. L.-F.

$f_L(u) = u^2/2, f_R(u) = (u+3)^2/2$, same CRP with $u_L = 3, u_R = -6$
data such that $f_L'(u_L) > 0, f_L'(u_R) < 0, f_R'(u_L) > 0, f_R'(u_R) < 0,$



mod. LF computes a compound discontinuity with boundary layer: $L-$shock $u_L \to u(0-)$, discontinuity $u(0-) \to u(0+)$, $R-$shock $u(0+) \to u_R$

Numerical flux coupling for Euler

For Euler, $\mathbf{u} = (\varrho, \varrho u, \varrho e)$, flux $\mathbf{f}(\mathbf{u}) = (\varrho u, \varrho u^2 + p, (\varrho e + p)u)$, two gamma laws γ_L, γ_R. The eigenvalues may change sign, the flux is not an admissible change of variables.
Numerical flux coupling via a *global relaxation coupling solver*

- a larger relaxation system relaxing towards Euler as $\epsilon \to 0$
- a numerical coupling of the convective part of the relaxation systems with judicious choice of CC
- a splitting method: convection + instantaneous relaxation $\epsilon = 0$

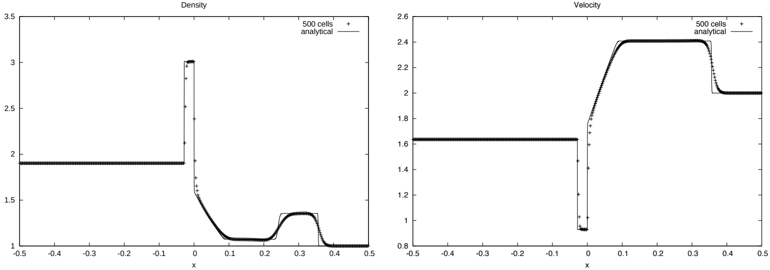
Results in a standard finite volume method: if it 'converges' to $\mathbf{u}$, $\mathbf{u}$ is solution of a *coupled problem* with *continuous flux*, and entropy solution in $x < 0$ and $x > 0$
Extends to general fluid models

Euler (barotropic case): $\mathbf{u} = (\varrho, \varrho u)$, flux $\mathbf{f}(\mathbf{u}) = (\varrho u, \varrho u^2 + p)$
Relaxation system (Suliciu):

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) = 0 \\ \partial_t(\varrho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) = \lambda \varrho(\tau - \mathcal{T}) \end{cases}$$

with $\Pi_\alpha = \tilde{\Pi}(\tau, \mathcal{T}) \equiv \tilde{p}_\alpha(\mathcal{T}) + a^2(\mathcal{T} - \tau)$, $\tau = 1/\varrho, \tilde{p}(\tau) = p(\varrho)$.
Formally $\mathcal{T} \to \tau$, $\Pi \to p$ as $\lambda \to \infty$.
3 LD fields, RP are easily computed $\to$ Godunov's scheme:
$j \neq 0$ (left and right) $\mathbf{g}^n_{\alpha,j} = \mathbf{f}_\alpha(\mathbf{W}_\alpha(0; \mathbf{u}^n_{j-1/2}, \mathbf{u}^n_{j+1/2}))$
$j = 0$, solve a CRP with transmission of $\mathbf{v} = (\tau, u, \Pi)$ then
$\mathbf{g}^n_{\alpha,0} = \mathbf{f}_\alpha(\mathbf{W}_c(0; \mathbf{v}^n_{-1/2}, \mathbf{v}^n_{+1/2}))$, $\mathbf{g}^n_{L,0} = \mathbf{g}^n_{R,0}$
$\to$ results in a *conservative consistent scheme* for Euler, entropy scheme in $x < 0, x > 0$

Example of a (numerical) flux coupling:



CRP for Euler: data $\varrho_L = 1.902, u_L = 1.6361,\ p_L = 2.4598$;
$\varrho_R = 1, u_R = 2, p_R = 1$ *(computation by Thomas Galié)*
exact solution: $1L-$shock, stationary (coupling) wave, $1R-$ sonic rarefaction, $2R-$CD and $3R-$shock

Conclusion

- This analysis was necessary: it gives in many cases
  - a theoretical model for interface coupling
  - a better understanding of what can be transmitted
  - a robust coupling scheme

  and useful tools (even for other approaches)
- Some questions left
- It is not the ultimate approach
  - thickened interface
- Related topics of interest are
  - interface coupling with small scale phenomena
  - coupling more complex fluid systems (multiphysics)
  - model adaptation