TSAPPICC
Two-Scale Asymptotic-Preserving Particle-In-Cell Code for a beam in a focusing channel

Emmanuel Frénod *

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1 Introduction

On the one hand, Two-Scale Numerical Methods have exhibited good behaviors for simulation of problems where strong oscillations arise (see: Ailliot, Frénod & Monbet [1], Frénod, Mouton & Sonnendrcker [4], Frénod, Salvarani & Sonnendrücker [6] and Mouton [10]). On the other hand, Asymptotic Preserving Schemes, based on a Macro-Micro decomposition, have exhibited good behaviors to simulate phenomena which are modeled by a singularly perturbed equation (see: Degond, Deluzet & Negulescu [2], Degond, Deluzet, Sangam & Vignal [3], and Lemou & Mieussens [9]).

Tokamak Plasma Physic and Beam Physic are concerned by both strong oscillations and singular perturbation (see Frénod & Sonnendrücker [7, 8] and Frénod, Raviart & Sonnendrücker [5]). Hence, it is cleaver to investigate for those physical questions ways between Two-Scale Numerical Methods and Asymptotic Preserving Schemes. This is the topic of this Cemracs Project.

The following Vlasov equation, set in a bi-dimensional position-velocity space,

$$\begin{cases} \dfrac{\partial f^\varepsilon}{\partial t} + \dfrac{1}{\varepsilon} v \dfrac{\partial f^\varepsilon}{\partial r} + \Big(\mathbf{E}^\varepsilon - \dfrac{r}{\varepsilon}\Big) \dfrac{\partial f^e}{\partial v} = 0, \\ \dfrac{1}{r}\dfrac{\partial (r\mathbf{E}^\varepsilon)}{\partial r} = \rho^\varepsilon(t,r), \qquad \rho^\varepsilon(t,r) = \displaystyle\int_{\mathbb{R}} f^\varepsilon(t,r,v)\,dv, \\ f^\varepsilon(t=0,r,v) = f_0, \end{cases} \tag{1.1}$$

*Université Européenne de Bretagne, Lab-STICC (UMR CNRS 3192), Université de Bretagne-Sud, Centre Yves Coppens, Campus de Tohannic, F-56017, Vannes & Projet INRIA Calvi, Université de Strasbourg, IRMA, 7 rue René Descartes, F-67084 Strasbourg Cedex, France

where $f^\varepsilon = f^\varepsilon(t,r,v)$ for $t \in [0,T)$, $r \in \mathbb{R}^+$ and $v \in \mathbb{R}$, is a simplified model for a beam in a focusing channel.

The goal of this project is to test on this model Two-Scale Asymptotic-Preserving Particle In Cell Discretization, or TSAPPIC Discretization, before going further on more complex equations of tokamak plasma physic.

2 Recalling Asymptotic Behavior of the Model when $\varepsilon \to 0$

In [6], it was proven that under suitable assumptions,

$$f^\varepsilon \text{ Two-Scale converges to profile } F \in L^\infty([0,T]\times[0,2\pi]; L^2(\mathbb{R}^2; r dr dv)), \tag{2.1}$$

$$\mathbf{E}^\varepsilon \text{ Two-Scale converges to } \mathcal{E} \in L^\infty([0,T]\times[0,2\pi]; W^{1,3/2}(\mathbb{R}; r dr)), \tag{2.2}$$

and that

$$F(t,\tau,r,v) = G(t, \cos(\tau)r - \sin(\tau)v, \sin(\tau)r + \cos(\tau)v), \tag{2.3}$$

where $G = G(t,q,u) \in L^\infty([0,T]; L^2(\mathbb{R}^2; q dq du))$ is solution to

$$\begin{cases} \dfrac{\partial G}{\partial t} + \displaystyle\int_0^{2\pi} -\sin(\sigma)\,\mathcal{E}(t,\sigma,\cos(\sigma)q + \sin(\sigma)u)\,d\sigma\;\dfrac{\partial G}{\partial q} \\ \qquad\qquad + \displaystyle\int_0^{2\pi} \cos(\sigma)\,\mathcal{E}(t,\sigma,\cos(\sigma)q + \sin(\sigma)u)\,d\sigma\;\dfrac{\partial G}{\partial u} = 0, \\ G(t=0) = f_0, \end{cases} \tag{2.4}$$

with $\mathcal{E}$ given by

$$\frac{1}{r}\frac{\partial(r\mathcal{E})}{\partial r} = \Upsilon(t,\tau,r) = \int_{\mathbb{R}} G\big(t, \cos(\tau)r - \sin(\tau)v, \sin(\tau)r + \cos(\tau)v\big)\,dv\,. \tag{2.5}$$

3 Two-Scale Macro-Micro Decomposition : Preliminaries

Equation (2.4) and computations that follow involve mappings:

$$(r,v) \mapsto (q,u) = \mathcal{P}^+(\tau,r,v) = (\mathcal{P}^+_q(\tau,r,v), \mathcal{P}^+_u(\tau,r,v)) = \\ (\cos(\tau)r - \sin(\tau)v, \sin(\tau)r + \cos(\tau)v), \tag{3.1}$$

$$(q,u) \mapsto (r,v) = \mathcal{P}^-(\tau,q,u) = (\mathcal{P}^-_r(\tau,q,u), \mathcal{P}^-_v(\tau,q,u)) = \\ (\cos(\tau)q + \sin(\tau)u, -\sin(\tau)q + \cos(\tau)u) = \mathcal{P}^+(-\tau,q,u). \tag{3.2}$$

Writing

$$(G\circ\mathcal{P}^+)(t,\tau,r,v) = G(t, \cos(\tau)r - \sin(\tau)v, \sin(\tau)r + \cos(\tau)v), \tag{3.3}$$

the solution f^ε is sought in the following form:

$$f^\varepsilon(t,r,v) = (G\circ\mathcal{P}^+)(t,\frac{t}{\epsilon},r,v) + \varepsilon\,(G_1^\varepsilon\circ\mathcal{P}^+)(t,\frac{t}{\varepsilon},r,v) - \varepsilon\, l(t,\frac{t}{\varepsilon},r,v),$$
$$+\frac{\partial k^\varepsilon}{\partial\tau}(t,\frac{t}{\varepsilon},r,v) + v\frac{\partial k^\varepsilon}{\partial r}(t,\frac{t}{\varepsilon},r,v) - r\frac{\partial k^\varepsilon}{\partial v}(t,\frac{t}{\varepsilon},r,v), \quad (3.4)$$

where G is the function linked with the Two-Scale limit F of f^ε by (2.3) and where l is the function such that

$$\begin{aligned}(l\circ\mathcal{P}^-)(t,\tau,q,u) &= l(t,\tau,\mathcal{P}_r^-(\tau,q,u),\mathcal{P}_v^-(\tau,q,u))\\ &= \left(\int_0^\tau -\sin(\sigma)\mathcal{E}(t,\sigma,\mathcal{P}_r^-(\sigma,q,u))\,d\sigma - \tau\int_0^{2\pi} -\sin(\sigma)\mathcal{E}(t,\sigma,\mathcal{P}_r^-(\sigma,q,u))\,d\sigma\right)\frac{\partial G}{\partial q}\\ &\quad+\left(\int_0^\tau \cos(\sigma)\mathcal{E}(t,\sigma,\mathcal{P}_r^-(\sigma,q,u))\,d\sigma - \tau\int_0^{2\pi} \cos(\sigma)\mathcal{E}(t,\sigma,\mathcal{P}_r^-(\sigma,q,u))\,d\sigma\right)\frac{\partial G}{\partial u}.\end{aligned} \quad (3.5)$$

By construction

$$(G\circ\mathcal{P}^+)\in \mathrm{Ker}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right), \quad (3.6)$$
$$(G_1^\varepsilon\circ\mathcal{P}^+)\in \mathrm{Ker}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right), \quad (3.7)$$
$$\frac{\partial k^\varepsilon}{\partial\tau}+v\frac{\partial k^\varepsilon}{\partial r}-r\frac{\partial k^\varepsilon}{\partial v}\in \mathrm{Im}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right) = \left(\mathrm{Ker}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right)\right)^\perp, \quad (3.8)$$
$$(3.9)$$

and it is obvious to see:

$$l\in\left(\mathrm{Ker}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right)\right)^\perp = \mathrm{Im}\left(\frac{\partial}{\partial\tau}+v\frac{\partial}{\partial r}-r\frac{\partial}{\partial v}\right). \quad (3.10)$$

In what concerns initial data, it is gotten:

$$\begin{aligned}f^\varepsilon(0,r,v) &= f_0(r,v)\\ &= G(0,r,v)+\varepsilon\,G_1^\varepsilon(0,r,v)+\frac{\partial k^\varepsilon}{\partial\tau}(0,0,r,v)+v\frac{\partial k^\varepsilon}{\partial r}(0,0,r,v)-r\frac{\partial k^\varepsilon}{\partial v}(0,0,r,v),\end{aligned} \quad (3.11)$$

which implies

$$\varepsilon\,G_1^\varepsilon(0,r,v)+\frac{\partial k^\varepsilon}{\partial\tau}(0,0,r,v)+v\frac{\partial k^\varepsilon}{\partial r}(0,0,r,v)-r\frac{\partial k^\varepsilon}{\partial v}(0,0,r,v)=0. \quad (3.12)$$

4 Weak Formulation with Oscillating Test Functions

Writing a weak formulation of (1.1) with oscillating test functions

$$(\psi)^\epsilon = (\psi)^\epsilon(t,\mathbf{x},\mathbf{v}) = \psi(t,\frac{t}{\epsilon},\mathbf{x},\mathbf{v})\,, \tag{4.1}$$

where $\psi = \psi(t,\tau,r,v)$ is regular, 2π-periodic in τ and with compact support on $[0,T]\times\mathbb{R}^2$ for any fixed τ, which reads:

$$\int_0^T \int_{\mathbb{R}^2} f^\varepsilon \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ = \int_{\mathbb{R}^2} f_0\,\psi(0,0,.,.,)\;drdv, \tag{4.2}$$

taking (3.11) into account, gives

$$\int_0^T \int_{\mathbb{R}^2} (G\circ\mathcal{P}^+)^\varepsilon \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ + \varepsilon \int_0^T \int_{\mathbb{R}^2} (G_1^\varepsilon\circ\mathcal{P}^+)^\varepsilon \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ - \varepsilon \int_0^T \int_{\mathbb{R}^2} (l)^\varepsilon \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ + \int_0^T \int_{\mathbb{R}^2} \left((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon + v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon \right) \\ \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ = \int_{\mathbb{R}^2} f_0\,\psi(0,0,.,.,)\;drdv. \tag{4.3}$$

Integrating by parts the first term of (4.3), using the initial condition for G given in (2.4) and using the change of variables $(r,v)\mapsto(q,u)=\mathcal{P}^+(t/\varepsilon,r,v)$, we get:

$$\int_0^T \int_{\mathbb{R}^2} (G\circ\mathcal{P}^+)^\varepsilon \left((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon \right) dtdrdv \\ = -\int_0^T \int_{\mathbb{R}^2} \left(-\bigg(\int_0^{2\pi} -\sin(\sigma)\,(\mathcal{E}\circ\mathcal{P}_r^-(\sigma,.,.))\,d\sigma \bigg) \frac{\partial G}{\partial q} - \bigg(\int_0^{2\pi} \cos(\sigma)\,(\mathcal{E}\circ\mathcal{P}_r^-(\sigma,.,.))\,d\sigma \bigg) \frac{\partial G}{\partial u} \right. \\ \left. + (\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon \Big(-\sin(\frac{t}{\varepsilon})\frac{\partial G}{\partial q} + \cos(\frac{t}{\varepsilon})\frac{\partial G}{\partial u}\; \Big) \right) (\psi\circ\mathcal{P}^-)^\varepsilon dtdqdu \\ + \int_{\mathbb{R}^2} f_0\,\psi(0,0,.,.,)\;drdv. \tag{4.4}$$

Making something similar for the second term gives:

$$\varepsilon \int_0^T \int_{\mathbb{R}^2} (G_1^\varepsilon \circ \mathcal{P}^+)^\varepsilon \left((\frac{\partial \psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi}{\partial \tau})^\varepsilon + \frac{1}{\varepsilon} v (\frac{\partial \psi}{\partial r})^\varepsilon + \left(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\right) (\frac{\partial \psi}{\partial v})^\varepsilon \right) dtdrdv$$
$$= -\varepsilon \int_0^T \int_{\mathbb{R}^2} \left(\frac{\partial G_1^\varepsilon}{\partial t} + (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial u} \big) \right) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu$$
$$+ \varepsilon \int_{\mathbb{R}^2} G_1^\varepsilon(0,.,.)\, \psi(0,0,.,.,)\, dqdu \quad (4.5)$$

Concerning the third term, noticing

$$\frac{\partial (l \circ \mathcal{P}^-)}{\partial \tau} = (\frac{\partial l}{\partial \tau} \circ \mathcal{P}^-) + (-\sin(\tau) q + \cos(\tau) u)(\frac{\partial l}{\partial r} \circ \mathcal{P}^-) + (-\cos(\tau) q - \sin(\tau) u)(\frac{\partial l}{\partial v} \circ \mathcal{P}^-)$$
$$= (\frac{\partial l}{\partial \tau} \circ \mathcal{P}^-) + \mathcal{P}_v^-(\tau,.,.)(\frac{\partial l}{\partial r} \circ \mathcal{P}^-) - \mathcal{P}_r^-(\tau,.,.)(\frac{\partial l}{\partial v} \circ \mathcal{P}^-), \quad (4.6)$$

we obtain:

$$\varepsilon \int_0^T \int_{\mathbb{R}^2} (l)^\varepsilon \left((\frac{\partial \psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi}{\partial \tau})^\varepsilon + \frac{1}{\varepsilon} v (\frac{\partial \psi}{\partial r})^\varepsilon + \left(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\right) (\frac{\partial \psi}{\partial v})^\varepsilon \right) dtdrdv$$
$$= -\int_0^T \int_{\mathbb{R}^2} \left(\varepsilon (\frac{\partial l}{\partial t})^\varepsilon + (\frac{\partial l}{\partial \tau})^\varepsilon + v (\frac{\partial l}{\partial r})^\varepsilon + (\varepsilon \mathbf{E}^\varepsilon - r) (\frac{\partial l}{\partial v})^\varepsilon \right) (\psi)^\varepsilon dtdrdv, \quad (4.7)$$

Making here again change of variables $(r, v) \mapsto (q, u) = \mathcal{P}^+(t/\varepsilon, r, v)$, equality (4.7) becomes

$$-\int_0^T \int_{\mathbb{R}^2} \Bigg(\varepsilon (\frac{\partial l}{\partial t} \circ \mathcal{P}^-)^\varepsilon + (\frac{\partial l}{\partial \tau} \circ \mathcal{P}^-)^\varepsilon + v (\frac{\partial l}{\partial r} \circ \mathcal{P}^-)^\varepsilon$$
$$+ \big(\varepsilon (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-) - \mathcal{P}_r^- \big) (\frac{\partial l}{\partial v} \circ \mathcal{P}^-)^\varepsilon \Bigg) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu$$
$$= -\int_0^T \int_{\mathbb{R}^2} \left(\varepsilon (\frac{\partial l}{\partial t} \circ \mathcal{P}^-)^\varepsilon + (\frac{\partial (l \circ \mathcal{P}^-)}{\partial \tau})^\varepsilon + \varepsilon (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon (\frac{\partial l}{\partial v} \circ \mathcal{P}^-)^\varepsilon \right) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu, \quad (4.8)$$

where we used (4.6). Beside this from (3.5) we get,

$$\frac{\partial (l \circ \mathcal{P}^-)}{\partial \tau} = \left(-\sin(\tau)(\mathcal{E} \circ \mathcal{P}_r^-)(\tau,.,.)) - \int_0^{2\pi} -\sin(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma,.,.)\, d\sigma \right) \frac{\partial G}{\partial q}$$
$$+ \left(\cos(\tau)(\mathcal{E} \circ \mathcal{P}_r^-)(\tau,.,.)) - \int_0^{2\pi} \cos(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma,.,.)\, d\sigma \right) \frac{\partial G}{\partial u}. \quad (4.9)$$

Injecting this in (4.8) gives the following form to the third term of (4.3):

$$
\begin{aligned}
&\varepsilon \int_0^T \int_{\mathbb{R}^2} (l)^\varepsilon \bigg((\frac{\partial \psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi}{\partial \tau})^\varepsilon + \frac{1}{\varepsilon} v (\frac{\partial \psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial \psi}{\partial v})^\varepsilon \bigg)\, dtdrdv \\
&= - \int_0^T \int_{\mathbb{R}^2} \Bigg(\varepsilon (\frac{\partial l}{\partial t} \circ \mathcal{P}^-)^\varepsilon + \Big(-\sin(\frac{t}{\varepsilon})(\mathcal{E} \circ \mathcal{P}_r^-)^\varepsilon - \int_0^{2\pi} -\sin(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma, ., .)\, d\sigma \Big) \frac{\partial G}{\partial q} \\
&\qquad + \Big(\cos(\frac{t}{\varepsilon})(\mathcal{E} \circ \mathcal{P}_r^-)^\varepsilon - \int_0^{2\pi} \cos(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma, ., .)\, d\sigma \Big) \frac{\partial G}{\partial u} \\
&\qquad\qquad + \varepsilon (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon (\frac{\partial l}{\partial v} \circ \mathcal{P}^-)^\varepsilon \Bigg) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu. \qquad (4.10)
\end{aligned}
$$

Using (4.4), (4.5) and (4.10) in (4.3) gives:

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^2} \Bigg(- \Big(\int_0^{2\pi} -\sin(\sigma)\, (\mathcal{E} \circ \mathcal{P}_r^-(\sigma, ., .))\, d\sigma \Big) \frac{\partial G}{\partial q} - \Big(\int_0^{2\pi} \cos(\sigma)\, (\mathcal{E} \circ \mathcal{P}_r^-(\sigma, ., .))\, d\sigma \Big) \frac{\partial G}{\partial u} \\
&+ (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G}{\partial u} \big) \Bigg) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu + \int_{\mathbb{R}^2} f_0\, \psi(0, 0, ., .,)\, drdv \\
&\quad - \varepsilon \int_0^T \int_{\mathbb{R}^2} \Big(\frac{\partial G_1^\varepsilon}{\partial t} + (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial u} \big) \Big) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu \\
&\qquad + \varepsilon \int_{\mathbb{R}^2} G_1^\varepsilon(0, ., .)\, \psi(0, 0, ., .,)\, dqdu \\
&+ \int_0^T \int_{\mathbb{R}^2} \Bigg(\varepsilon (\frac{\partial l}{\partial t} \circ \mathcal{P}^-)^\varepsilon + \Big(-\sin(\frac{t}{\varepsilon})(\mathcal{E} \circ \mathcal{P}_r^-)^\varepsilon - \int_0^{2\pi} -\sin(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma, ., .)\, d\sigma \Big) \frac{\partial G}{\partial q} \\
&\qquad + \Big(\cos(\frac{t}{\varepsilon})(\mathcal{E} \circ \mathcal{P}_r^-)^\varepsilon - \int_0^{2\pi} \cos(\sigma)(\mathcal{E} \circ \mathcal{P}_r^-)(\sigma, ., .)\, d\sigma \Big) \frac{\partial G}{\partial u} \\
&\qquad + \varepsilon (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon (\frac{\partial l}{\partial v} \circ \mathcal{P}^-)^\varepsilon \Bigg) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu \\
&\quad + \int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial k^\varepsilon}{\partial \tau})^\varepsilon + v (\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r (\frac{\partial k^\varepsilon}{\partial v})^\varepsilon \Big) \\
&\qquad \bigg((\frac{\partial \psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial \psi}{\partial \tau})^\varepsilon + \frac{1}{\varepsilon} v (\frac{\partial \psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial \psi}{\partial v})^\varepsilon \bigg)\, dtdrdv \\
&\qquad\qquad = \int_{\mathbb{R}^2} f_0\, \psi(0, 0, ., .,)\, drdv, \qquad (4.11)
\end{aligned}
$$

5 Two-Scale Macro-Micro Decomposition : Expanded Weak Formulation

Rewriting the Poisson equation of (1.1) gives

$$\frac{1}{r}\frac{\partial(r\mathbf{E}^\varepsilon)}{\partial r} = \int_{\mathbb{R}} (G\circ\mathcal{P}^+)(t,\frac{t}{\epsilon},r,v)\,dv + \varepsilon\int_{\mathbb{R}} (G_1^\varepsilon\circ\mathcal{P}^+)(t,\frac{t}{\varepsilon},r,v)\,dv - \varepsilon\int_{\mathbb{R}} l(t,\frac{t}{\varepsilon},r,v)\,dv$$
$$+\int_{\mathbb{R}} \frac{\partial k^\varepsilon}{\partial\tau}(t,\frac{t}{\varepsilon},r,v) + v\frac{\partial k^\varepsilon}{\partial r}(t,\frac{t}{\varepsilon},r,v) - r\frac{\partial k^\varepsilon}{\partial v}(t,\frac{t}{\varepsilon},r,v)\,dv, \quad (5.1)$$

and rewriting (5.2) incorporating (3.1) yields

$$\frac{1}{r}\frac{\partial(r\mathcal{E})}{\partial r} = \int_{\mathbb{R}} (G\circ\mathcal{P}^+)(t,\tau,r,v)\,dv \text{ and } \frac{1}{r}\frac{\partial(r(\mathcal{E})^\varepsilon)}{\partial r} = \int_{\mathbb{R}} (G\circ\mathcal{P}^+)(t,\frac{t}{\epsilon},r,v)\,dv. \quad (5.2)$$

Hence, defining $\mathcal{E}_1^\varepsilon$ as

$$\mathcal{E}_1^\varepsilon(t,x,v) = \mathbf{E}^\varepsilon(t,x,v) - (\mathcal{E})^\varepsilon(t,x,v) = \mathbf{E}^\varepsilon(t,x,v) - \mathcal{E}(t,\frac{t}{\epsilon},r,v), \quad (5.3)$$

which is obviously solution to

$$\frac{1}{r}\frac{\partial(r\mathcal{E}_1^\varepsilon)}{\partial r} = \varepsilon\int_{\mathbb{R}} (G_1^\varepsilon\circ\mathcal{P}^+)(t,\frac{t}{\varepsilon},r,v)\,dv - \varepsilon\int_{\mathbb{R}} l(t,\frac{t}{\varepsilon},r,v)\,dv$$
$$+\int_{\mathbb{R}} \frac{\partial k^\varepsilon}{\partial\tau}(t,\frac{t}{\varepsilon},r,v) + v\frac{\partial k^\varepsilon}{\partial r}(t,\frac{t}{\varepsilon},r,v) - r\frac{\partial k^\varepsilon}{\partial v}(t,\frac{t}{\varepsilon},r,v)\,dv, \quad (5.4)$$

making the simplifications that need to be done in (4.11) yields

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^2} (\mathcal{E}_1^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\Big(-\sin(\frac{t}{\varepsilon})\frac{\partial G}{\partial q} + \cos(\frac{t}{\varepsilon})\frac{\partial G}{\partial u}\Big)(\psi\circ\mathcal{P}^-)^\varepsilon dtdqdu\\
&\quad -\varepsilon\int_0^T\int_{\mathbb{R}^2}\Big(\frac{\partial G_1^\varepsilon}{\partial t} + (\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\big(-\sin(\frac{t}{\varepsilon})\frac{\partial G_1^\varepsilon}{\partial q} + \cos(\frac{t}{\varepsilon})\frac{\partial G_1^\varepsilon}{\partial u}\,\big)\Big)(\psi\circ\mathcal{P}^-)^\varepsilon dtdqdu\\
&\qquad +\varepsilon\int_{\mathbb{R}^2} G_1^\varepsilon(0,.,.)\,\psi(0,0,.,.,)\,dqdu\\
&\quad +\int_0^T\int_{\mathbb{R}^2}\Big(\varepsilon(\frac{\partial l}{\partial t}\circ\mathcal{P}^-)^\varepsilon + \varepsilon(\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon(\frac{\partial l}{\partial v}\circ\mathcal{P}^-)^\varepsilon\Big)(\psi\circ\mathcal{P}^-)^\varepsilon dtdqdu\\
&\quad +\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon + v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)\\
&\qquad\Big((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon\Big)\,dtdrdv = 0.
\end{aligned} \quad (5.5)$$

6 The Two-Scale Macro Equation

Using in (5.5) oscillating test functions ψ which are in Ker $\big(\frac{\partial}{\partial\tau} + v\frac{\partial}{\partial r} - r\frac{\partial}{\partial v}\big)$ or, in other words, which write

$$\psi(t,\tau,r,v) = (\gamma\circ\mathcal{P}^+)(t,\tau,r,v) = \gamma(t,\mathcal{P}_q^+(\tau,r,v),\mathcal{P}_u^+(\tau,r,v)), \quad (6.1)$$

for regular functions $\gamma(t,q,u)$, and since, in this case $(\psi\circ\mathcal{P}^-)^\varepsilon = ((\gamma\circ\mathcal{P}^+)\circ\mathcal{P}^-)^\varepsilon = (\gamma)^\varepsilon = \gamma$ and

$$\frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon - \frac{1}{\varepsilon}r(\frac{\partial\psi}{\partial v})^\varepsilon = 0, \tag{6.2}$$

we get:

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^2}(\mathcal{E}_1^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\big(-\sin(\frac{t}{\varepsilon})\frac{\partial G}{\partial q}+\cos(\frac{t}{\varepsilon})\frac{\partial G}{\partial u}\ )\Big)\gamma\ dtdqdu\\
&\quad-\varepsilon\int_0^T\int_{\mathbb{R}^2}\Big(\frac{\partial G_1^\varepsilon}{\partial t}+(\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\big(-\sin(\frac{t}{\varepsilon})\frac{\partial G_1^\varepsilon}{\partial q}+\cos(\frac{t}{\varepsilon})\frac{\partial G_1^\varepsilon}{\partial u}\ )\Big)\gamma\ dtdqdu\\
&\qquad+\varepsilon\int_{\mathbb{R}^2}G_1^\varepsilon(0,.,.)\,\gamma(0,.,.,)\ dqdu\\
&\qquad+\int_0^T\int_{\mathbb{R}^2}\Big(\varepsilon(\frac{\partial l}{\partial t}\circ\mathcal{P}^-)^\varepsilon+\varepsilon(\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon(\frac{\partial l}{\partial v}\circ\mathcal{P}^-)^\varepsilon\Big)\gamma\ dtdqdu\\
&\qquad\quad+\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)\\
&\qquad\qquad\Big((\frac{\partial(\gamma\circ\mathcal{P}^+)}{\partial t})^\varepsilon+\mathbf{E}^\varepsilon(\frac{\partial(\gamma\circ\mathcal{P}^+)}{\partial v})^\varepsilon\Big)\ dtdrdv=0.
\end{aligned}\tag{6.3}$$

To treat the last term of (6.3), we first provide some calculations, then we make the change of variables $(r,v)\mapsto(q,u)=\mathcal{P}^+(t/\varepsilon,r,v)$ and finally we integrate by parts in time one of its terms,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)\Big((\frac{\partial(\gamma\circ\mathcal{P}^+)}{\partial t})^\varepsilon+\mathbf{E}^\varepsilon(\frac{\partial(\gamma\circ\mathcal{P}^+)}{\partial v})^\varepsilon\Big)\ dtdrdv\\
&\ =\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)\\
&\qquad\Big((\frac{\partial\gamma}{\partial t}\circ\mathcal{P}^+)^\varepsilon+\mathbf{E}^\varepsilon\big(-\sin(\frac{t}{\varepsilon})(\frac{\partial\gamma}{\partial q}\circ\mathcal{P}^+)^\varepsilon+\cos(\frac{t}{\varepsilon})(\frac{\partial\gamma}{\partial u}\circ\mathcal{P}^+)^\varepsilon\big)\Big)\ dtdrdv\\
&\quad=\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau}\circ\mathcal{P}_r^-)^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r}\circ\mathcal{P}_r^-)^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v}\circ\mathcal{P}_r^-)^\varepsilon\Big)\\
&\qquad\qquad\Big(\frac{\partial\gamma}{\partial t}+(\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\big(-\sin(\frac{t}{\varepsilon})\frac{\partial\gamma}{\partial q}+\cos(\frac{t}{\varepsilon})\frac{\partial\gamma}{\partial u})\big)\Big)\ dtdqdu\\
&\quad\ =\int_0^T\int_{\mathbb{R}^2}\Big((\frac{\partial k^\varepsilon}{\partial\tau}\circ\mathcal{P}_r^-)^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r}\circ\mathcal{P}_r^-)^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v}\circ\mathcal{P}_r^-)^\varepsilon\Big)\\
&\qquad\qquad\Big((\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon\big(-\sin(\frac{t}{\varepsilon})\frac{\partial\gamma}{\partial q}+\cos(\frac{t}{\varepsilon})\frac{\partial\gamma}{\partial u})\big)\Big)\ dtdqdu\\
&\quad-\int_0^T\int_{\mathbb{R}^2}\frac{\partial\Big((\frac{\partial k^\varepsilon}{\partial\tau}\circ\mathcal{P}_r^-)^\varepsilon+v(\frac{\partial k^\varepsilon}{\partial r}\circ\mathcal{P}_r^-)^\varepsilon-r(\frac{\partial k^\varepsilon}{\partial v}\circ\mathcal{P}_r^-)^\varepsilon\Big)}{\partial t}\gamma\ dtdqdu\\
&\quad+\int_{\mathbb{R}^2}\Big(\frac{\partial k^\varepsilon}{\partial\tau}(0,0,.,.,)+v\frac{\partial k^\varepsilon}{\partial r}(0,0,.,.,)-r\frac{\partial k^\varepsilon}{\partial v}(0,0,.,.,)\Big)\gamma(0,.,.,)\ dqdu.
\end{aligned}\tag{6.4}$$

Hence, using (3.12), equation (6.3) finally reads

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \Big(\frac{\partial G_1^\varepsilon}{\partial t} + (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial u} \big) \Big) \gamma \, dtdqdu \\
&\qquad = \int_0^T \int_{\mathbb{R}^2} (\mathcal{E}_1^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\frac{1}{\varepsilon} \sin(\frac{t}{\varepsilon}) \frac{\partial G}{\partial q} + \frac{1}{\varepsilon} \cos(\frac{t}{\varepsilon}) \frac{\partial G}{\partial u} \big) \Big) \gamma \, dtdqdu \\
&\qquad - \int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial l}{\partial t} \circ \mathcal{P}^-)^\varepsilon + (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon (\frac{\partial l}{\partial v} \circ \mathcal{P}^-)^\varepsilon \Big) \gamma \, dtdqdu \\
&\qquad - \int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial k^\varepsilon}{\partial \tau} \circ \mathcal{P}_r^-)^\varepsilon + v (\frac{\partial k^\varepsilon}{\partial r} \circ \mathcal{P}_r^-)^\varepsilon - r (\frac{\partial k^\varepsilon}{\partial v} \circ \mathcal{P}_r^-)^\varepsilon \Big) \\
&\qquad\qquad \Big((\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\frac{1}{\varepsilon} \sin(\frac{t}{\varepsilon}) \frac{\partial \gamma}{\partial q} + \frac{1}{\varepsilon} \cos(\frac{t}{\varepsilon}) \frac{\partial \gamma}{\partial u} \big) \Big) \Big) \, dtdqdu \\
&\qquad + \int_0^T \int_{\mathbb{R}^2} \frac{1}{\varepsilon} \frac{\partial \Big((\frac{\partial k^\varepsilon}{\partial \tau} \circ \mathcal{P}_r^-)^\varepsilon + v (\frac{\partial k^\varepsilon}{\partial r} \circ \mathcal{P}_r^-)^\varepsilon - r (\frac{\partial k^\varepsilon}{\partial v} \circ \mathcal{P}_r^-)^\varepsilon \Big)}{\partial t} \gamma \, dtdqdu \qquad (6.5)
\end{aligned}
$$

which is the Two-Scale Macro Equation.

7 The Two-Scale Micro Equation

Now, in in (5.5) we chose oscillating test functions ψ which are in $\mathrm{Im}\big(\frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r} - r \frac{\partial}{\partial v}\big)$ i.e., which write

$$
\psi(t, \tau, r, v) = \frac{\partial \kappa}{\partial \tau}(t, \tau, r, v) + v \frac{\partial \kappa}{\partial r}(t, \tau, r, v) - r \frac{\partial \kappa}{\partial v}(t, \tau, r, v), \qquad (7.1)
$$

for regular functions $\kappa(t, \tau, r, v)$. Making change of variables $(q, u) \mapsto (r, v) = \mathcal{P}^-(t/\varepsilon, r, v)$, the first term of (5.5) reads

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^2} (\mathcal{E}_1^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \Big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G}{\partial u} \Big) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu \\
&\qquad = - \int_0^T \int_{\mathbb{R}^2} \mathcal{E}_1^\varepsilon (\frac{\partial (G \circ \mathcal{P}^+)}{\partial v})^\varepsilon \Big((\frac{\partial \kappa}{\partial \tau})^\varepsilon + v (\frac{\partial \kappa}{\partial r})^\varepsilon - r (\frac{\partial \kappa}{\partial v})^\varepsilon \Big) dtdrdv, \qquad (7.2)
\end{aligned}
$$

the second one gives:

$$
\begin{aligned}
&- \varepsilon \int_0^T \int_{\mathbb{R}^2} \Big(\frac{\partial G_1^\varepsilon}{\partial t} + (\mathbf{E}^\varepsilon \circ \mathcal{P}_r^-)^\varepsilon \big(-\sin(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial q} + \cos(\frac{t}{\varepsilon}) \frac{\partial G_1^\varepsilon}{\partial u} \big) \Big) (\psi \circ \mathcal{P}^-)^\varepsilon dtdqdu \\
&= -\varepsilon \int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial G_1^\varepsilon}{\partial t} \circ \mathcal{P}^+)^\varepsilon + \mathbf{E}^\varepsilon \big(\frac{\partial (G_1^\varepsilon \circ \mathcal{P}^+)}{\partial v} \big)^\varepsilon \Big) \Big((\frac{\partial \kappa}{\partial \tau})^\varepsilon + v (\frac{\partial \kappa}{\partial r})^\varepsilon - r (\frac{\partial \kappa}{\partial v})^\varepsilon \Big) \, dtdrdv.
\end{aligned}
$$

(7.3)

The forth one reads

$$\int_0^T \int_{\mathbb{R}^2} \Big(\varepsilon(\frac{\partial l}{\partial t}\circ\mathcal{P}^-)^\varepsilon + \varepsilon(\mathbf{E}^\varepsilon\circ\mathcal{P}_r^-)^\varepsilon(\frac{\partial l}{\partial v}\circ\mathcal{P}^-)^\varepsilon\Big)(\psi\circ\mathcal{P}^-)^\varepsilon\,dtdqdu$$
$$= \int_0^T \int_{\mathbb{R}^2} \Big(\varepsilon(\frac{\partial l}{\partial t})^\varepsilon + \varepsilon\mathbf{E}^\varepsilon(\frac{\partial l}{\partial v})^\varepsilon\Big)\Big((\frac{\partial\kappa}{\partial\tau})^\varepsilon + v(\frac{\partial\kappa}{\partial r})^\varepsilon - r(\frac{\partial\kappa}{\partial v})^\varepsilon\Big)dtdrdv, \quad (7.4)$$

and finally gives

$$\int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon + v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)$$
$$\Big((\frac{\partial\psi}{\partial t})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial\psi}{\partial\tau})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial\psi}{\partial r})^\varepsilon + \Big(\mathbf{E}^\varepsilon - \frac{r}{\varepsilon}\Big)(\frac{\partial\psi}{\partial v})^\varepsilon\Big)\,dtdrdv$$
$$= -\int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial^2 k^\varepsilon}{\partial t\partial\tau})^\varepsilon + v(\frac{\partial^2 k^\varepsilon}{\partial t\partial r})^\varepsilon - r(\frac{\partial^2 k^\varepsilon}{\partial t\partial v})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial^2 k^\varepsilon}{\partial\tau^2})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial^2 k^\varepsilon}{\partial\tau\partial r})^\varepsilon - \frac{1}{\varepsilon}r(\frac{\partial^2 k^\varepsilon}{\partial\tau\partial v})^\varepsilon\Big)$$
$$\Big((\frac{\partial\kappa}{\partial\tau})^\varepsilon + v(\frac{\partial\kappa}{\partial r})^\varepsilon - r(\frac{\partial\kappa}{\partial v})^\varepsilon\Big)dtdrdv$$
$$+\int_{\mathbb{R}^2}\Big(\frac{\partial k^\varepsilon}{\partial\tau}(0,0,.,.,) + v\frac{\partial k^\varepsilon}{\partial r}(0,0,.,.,) - r\frac{\partial k^\varepsilon}{\partial v}(0,0,.,.,)\Big)$$
$$\Big(\frac{\partial\kappa}{\partial\tau}(0,0,.,.,) + v\frac{\partial\kappa}{\partial r}(0,0,.,.,) - r\frac{\partial\kappa}{\partial v}(0,0,.,.,)\Big)\,drdv$$
$$+\int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon + v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)$$
$$\Big(v\Big((\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial\tau\partial r})^\varepsilon + v(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r^2})^\varepsilon - r(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r\partial v})^\varepsilon\Big) - r\Big((\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial\tau\partial v})^\varepsilon + v(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r\partial v})^\varepsilon - r(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial v^2})^\varepsilon\Big)\Big)dtdrdv. \quad (7.5)$$

Taking (7.2), (7.3), (7.4) and (7.5) into account, remembering (3.12), we get from (5.5) the weak formulation of the Two-Scale Micro equation:

$$-\int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial^2 k^\varepsilon}{\partial t\partial\tau})^\varepsilon + v(\frac{\partial^2 k^\varepsilon}{\partial t\partial r})^\varepsilon - r(\frac{\partial^2 k^\varepsilon}{\partial t\partial v})^\varepsilon + \frac{1}{\varepsilon}(\frac{\partial^2 k^\varepsilon}{\partial\tau^2})^\varepsilon + \frac{1}{\varepsilon}v(\frac{\partial^2 k^\varepsilon}{\partial\tau\partial r})^\varepsilon - \frac{1}{\varepsilon}r(\frac{\partial^2 k^\varepsilon}{\partial\tau\partial v})^\varepsilon\Big)$$
$$\Big((\frac{\partial\kappa}{\partial\tau})^\varepsilon + v(\frac{\partial\kappa}{\partial r})^\varepsilon - r(\frac{\partial\kappa}{\partial v})^\varepsilon\Big)dtdrdv$$
$$+\int_0^T \int_{\mathbb{R}^2} \Big((\frac{\partial k^\varepsilon}{\partial\tau})^\varepsilon + v(\frac{\partial k^\varepsilon}{\partial r})^\varepsilon - r(\frac{\partial k^\varepsilon}{\partial v})^\varepsilon\Big)$$
$$\Big(v\Big((\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial\tau\partial r})^\varepsilon + v(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r^2})^\varepsilon - r(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r\partial v})^\varepsilon\Big) - r\Big((\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial\tau\partial v})^\varepsilon + v(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial r\partial v})^\varepsilon - r(\frac{1}{\varepsilon}\frac{\partial^2\kappa}{\partial v^2})^\varepsilon\Big)\Big)dtdrdv$$
$$= \int_0^T \int_{\mathbb{R}^2} \mathcal{E}_1^\varepsilon(\frac{\partial(G\circ\mathcal{P}^+)}{\partial v})^\varepsilon\Big((\frac{\partial\kappa}{\partial\tau})^\varepsilon + v(\frac{\partial\kappa}{\partial r})^\varepsilon - r(\frac{\partial\kappa}{\partial v})^\varepsilon\Big)dtdrdv$$
$$-\int_0^T \int_{\mathbb{R}^2} \Big(\varepsilon(\frac{\partial l}{\partial t})^\varepsilon + \varepsilon\mathbf{E}^\varepsilon(\frac{\partial l}{\partial v})^\varepsilon\Big)\Big((\frac{\partial\kappa}{\partial\tau})^\varepsilon + v(\frac{\partial\kappa}{\partial r})^\varepsilon - r(\frac{\partial\kappa}{\partial v})^\varepsilon\Big)dtdrdv. \quad (7.6)$$

8 Work Program : Towards TSAPS

Calculation verification - The first thing to do is to check the calculations just led. A formal calculation software routine will be realized for this purpose. The routine will be done in a way to be reusable for more complicated frameworks (4D and 6D Vlasov-Poisson systems)

Fourier expansion in τ - In (6.3) and (7.6), every function depending on τ will be expanded into finite Fourier sums, for instance,

$$\kappa(t,\tau,r,v) = \kappa_0 + \sum_{k=1}^{N_F} \kappa_k^{\mathsf{c}}(t,r,v)\cos(k\tau) + \kappa_k^{\mathsf{s}}(t,r,v)\sin(k\tau). \tag{8.1}$$

The resulting computation will be led using a formal calculation software.

Discretization - The time discretization will be done having in mind that

$$\int_0^T \frac{1}{\varepsilon}\sin(\frac{t}{\varepsilon})\,dt = \mathcal{O}(1) \text{ (and not } \mathcal{O}(\frac{1}{\varepsilon})), \tag{8.2}$$

so that several terms with a factor $1/\varepsilon$ are in fact not so great.

Two-Scale Macro Equation implementation - Once the time discretization done, a space discretization, based on a PIC method will be led. The code developed for the simulation done in [6] will be reused and inserted in SeLaLib for this.

Two-Scale Micro Equation implementation - This implementation will be highly linked with computations involving the Fourier Expansion and and the discretization.

Poisson Solver implementation - There are in fact two Poisson Equations to solve : (5.2) to get $\mathcal{E}$ and (5.4) for $\mathcal{E}_1^\varepsilon$, ($\mathbf{E}^\varepsilon = (\mathcal{E})^\varepsilon + \mathcal{E}_1^\varepsilon$).

Tests - The method will be tested on several examples

- with ε constant over the time interval with worth 10^{-2},
- with ε constant over the time interval with worth 0.5,
- with ε constant over the time interval with worth 1,
- with ε varying linearly over the time interval from 10^{-2} to 1
- with ε varying smoothly over the time interval with worth 10^{-2} at the extremities of the time interval and 0.5 in the middle.
- with ε varying not smoothly with worth 10^{-2} in the first part of the the time interval and 0.5 in the second part.

The results will be compared with results gotten with a One-Scale PIC Code. The one used for [6] can be a good choice.

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