Cemracs '11 : IDSA Project
The IDSA in
Supernovae Modelling

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Abstract

Modelling of core collapse supernovae leads to a high-dimensional coupled non-linear problem. A valid model is given by the hydrodynamic equations coupled with Boltzmann neutrino transport. The numerical cost of a direct resolution of such a coupled system is prohibitive in 3D because the treatment of Boltzmann's equation is too expensive. Therefore, one strives for a good approximation of the transport part. The IDSA (Isotropic Diffusion Source Approximation) [1] is a candidate for such an approximation.

This project has two aims : first to study analytically the quality of the approximation compared to the Boltzmann equation in 1D and second to implement a stable and efficient solver for the IDSA in 1D.

1 Reference model

The system of equations that we consider as a starting point reads

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{u}) = \mathbf{S}(\mathcal{F}(f), \mathbf{u}), \tag{1}$$

$$\underbrace{\frac{1}{c}\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial r} + \frac{1}{r}(1-\mu^2)\frac{\partial f}{\partial \mu}}_{\mathcal{D}(f)} = j(\mathbf{u}) - \tilde{\chi}(\mathbf{u}) f + \mathcal{C}(f). \tag{2}$$

Equation (1) is the ideal hydrodynamic equation with source term $\mathbf{S}$ accounting for coupling with Boltzmann transport. Equation (2) is the Boltzmann equation in laboratory frame for the distribution function $f(r, t, \mu, \omega)$. Here, r is the radius, t the time, μ the cosine of the angle between the radius and the direction of neutrino emission and ω the neutrino energy, $j(r, t, \mu, \omega, \mathbf{u})$ is the emissivity, $\tilde{\chi}(r, t, \mu, \omega, \mathbf{u})$ the absorptivity and c the speed of light. $\mathcal{C}(f)$ is the isoenergetic scattering collision integral and linear in f.

2 IDSA [1]

Basic idea : We suppose a decomposition of $f = f^t + f^s$ with distribution functions f^t and f^s accounting for trapped and streaming neutrinos in the **whole domain**. Physically, this is motivated by the very high matter density gradient existing in a supernova. At high densities, the neutrinos are trapped and at low densities, they are freely streaming.

By linearity of $\mathcal{D}$ and $\mathcal{C}$ we obtain

$$\mathcal{D}(f^s) - \mathcal{C}(f^s) + \tilde{\chi} f^s = j - \tilde{\chi} f^t + \mathcal{C}(f^t) - \mathcal{D}(f^t) =: \Sigma, \tag{3}$$

with the **diffusion source** $\Sigma(r, t, \mu, \omega, f, \mathbf{u})$.

Assumptions

1. **Isotropy :** f^t, Σ, j and $\tilde{\chi}$ are independent of μ and, therefore, $\mathcal{C}(f^t) = 0$. Since f^s is not independent of μ, we consider the angular mean $I_f = \frac{1}{2}\int_{-1}^{1} f^s d\mu$ instead, see Section 2.2.

2. f^s is in the **free streaming limit** : in this limit we have $\mathcal{C}(f^s) \approx 0$ and we neglect it.

3. f^s is in the **stationary state limit** : the evolution equation for f^s reduces to a Poisson equation.

4. f^t is in the **diffusion limit** : this will give a definition to the source term Σ.

2.1 Trapped particles

Integrating the trapped part of (3) w.r.t. μ and rewriting $\mathcal{D}$ in the comoving frame gives, using the **isotropy** assumption,

$$\frac{1}{2}\int_{-1}^{1} \mathcal{D}(f^t) d\mu = \frac{1}{c}\frac{df^t}{dt} + \frac{1}{3c}\frac{\partial \ln \rho}{\partial t}\omega \frac{\partial f^t}{\partial \omega} = j - \tilde{\chi} f^t - \Sigma. \tag{4}$$

Here, ρ is the matter density.

To compute the **diffusion limit**, we use a first order approximation of $\mathcal{C}(f)$ w.r.t. μ, integrate (2) w.r.t. μ and perform a **Chapman–Enskog expansion** [4] of $\mathcal{D}(f)$.

The **zeroth order** term gives the equilibrium distribution function $f_0 = \frac{j}{\tilde{\chi}} \overset{!}{=} f^t$ since f^t is assumed to be in the **diffusion limit**.

The **first order** equation, with Assumptions 1. and 2., then reads

$$\frac{1}{2}\int_{-1}^{1} \mathcal{D}(\underbrace{f^t + \varepsilon f_1}_{\doteq f}) d\mu \doteq \frac{1}{c}\frac{df^t}{dt} + \frac{1}{3c}\frac{\partial \ln \rho}{\partial t}\omega \frac{\partial f^t}{\partial \omega} + \alpha \doteq j - \tilde{\chi} \cdot \left(f^t + I_f\right). \tag{5}$$

Here the sign $\doteq$ indicates equality up to the order $\mathcal{O}(\varepsilon^2)$.

The comparison of (4) and (5) leads to the definition of Σ, namely

$$\Sigma := \mathcal{M}\Big(\underbrace{\nabla \cdot \Big(-\frac{\lambda(\mathbf{u})}{3}\nabla f^t\Big)}_{=:\alpha} + \tilde{\chi}(\mathbf{u}) I_f\Big), \tag{6}$$

where we introduce the limiter $\mathcal{M} := \min\{\max[\cdot, 0], j(\mathbf{u})\}$ to account for the **global assumption** that we have f^t on the **whole domain**. Here, $\lambda(\mathbf{u})$ is the mean free path.

2.2 Streaming particles

Integrating the streaming part of (3) w.r.t. μ and applying the **stationary limit** and the **free streaming limit** assumptions lead to a Poisson equation for a new potential ψ whose gradient is $\frac{\partial\psi}{\partial r} = \frac{1}{2}\int_{-1}^{1} f^s \mu d\mu$.

In 1D one has

$$I_f = H\Big(\underbrace{\frac{1}{r^2}\int_0^r (\Sigma - \tilde{\chi} I_f) r'^2 dr'}_{=\frac{\partial\psi}{\partial r}}\Big), \tag{7}$$

where H is given by a **geometrical** calculation.

2.3 Reduced model after IDSA

The system of equations that we get from the reference model after approximation of (2) is

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla\cdot\mathbf{F}(\mathbf{u}) = \mathbf{S}\left(\mathcal{F}(f)\right), \tag{8}$$

$$\frac{1}{c}\frac{df^t}{dt} + \frac{1}{3c}\frac{\partial \ln\rho(\mathbf{u})}{\partial t}\omega\frac{\partial f^t}{\partial\omega} = j(\mathbf{u}) - \tilde{\chi}(\mathbf{u})f^t - \Sigma(\mathbf{u}, f^t, I_f), \tag{9}$$

$$I_f = H\Big(\frac{1}{r^2}\int_0^r (\Sigma(\mathbf{u}, f^t, I_f) - \tilde{\chi}(\mathbf{u}) I_f) r'^2 dr'\Big). \tag{10}$$

3 Project aims

3.1 First aim

Although numerical experiments in [1] suggest good agreement of IDSA and the full Boltzmann model in 1D [2], so far no analytical results are at hand that confirm these findings theoretically. The first aim of this project is to understand analytically the quality of this approximation. Alternatively, other approximations of the full Boltzmann model [4] should be searched for and investigated.

3.2 Second aim

Using the hydrodynamic part as a black box, we seek an appropriate discretization and solution strategy for the IDSA equations (9) and (10) in 1D which shall be implemented in a prototype Matlab code. In particular, we face the following challenges :

- The **limiter** $\mathcal{M}$ in Σ leads to **non-smoothness**. Therefore, we need to look for a model without $\mathcal{M}$ or a solver that can handle non-smoothness [5, 6, 7].
- Equation (9) may be **stiff** in some region. We intend to consider well-known implicit Runge–Kutta methods like RADAU5 [3] that can tackle stiffness.

References

[1] M. Liebendörfer, S.C. Whitehouse & T. Fischer: *The Isotropic Diffusion Source Approximation for Supernova Neutrino Transport*, ApJ **698**, 1174 (2009).

[2] M. Liebendörfer, O.E.B. Messer, A. Mezzacapa, S.W. Bruenn, C.Y. Cardall & F.-K. Thielmann: *A Finite Difference Representation of Neutrino Radiation Hydrodynamics in Spherically Symmetric General Relativistic Spacetime*, ApJS **150**, 263 (2004).

[3] E. Hairer, G. Wanner: *Solving Ordinary Differential Equations II : Stiff and Differential-Algebraic Problems.* Springer, second edition, 1996.

[4] S. Chapman, T.G. Cowling :*The Mathematical Theory of Non-Uniform Gases.* Cambridge, third edition, 1970.

[5] L. Qi, J. Sun : *A Nonsmooth Version of Newton's Method,* Mathematical Programming **58**, 353 (1993).

[6] M. Ulbrich : *Nonsmooth Newton-like Methods for Variational Inequalities and Constrained Optimization Problems in Function Spaces,* Habilitation, TU München, 2001.

[7] F.H. Clarke : *Optimization and Nonsmooth Analysis,* Wiley & Sons, 1983.