Finite volume schemes for multiscale models for multiphase flow on surfaces

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In the modelling of phase separation or demixing of different substances phase field models are a useful tool. In these models the interface between the substances/phases is given by a thin layer where a function, indicating the phase/substance, changes rapidly but smoothly. Their main advantage compared to sharp interface models is that the tracking of the interface is avoided and topological changes of the interface, e.g. coalescence or vanishing of bubbles, are generically included in the model. In certain situations these processes do not take place in Euclidean space but on curved surfaces. Examples include demixing of different surfactants moving on the surface given by the interface between two fluids and the separation of different sorts of receptors or lipids on the surface/membrane of a biological cell. This leads Allen-Cahn or Cahn-Hilliard type equations on the surface $\Gamma \subset \mathbb{R}^3$, the latter having the form,

$$u_t = -\nabla_\Gamma \left(\mathcal{D} \nabla_\Gamma \left(\varepsilon \Delta_\Gamma u - \frac{1}{\varepsilon} \Psi'(u) \right) \right) \qquad \text{on } \Gamma \times [0, \infty), \tag{1}$$ 1

where $u(x,t) \in \mathbb{R}$ is the phase field, $\mathcal{D}$ is the diffusivity tensor, $\varepsilon > 0$ is a regularisation parameter and $\Psi = \Psi(u)$ is the free energy density, having a double-well shape. In addition $\nabla_\Gamma, \Delta_\Gamma$ are the surface gradient and Laplace Beltrami-Operator, generalzing the gradient and the Laplace-Operator from Euclidean space. Classically phase field equations of Allen-Cahn or Cahn-Hilliard type are solved using finite element methods but at the interface between the two phases the phase field, although it is smooth, exhibits steep gradients, thus a discretisation using finite volume schemes seems reasonable.

The idea is to develop and implement a finite volume discretisation of (1) on the sphere and if there is enough time also on more complex surfaces. Several grids on the sphere are at hand.