3D-PRISM project

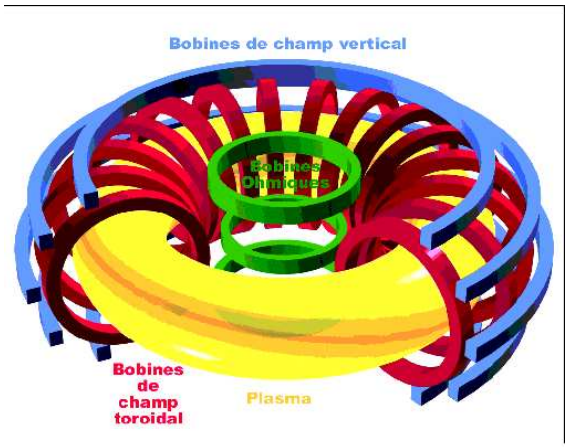
Leading participants : H. Guillard, B. Nkonga

Other participants : A. Bonnement, T. Fajraoui, M. Martin, A. Mouton, A. Sangam

INRIA Sophia-Antipolis, équipe PUMAS, AE Fusion
UNS Laboratoire JAD, UMR CNRS 6621

Contents

Introduction



- Tokamak with cylindrical symmetry
- Magnetic field following the geometry
- Anisotropic diffusion depends on the magnetic field and so on the geometry

Aim of this work :

- How develop a Finite Volume method on a variable system coordinates ?
- Advantages and drawbacks compared with cartesian formulation

Introduction to curvilinear coordinates

Physic space : $\Omega(\overrightarrow{x})$ Computational space : $\Omega(\overrightarrow{\xi})$

Covariant basis

$$\forall\, i = 1, \ldots, n, \qquad \underline{e_{\xi i}} = \frac{\partial x_k}{\partial \xi^i}\,\underline{e_{x_k}}$$

For example in polar coordinates :

- covariant basis :

$$\underline{e_r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \underline{e_\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

- contravariant basis : $\underline{e^{\xi^i}} \cdot \underline{e_{\xi j}} = \delta_{i,j}$

Operators in curvilinear coordinates

Gradient :

$$\frac{\partial \underline{e^{\xi_i}}}{\partial \xi^j} = \Gamma^k_{i,j}\, \underline{e_{\xi^k}}$$

Divergence operator :

$$\nabla_{\underline{x}} \cdot \underline{f} = \frac{1}{J} \frac{\partial}{\partial \xi^i} \left(J\, f^{\xi^i} \right),$$

Divergence of a tensor field :

$$\nabla_{\underline{x}} \cdot \underline{\underline{T}} = \frac{1}{J} \frac{\partial}{\partial \xi^i} \left(J\, \underline{\underline{T}} \cdot \underline{e^{\xi^i}} \right).$$

Contents

Usual polar method

Vectorial equation in polar coordinates

$$\partial_t(r\,\underline{S}) + \partial_r(r\,T^{r,r}\,\underline{e_r} + r\,T^{r,\theta}\,\underline{e_\theta}) + \partial_\theta(r\,T^{\theta,r}\,\underline{e_r} + r\,T^{\theta,\theta}\,\underline{e_\theta}) = 0\,.$$

How approach this equation ?

- Classical method : Projection before numerical integration
  $\Rightarrow$ Drawback : Source terms appear from hyperbolic term
- Other method : Numerical integration before projection
  $\Rightarrow$ Drawback : Projection choice needed

Source terms

Vectorial equation in polar coordinates :

$$\partial_t(r\,\underline{S}) + \partial_r(r\,T^{r,r}\,\underline{e_r} + r\,T^{r,\theta}\,\underline{e_\theta}) + \partial_\theta(r\,T^{\theta,r}\,\underline{e_r} + r\,T^{\theta,\theta}\,\underline{e_\theta}) = 0\,.$$

Projection

$$\begin{cases} \partial_t(r\,S_r) + \partial_r(rT^{r,r}) + \partial_\theta T^{r,\theta} = T^{\theta,\theta} \\ \partial_t(r\,S_\theta) + \partial_r(r\,T^{r,\theta}) + \partial_\theta T^{\theta,\theta} = -T^{r,\theta} \end{cases}$$

Source terms $T^{\theta,\theta}$ and $-T^{r,\theta}$ appear.

$\Rightarrow$ How discretize source terms to have equivalent methods ?

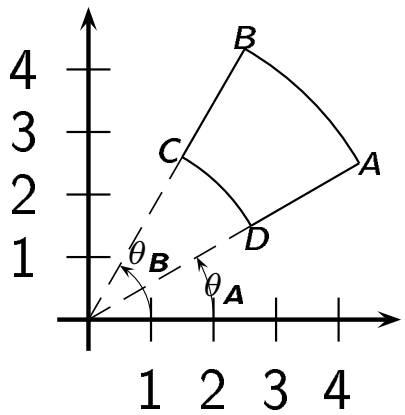
Source term discretization

Comparison between the two methods

Equivalent methods if :

$$\int_{\Omega_i} T^{\theta,\theta}(r,\theta)\, dr\, d\theta$$

$$= \Delta\theta\, \Delta r\, \frac{\hat{T}^{\theta,\theta}_{|\theta_B} + \hat{T}^{\theta,\theta}_{|\theta_A}}{2} + (1 - \frac{\Delta\theta}{2}\frac{\sin\Delta\theta}{1-\cos\Delta\theta})\, \Delta r\, (\hat{T}^{\theta,\theta}_{|\theta_B} - \hat{T}^{\theta,\theta}_{|\theta_A})$$



- Usual center term
- Viscosity term

Differences between the polar and cartesian methods

- Meshes
- Geometrical parameters : cell areas, normals
- Fluxes computing : Constants in cell, post-processing

Polar and Cartesian meshes

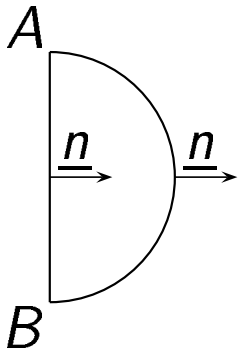
Normals

Normals in curvilinear coordinates

$$\int_{\partial\Omega_{i,j}(\underline{x})} \underline{n}(\underline{x}) dl(\underline{x}) = \int_{\partial\Omega_{i,j}(\underline{\xi})} J \underline{n}_k(\underline{\xi}) \underline{e}^k \, dl(\underline{\xi})$$

Divergence theorem on a closed outline :

$$\oint \underline{n} dl = 0 \text{ i.e. } \int_{\widehat{AB}} \underline{n} dl = \int_{\overline{AB}} \underline{n} dl.$$



- Same normal in polar and cartesian coordinates
- The normal depends only on A and B

Areas

- Cartesian Coordinates :

$$|\Omega_i|_{x,y} = (r + \frac{\Delta r}{2})\,\Delta r\,\sin\Delta\theta$$

- Polar coordinates :

$$|\Omega_i|_{r,\theta} = \int_{\Omega_i} r\,dr\,d\theta = (r + \frac{\Delta r}{2})\,\Delta r\,\Delta\theta$$

Equivalence between polar and cartesian

When $\Delta\theta$ is small, $\sin\Delta\theta \approx \Delta\theta$, so $|\Omega_i|_{x,y} \approx |\Omega_i|_{r,\theta}$.

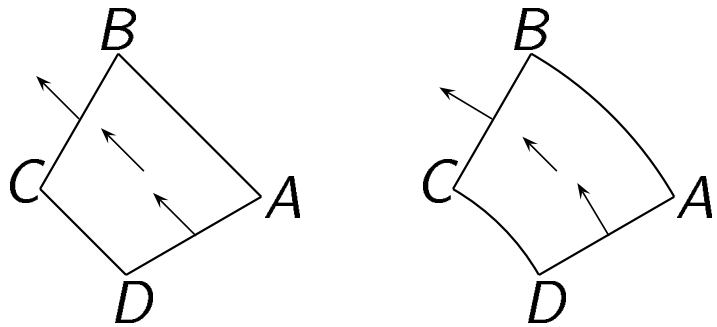
Constants on a cell

- Cartesian Coordinates : $\underline{V} = V_x \underline{e}_x + V_y \underline{e}_y$

  $\Rightarrow$ Velocity vector constant

- Polar coordinates : $\underline{V} = V_r \underline{e}_r + V_\theta \underline{e}_\theta$

  $\Rightarrow$ Only V_r and V_θ constant, Velocity vector no constant



Equivalence between polar and cartesian when $\Delta\theta \longrightarrow 0$

Post-processing before Riemann Solver

- Cartesian coordinates :
  Same flux given to two neighbour points i and j

$$\underline{\Phi}_i = \Phi^x \underline{e}_x + \Phi^y \underline{e}_y \text{ and } \underline{\Phi}_j = \Phi^x \underline{e}_x + \Phi^y \underline{e}_y$$

- Polar coordinates :
  Fluxes depend on the local basis $(\underline{e_r}, \underline{e_\theta})$ so different fluxes given to two neighbour points i and j

$$\underline{\Phi}_i = \Phi^r \underline{e}_{r_i} + \Phi^\theta \underline{e}_{\theta_i} \text{ and } \underline{\Phi}_j = \Phi^r \underline{e}_{r_j} + \Phi^\theta \underline{e}_{\theta_j}$$

Contents

Scalar test

Advection equation

$$\partial_t n + \nabla.(nu) = 0$$

Initial conditions

$n = n_0$, $u_r = 0$ and $u_\theta = u_0$ where n_0 and u_0 are constant

$\Rightarrow$ Same results for cartesian and polar approaches : the solution is preserved.

$\Rightarrow$ In scalar test, only the cell areas are different between the cartesian and polar approaches.

Vectorial test

Isoterm Euler System

$$\begin{cases} \partial_t n + div(nu) = 0 \\ \partial_t (nu) + div(nu \otimes u) + \nabla n = 0 \end{cases}$$

Initial conditions

$n = n_0$, $u_r = 0$ and $u_\theta = u_0$ where n_0 and u_0 are constant

$\Rightarrow$ In vectorial test, the computation of the fluxes are different as well as the cell areas.

Vectorial test

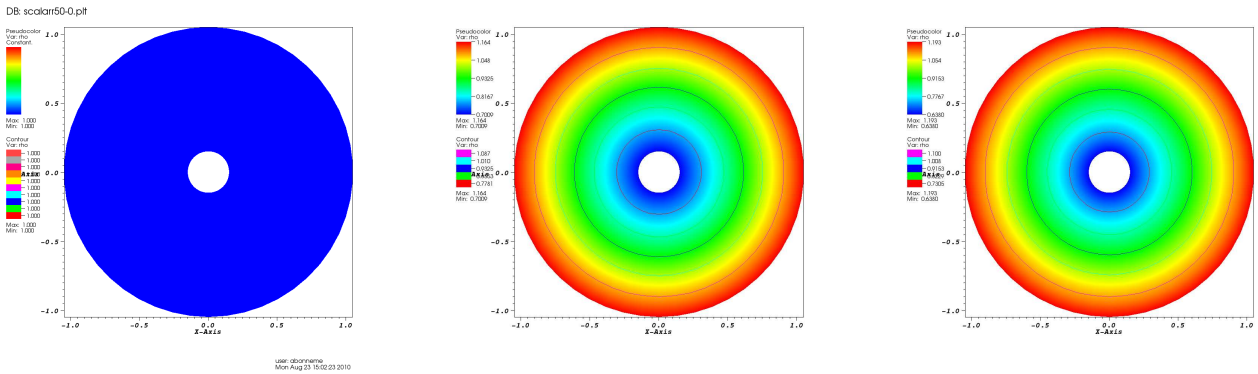


Fig.: IC, Cartesian results at $t = 0.83$; Polar results at $t = 0.82$

$\Rightarrow$ Same results for cartesian and polar approaches

$\Rightarrow$ Different fluxes for the two methods but fluxes balance is the same

Gresho test

Isoterm Euler System

$$\begin{cases} \partial_t n + div(nu) = 0 \\ \partial_t(nu) + div(nu \otimes u) + \nabla n = 0 \end{cases}$$

Stationary solution

$$\text{Stationary solution if } \partial_r(\ln n) = \frac{u_\theta^2}{r}$$

For example : $n(r) = n_0 exp(r)$, $u_r = 0$ and $u_\theta(r) = \sqrt{r}$

Gresho test

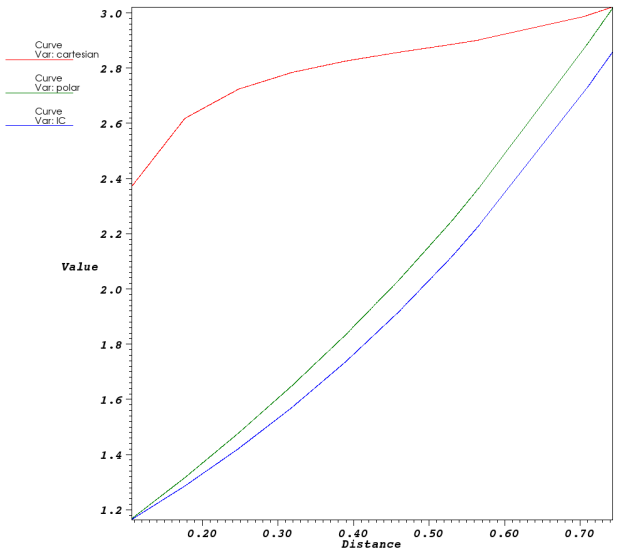


Fig.: Density profiles for IC, cartesian and polar : $N_\theta = 4$

- Even if N_θ is small, polar results close to stationary solution.
- Cartesian does not show good results when N_θ is small : profile different to stationary condition.

  $\Rightarrow$ Cartesian and polar methods are only equivalent when N_θ is large

Gresho test

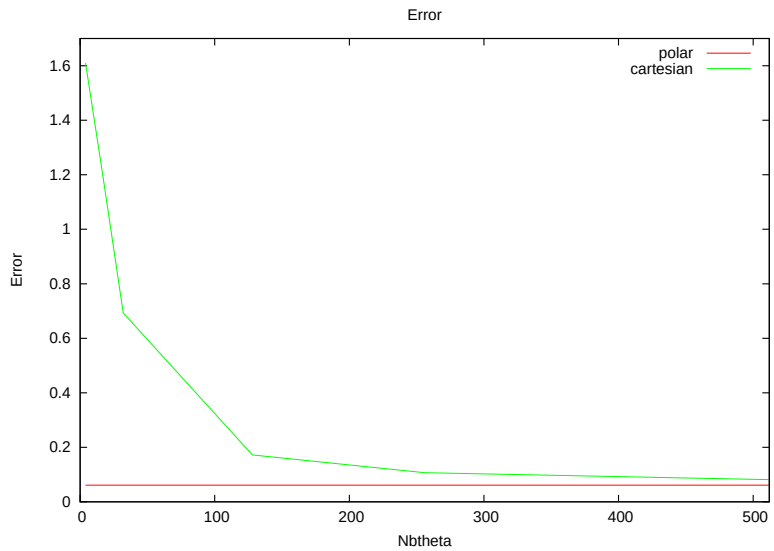


Fig.: Error profiles in function of N_θ

- Polar error depends only on N_r and not on N_θ
- Cartesian error decreases and comes closer to polar curve when N_θ raises up.
  $\Rightarrow$ These results confirm that the two methods are equivalent when N_θ is large

Contents

Conclusions

- Main difference between cartesian and polar methods is the choice of the constants in the cell
- Fluxes computing different between the two methods but equivalent methods when N_θ is large
- Polar method independant on N_θ, good results even if mesh not refined in θ

Perspectives

- Application to toroidal coordinates (3D)
- Application to Element Finite method