Cemracs project: HYDRO-ALE
Extension of ALE to quadric cells

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OUTLINE

1. Discontinuous ALE on polygonal cells
   - Equations and splitting
   - Mesh adaptation by conservative Discontinuous ALE
2. CEMRACS : Extension to conic cells
   - explicit formula for area
   - Consequence : extension of polygonal remapping step

Motivation

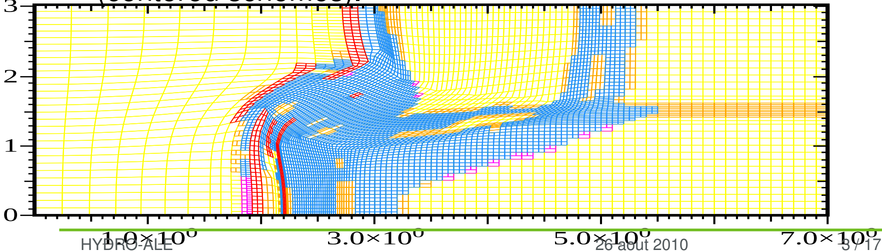
Extension of existing ALE frame for mesh adaptation

Lagrange

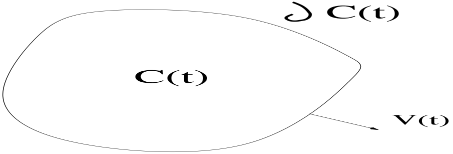
(Refinement/ Derefinement/ Edge Swapping/ Sliding)

Rezone

by a modeling of mesh connectivity evolution by a set of PDE (extended kinematic equations) in a finite volume context (centered schemes).



Equations for ALE, splitting



General ALE framework, conservation Laws.
For a control volume $C(t)$, let V an arbitrary speed of $\partial C(t)$:

$$
\left\{
\begin{array}{l}
D_t \int_{C(t)} 1 dx - \int_{\partial C(t)} V.N dl = 0, \quad \textit{Geometric Conservation Law}. \\
D_t \int_{C(t)} \rho dx + \int_{\partial C(t)} \rho (U - V).N dl = 0, \\
D_t \int_{C(t)} \rho U dx + \int_{\partial C(t)} \rho U (U - V).N + P N dl = 0, \\
D_t \int_{C(t)} \rho E dx + \int_{\partial C(t)} \rho E (U - V).N + P U.N dl = 0. \\
+EOS \qquad P(\rho, \epsilon) \qquad \epsilon \quad internal \quad energy.
\end{array}
\right.
$$

Solved by splitting ($V := U + U^{grid}$)

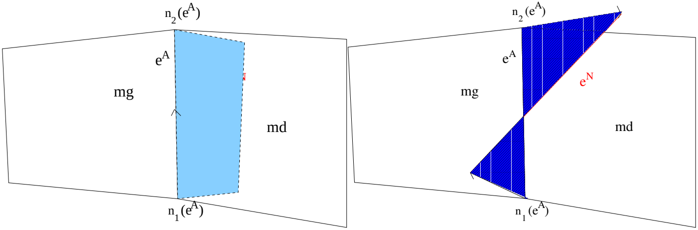
- Pure Lagrangian phase ($V^{lag} := U$) No advection
- Pure Advection phase ($V^{Grid} := U^{grid}$) No pressure force.

Remapping a first approximation

For the computation of conservative quantities on a new mesh, the exact problem need an exact polygon/polygon intersection!! [Margolin-Shashkov].

Definition

Swept area flux scheme given by the computation of algebraic volume given by region swept by edge.



area swept by edge movement δF_{ik} a cell is either donor nor receiver

$$F_{ik}^{sda,1} = max(0, V(\delta F_{ik}))\bar{Q}^A_{mv(i,k)} + min(0, V(\delta F_{ik}))\bar{Q}^A_i$$

Second approximation: Only one local intersection of edges

Definition

self intersection flux, *only compute (the eventual) intersection point in the edge mouvement.*

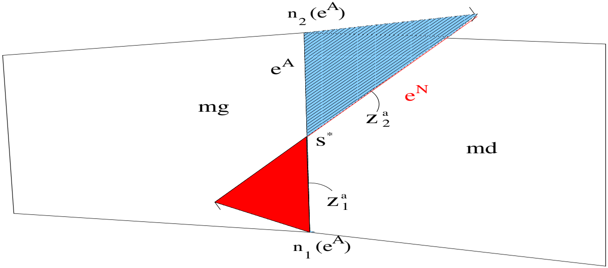
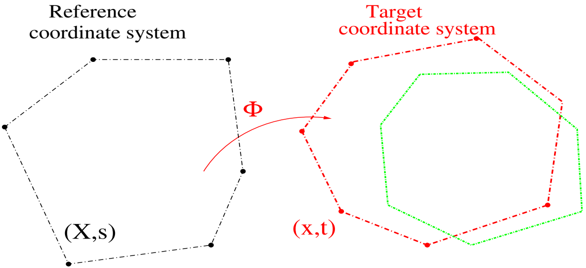


Figure: the two cells are simultaneously donor/receiver cells

$$F_{ik}^{self,1} = V^+(\delta Z_{ik}^+)\bar{Q}_{mv(i,k)}^A + V^-(\delta Z_{ik}^-)\bar{Q}_i^A$$

Remarks on limitation with basic ALE formulation

ALE framework:**homeomorphic map** (bijective continuous).



Hypothesis: Jac(Φ) =det(dX/dx) >0

Mesh connectivity can not change (refinement/derefinement)

- **Injection** do not permit collapsing of vertices.
- **Continuity** do not permit fission of vertices.

Mesh adaptation can not be modelized by continuous nodal movements.

Mesh adaptation by conservative Discontinuous ALE

Formulation : $V^{Grid} := V^{Discontinuous,Grid} + V^{Continuous,Grid}$, so that new kinematic equation now:

$$\frac{dx}{dt} = V^{Lag}(x,t) + V^{D,Grid}(x,t) + V^{C,Grid}(x,t)$$

Linear multi-dimensionnal advection with **Discontinuous** coefficients see [Bouchut,James], [Poupaud,Rascle]:

$$\boxed{V^{D,Grid}(x,t) \in L^1_{loc}((0,+\infty); L^\infty(\mathbb{R}^d))^d}$$

Sufficient condition OSLC (uniqueness of Filipov characteristics):

$$\Big(V^{D,Grid}(x,t) - V^{D,Grid}(y,t), x-y \Big) \leq |x-y| L(t,|x-y|),$$

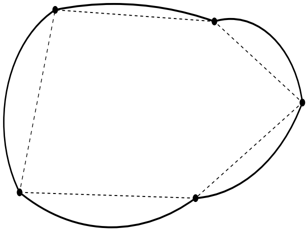
Compressives discontinuities are well posed ...

Results for "conservative" mesh adaptation

1. $V^{D,Grid}(t,.) \in L^{\infty}(\boldsymbol{R}^d) \Rightarrow$ CFL gives an automatic constraint on remeshing tools, must be local.
2. Tangential part of $V^{D,Grid}$ is **aligned with the boundary**.
3. In practice $V^{D,Grid} = V^{D,Grid,Translational} + V^{D,Grid,Rotational}$
   1. Translational : gives extension of AMR adaptation (refinement and derefinement) on arbitrary polygons.
   2. Rotational : gives extension of swapping operation on arbitrary polygons, sliding can be modelized.
4. **No exact polygon/polygon intersection** even when reconnection.
5. Fully Finite Volumes approach, conservative even if discontinuous flux, second order maximum principle scheme (except for hydrodynamic part).

Initial guess:

1. Find an analytical formula to compute the area of arbitrary parabolic and locally circular cells with respect to nodes on the boundary: EXACT formula.
2. Extend the remapping (algebraic swept area and self-intersection flux) on conic cells.



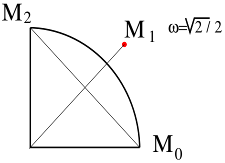
Rational quadratic bezier parametrization

$$M(q) = \frac{M_0(1-q)^2 + M_1 2\omega q(1-q) + M_2 q^2}{(1-q)^2 + 2\omega q(1-q) + q^2}, \qquad q \in [0,1].$$

(M_1, ω) : Control point and associated weight for edge (M_0, M_1).

Computation of area for cells with arbitrary conic section (chinese paper 1996 W. Guojin)):

1. $\omega > 1$: hyperbola parametrization
2. $\omega = 1$: parabola parametrization
3. $\omega < 1$: ellipse, in the case where M_1 is on the perpendicular bisector of (M_0, M_1) and $\omega = cos(\alpha/2)$, circle is recovered.



4. $\omega = 0$: straight line (recover polygonal mesh).

Conservative continuous advection on conic cells

Extension of the resolution of GCL, maximum principle can be establish for first order scheme.
Recall Continuous conservative advection is done by rezoning+remapping:

1. vertices + control points are moved.
2. weights of control points can change : $\omega(t)$.

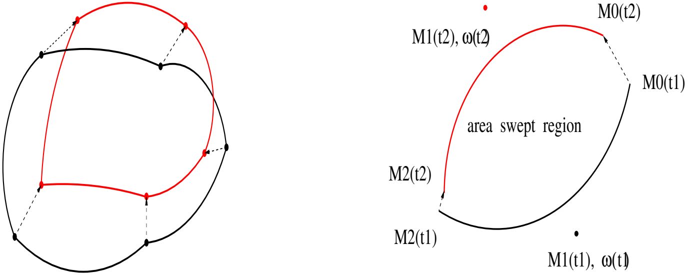


Figure: Old cell (black) move to new cell (red) and Local edge swept forms conics regions and then can be computed exactly !

Only one remapping step of characteristic function: same displacement of vertices and only weight is different.

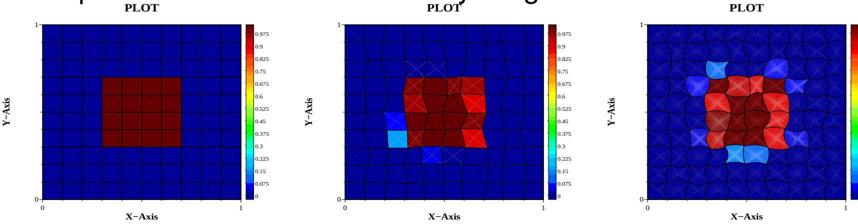


Figure: Initial data, polygons ($\omega = 0$), ellipse ($\omega = .5$)

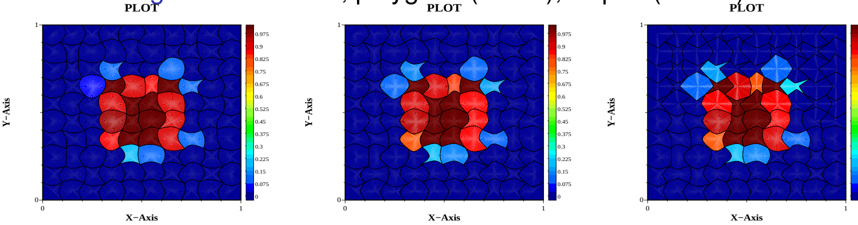


Figure: parabola ($\omega = 1$), hyperbola ($\omega = 2$), mixed

Large Displacement

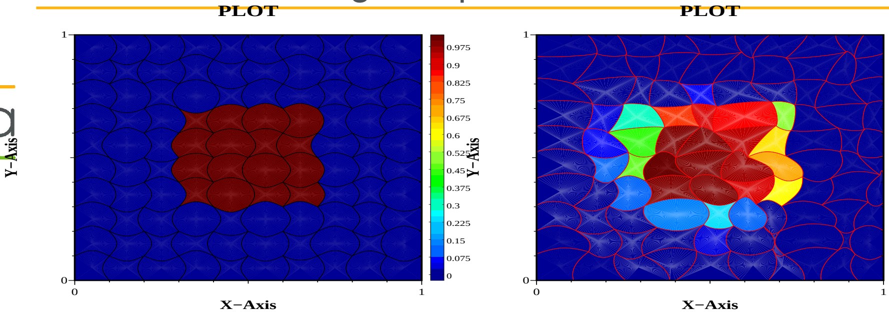
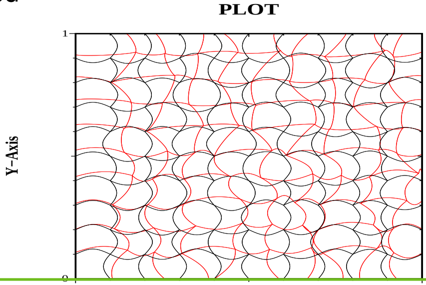
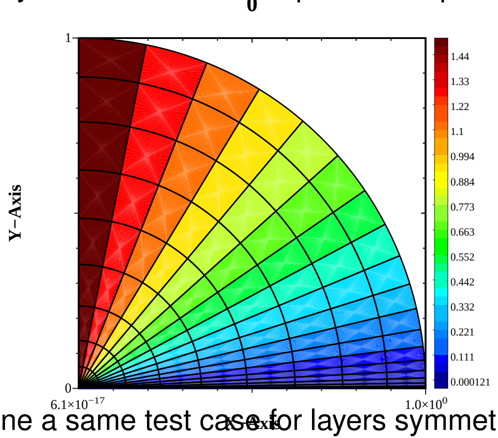


Figure: sub cycling is needed here (4 cycles), maximum principle is still maintained



Preservation of symmetry on polar mesh with non equi angular sector

Nodal displacement is done only in radial direction, the weight of control point is given by $cos((\alpha_{j+1} + \alpha_j)/2)$ and position is given by analytical formula with respect to endpoints nodes.



We must done a same test case for layers symmetry more challenging.

Conclusion

(1) We have extented the remapping step to arbitrary conic polygonal cells.

(2) All the test case gives correct results : maximum principle, but first order!.

(3) GO++NS code has been used (initially polygonal cell only): upgrade of

   1. geometry and mesh classes.
   2. first order remapping.
   3. visualization

Prospects for Discontinuous ALE on conic cells

(a) Extend the self-intersection flux (currently), local conic/conic intersection (at most fourth order equation). More remapping tests.

(b) Extend to second order.

(c) Extend the mesh rezoning step:

  Mesh representation: some curvature minimization of total edges impanging in a vertex, what is the optimal conic local localy ?

  Mesh smoothing: define a local objectif function to obtain better shaped cells than thoose given by hydrodynamic.

(d) Extend centered hydro scheme.