MHD Simulations for Fusion Applications

Lecture 4

Implicit Methods and the M3D-C^1 Approach

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Outline of Remainder of Lecture

Yesterday

- Galerkin method
- C^1 finite elements
- Examples
- Split implicit time differencing

Today

- Split implicit time differencing for MHD
- Accuracy, spectral pollution, and representation of the vector fields
- Projections of the split implicit equations
- Energy conserving subsets
- 3D solver strategy
- Examples

2-Fluid MHD Equations:

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$ continuity

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla \bullet \mathbf{B} = 0 \qquad \mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$ Maxwell

$$nM_i\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B} - \nabla \bullet \mathbf{\Pi}_{GV} + \mu \nabla^2 \mathbf{V}$$ momentum

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne}\left(\mathbf{J} \times \mathbf{B} - \nabla p_e\right)$$ Ohm's law

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_e \mathbf{V}\right) = -p_e \nabla \bullet \mathbf{V} + \eta J^2 - \nabla \bullet \mathbf{q}_e + Q_\Delta + S_{Fe}$$ electron energy

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_i \mathbf{V}\right) = -p_i \nabla \bullet \mathbf{V} + \mu \left|\nabla V\right|^2 - \nabla \bullet \mathbf{q}_i - Q_\Delta + S_{Fi}$$ ion energy

Ideal MHD

Resistive MHD

2-fluid MHD

n	number density	$\mathbf{V}$	fluid velocity	μ	viscosity
$\mathbf{B}$	magnetic field	p_e	electron pressure	η	resistivity
$\mathbf{J}$	current density	p_i	ion pressure	$\mathbf{q_i}, \mathbf{q_e}$	heat fluxes
$\mathbf{E}$	electric field	$p \equiv p_e + p_i$		Q_Δ	equipartition
$nM_i \equiv \rho$	mass density	e	electron charge	μ_0	permeability

The split-implicit time advance

Consider a simple 1-D Hyperbolic System of Equations (Wave Equation)

$$\frac{u_j^{n+1}-u_j^n}{\delta t}=c\left[\theta\left(\frac{v_{j+1/2}^{n+1}-v_{j-1/2}^{n+1}}{\delta x}\right)+(1-\theta)\left(\frac{v_{j+1/2}^{n}-v_{j-1/2}^{n}}{\delta x}\right)\right]$$

$$\frac{v_{j+1/2}^{n+1}-v_{j+1/2}^{n}}{\delta t}=c\left[\theta\left(\frac{u_{j+1}^{n+1}-u_j^{n+1}}{\delta x}\right)+(1-\theta)\left(\frac{u_{j+1}^{n}-u_j^{n}}{\delta x}\right)\right]$$

Substitute from second equation into first:

$$u_j^{n+1}=u_j^n+(\delta tc)^2\left[\theta^2\left(\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\delta x^2}\right)+\theta(1-\theta)\left(\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\delta x^2}\right)\right]+\delta tc\left(\frac{v_{j+1/2}^{n}-v_{j-1/2}^{n}}{\delta x}\right)$$

$$v_{j+1/2}^{n+1}=v_{j+1/2}^{n}+\frac{\delta tc}{\delta x}\left[\theta\left(u_{j+1}^{n+1}-u_j^{n+1}\right)+(1-\theta)\left(u_{j+1}^{n}-u_j^{n}\right)\right]$$

These two equations are completely equivalent to those on the previous page, but can be solved sequentially!

Only first involves Matrix Inversion … Diagonally Dominant

An alternate derivation:

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}$$
$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}$$

→

Expand RHS in Taylor series in time to time-center

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\frac{\partial v}{\partial t}\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

An alternate derivation:

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}$$
$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}$$

Expand RHS in Taylor series in time to time-center

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$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

Substitute from second equation into first

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\left(c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]\right)\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

An alternate derivation:

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}$$
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Expand RHS in Taylor series in time to time-center

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Substitute from second equation into first

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\left(c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]\right)\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

Use standard centered difference in time:

$$\left[1-\theta^2(\delta t)^2 c^2\frac{\partial^2}{\partial x^2}\right]u^{n+1} = \left[1+\theta(1-\theta)(\delta t)^2 c^2\frac{\partial^2}{\partial x^2}\right]u^n + \delta t c\frac{\partial}{\partial x}v^n$$

$$v^{n+1} = v^n + \delta t c\left[\theta\frac{\partial}{\partial x}u^{n+1} + (1-\theta)\frac{\partial}{\partial x}u^n\right]$$

An alternate derivation:

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}$$
$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}$$

Expand RHS in Taylor series in time to time-center

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\frac{\partial v}{\partial t}\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

Substitute from second equation into first

$$\frac{\partial u}{\partial t} = c\frac{\partial}{\partial x}\left[v^n + \theta\delta t\left(c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]\right)\right]$$
$$\frac{\partial v}{\partial t} = c\frac{\partial}{\partial x}\left[u^n + \theta\delta t\frac{\partial u}{\partial t}\right]$$

Use standard centered difference in time:

$$\left[1 - \theta^2(\delta t)^2 c^2\frac{\partial^2}{\partial x^2}\right]u^{n+1} = \left[1 + \theta(1-\theta)(\delta t)^2 c^2\frac{\partial^2}{\partial x^2}\right]u^n + \delta t c\frac{\partial}{\partial x}v^n$$

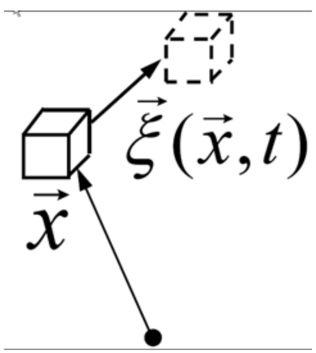
$$v^{n+1} = v^n + \delta t c\left[\theta\frac{\partial}{\partial x}u^{n+1} + (1-\theta)\frac{\partial}{\partial x}u^n\right]$$

This is the same operator as before when centered spatial differences used

δW

(next 3 vgs)

Linear ideal MHD response can be analyzed with δW approach



let $\mathbf{V}=\dfrac{\partial \xi}{\partial t}$ introduce a displacement vector ξ

linearize equations about $\mathbf{V}=0$

$$\rho\frac{\partial^2\xi}{\partial t^2}+\nabla p_1=\frac{1}{\mu_0}(\nabla\times\mathbf{B}_1)\times\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}_1$$

$$\mathbf{B}_1=\nabla\times(\xi\times\mathbf{B})$$

$$p_1+\nabla\bullet(p\xi)=-\frac{2}{3}p\nabla\bullet\xi$$

or,

$$\rho\frac{\partial^2\xi}{\partial t^2}=\mathbf{F}(\xi)$$ Linearized equation of motion

Ideal MHD operator →

$$\mathbf{F}(\xi)=\frac{1}{\mu_0}\left[(\nabla\times\mathbf{B})\times\mathbf{Q}+(\nabla\times\mathbf{Q})\times\mathbf{B}\right]+\nabla\left(\xi\bullet\nabla p+\tfrac{5}{3}p\nabla\bullet\xi\right)$$

$$\mathbf{Q}\equiv\mathbf{B}_1=\nabla\times(\xi\times\mathbf{B})$$

δW is the potential energy for a given displacement field

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi})$$

$$\mathbf{F}(\boldsymbol{\xi}) = \frac{1}{\mu_0}\left[\left(\nabla \times \mathbf{B}\right) \times \mathbf{Q} + \left(\nabla \times \mathbf{Q}\right) \times \mathbf{B}\right] + \nabla\left(\boldsymbol{\xi} \bullet \nabla p + \tfrac{5}{3} p \nabla \bullet \boldsymbol{\xi}\right)$$

$$\mathbf{Q} \equiv \mathbf{B}_1 = \nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}\right)$$

Assume $\boldsymbol{\xi}(x,t) = \boldsymbol{\xi}(x) e^{i\omega t}$ and take dot product with $-\tfrac{1}{2}\boldsymbol{\xi}^{\dagger}$,

and integrate over volume to get perturbed energy

$$\tfrac{1}{2} \rho \omega^2 \int d\tau \, \boldsymbol{\xi}^2 = -\tfrac{1}{2} \int d\tau \, \boldsymbol{\xi}^{\dagger} \bullet \mathbf{F}(\boldsymbol{\xi}) \equiv \delta W(\boldsymbol{\xi}^{\dagger}, \boldsymbol{\xi})$$

if $\delta W < 0$ for any displacement field $\boldsymbol{\xi} \rightarrow$ instability

The plasma motion will be such as to minimize δW

This is the term associated with the fast wave…it is positive definite with a large multiplier. Any unstable plasma motion will make this term small

$$\nabla \cdot \boldsymbol{\xi}_\perp \cong -2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \sim 0$$

$$\delta W = \tfrac{1}{2}\int d\tau \left[\begin{array}{l} \frac{1}{\mu_0}\mathbf{Q}_\perp^2 + \frac{1}{\mu_0}B^2\left[\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}\right]^2 + \tfrac{5}{3}p\left|\nabla \cdot \boldsymbol{\xi}\right|^2 \\ -2\left(\boldsymbol{\xi}_\perp \cdot \nabla p\right)\left(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp\right) - \sigma \boldsymbol{\xi}_\perp \times \mathbf{B} \cdot \mathbf{Q}_\perp \end{array} \right]$$

$\mathbf{b} = \mathbf{B} / B$ unit vector in direction of field

$\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b}$ curvature of magnetic field

$\sigma = \mathbf{J} \cdot \mathbf{B} / B^2$ parallel current density

$\mathbf{Q} = \nabla \times \left(\boldsymbol{\xi} \times \mathbf{B}\right)$ perturbed magnetic field

$\mathbf{Q}_\perp =$ perpendicular component of $\mathbf{Q}$

$\boldsymbol{\xi}_\perp =$ perpendicular component of $\boldsymbol{\xi}$

Now apply the split implicit advance to the basic 3D (ideal) MHD equations:

$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}\left[\nabla \times \mathbf{B}\right] \times \mathbf{B} - \nabla p$$

$$\dot{\mathbf{B}} = \nabla \times \left[\mathbf{V} \times \mathbf{B}\right]$$

$$\dot{p} = -\mathbf{V} \bullet \nabla p - \gamma p \nabla \bullet \mathbf{V}$$

Ideal MHD Equations for velocity, magnetic field, and pressure:

Symmetric Hyperbolic System

7-waves

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7-waves

$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}\left[\nabla \times \left(\mathbf{B} + \theta\delta t \dot{\mathbf{B}}\right)\right] \times \left(\mathbf{B} + \theta\delta t \dot{\mathbf{B}}\right) - \nabla\left(p + \theta\delta t \dot{p}\right)$$

$$\dot{\mathbf{B}} = \nabla \times \left[\left(\mathbf{V} + \theta\delta t \dot{\mathbf{V}}\right) \times \mathbf{B}\right]$$

$$\dot{p} = -\left(\mathbf{V} + \theta\delta t \dot{\mathbf{V}}\right) \bullet \nabla p - \gamma p \nabla \bullet \left(\mathbf{V} + \theta\delta t \dot{\mathbf{V}}\right)$$

Taylor Expand in Time as before

Now apply the split implicit advance to the basic 3D (ideal) MHD equations:

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Taylor Expand in Time as before

Substitute from 2nd and 3rd equation into first, finite difference in time:

$$\left\{\rho - \theta^2 (\delta t)^2 L\right\} \mathbf{V}^{n+1} = \left\{\rho - \theta(\theta - 1)(\delta t)^2 L\right\} \mathbf{V}^n + \delta t \left\{-\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}\right\}^n$$

MHD Operator: ⟶

$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\left\{\nabla \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right]\right\} \times \mathbf{B} + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right] + \nabla\left(\mathbf{V} \bullet \nabla p + \gamma p \nabla \bullet \mathbf{V}\right)$$

Now apply this technique to the basic 3D MHD equations:

$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}[\nabla \times \mathbf{B}] \times \mathbf{B} - \nabla p$$

$$\dot{\mathbf{B}} = \nabla \times [\mathbf{V} \times \mathbf{B}]$$

$$\dot{p} = -\mathbf{V} \bullet \nabla p - \gamma p \nabla \bullet \mathbf{V}$$

Ideal MHD Equations for velocity, magnetic field, and pressure:

Symmetric Hyperbolic System

7-waves

$$\rho_0 \dot{\mathbf{V}} = \frac{1}{\mu_0}\left[\nabla \times \left(\mathbf{B} + \theta \delta t \dot{\mathbf{B}}\right)\right] \times \left(\mathbf{B} + \theta \delta t \dot{\mathbf{B}}\right) - \nabla\left(p + \theta \delta t \dot{p}\right)$$

$$\dot{\mathbf{B}} = \nabla \times \left[\left(\mathbf{V} + \theta \delta t \dot{\mathbf{V}}\right) \times \mathbf{B}\right]$$

$$\dot{p} = -\left(\mathbf{V} + \theta \delta t \dot{\mathbf{V}}\right) \bullet \nabla p - \gamma p \nabla \bullet \left(\mathbf{V} + \theta \delta t \dot{\mathbf{V}}\right)$$

Taylor Expand in Time as before

Substitute from 2nd and 3rd equation into first, finite difference in time:

$$\left\{\rho - \theta^2 (\delta t)^2 L\right\} \mathbf{V}^{n+1} = \left\{\rho - \theta^2 (\delta t)^2 L\right\} \mathbf{V}^{n} + \delta t \left\{-\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}\right\}^{n+1/2}$$

note!

MHD Operator: ⟶

$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\left\{\nabla \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right]\right\} \times \mathbf{B} + \frac{1}{\mu_0}\left(\nabla \times \mathbf{B}\right) \times \left[\nabla \times \left(\mathbf{V} \times \mathbf{B}\right)\right] + \nabla\left(\mathbf{V} \bullet \nabla p + \gamma p \nabla \bullet \mathbf{V}\right)$$

Advantages of

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n+1/2}$$

note!

over:

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta(\theta-1)(\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n}$$

1. Gives correct steady-state physics (when $\mathbf{V}^{n+1}=\mathbf{V}^{n}$)

2. Gives stable numerical method for $\theta \geq 1/2$ even when plasma is unstable (L has negative eigenvalues)

3. Still second order accurate in time (obtained by evaluating p and $\mathbf{B}$ at the half-time level)

4. Effectively introduces a *k*-dependent mass term to provide numerical stability

MHD Operator: ⟶

$$L\{\mathbf{V}\}=\frac{1}{\mu_0}\left\{\nabla\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\right\}\times\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]$$
$$+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)$$

Summary of Split Implicit Time Advance

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n+1/2}$$

only implicit in $\mathbf{V}^{n+1}$!

$$\begin{aligned}p^{n+3/2}=\;&p^{n+1/2}-\delta t\mathbf{V}^{n+1}\bullet\left(\theta\nabla p^{n+3/2}+(1-\theta)\nabla p^{n+1/2}\right)\\&-\gamma\left(\theta p^{n+3/2}+(1-\theta)p^{n+1/2}\right)\delta t\nabla\bullet\mathbf{V}^{n+1}+\delta t S\end{aligned}$$

only implicit in $p^{n+3/2}$!

$$\mathbf{B}^{n+3/2}=\mathbf{B}^{n+1/2}+\delta t\nabla\times\begin{bmatrix}\mathbf{V}^{n+1}\times\left(\theta\mathbf{B}^{n+3/2}+(1-\theta)\mathbf{B}^{n+1/2}\right)\\-\eta\left(\theta\nabla\times\mathbf{B}^{n+3/2}+(1-\theta)\nabla\times\mathbf{B}^{n+1/2}\right)\\-\dfrac{1}{ne}\begin{bmatrix}(1-2\theta)\left(\nabla\times\mathbf{B}^{n+1/2}\right)\times\mathbf{B}^{n+1/2}\\+\theta\left(\nabla\times\mathbf{B}^{n+3/2}\right)\times\mathbf{B}^{n+1/2}+\theta\left(\nabla\times\mathbf{B}^{n+1/2}\right)\times\mathbf{B}^{n+3/2}\end{bmatrix}\\-\dfrac{1}{ne}\nabla\left(\theta p_e^{n+3/2}+(1-\theta)p_e^{n+1/2}\right)\end{bmatrix}$$

only implicit in $\mathbf{B}^{n+3/2}$!

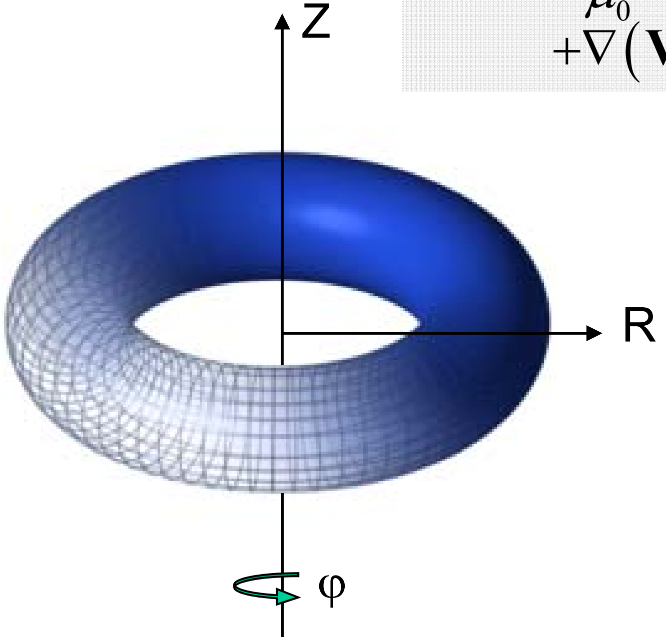
Hall terms

$$\begin{aligned}L\{\mathbf{V}\}=\;&\frac{1}{\mu_0}\left\{\nabla\times\left[\nabla\times(\mathbf{V}\times\mathbf{B})\right]\right\}\times\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\left[\nabla\times(\mathbf{V}\times\mathbf{B})\right]\\&+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)\end{aligned}$$

The Major Challenge in M3D-C^1 is in solving the Implicit Velocity Equation

$$\left\{\rho - \theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1} = \left\{\rho - \theta^2(\delta t)^2 L\right\}\mathbf{V}^{n} + \delta t\left\{-\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n+1/2}$$

$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\left\{\nabla\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\right\}\times\mathbf{B} + \frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right] + \nabla\left(\mathbf{V}\bullet\nabla p + \gamma p\nabla\bullet\mathbf{V}\right)$$



- Need to solve this in 3D torus with strong magnetic field in toroidal direction (φ) … anisotropy
- Wide range of wave speeds with differing polarizations leads to ill-conditioned matrices
- Gradients in (R,Z) plane much larger than in φ direction
- Also need to preserve $\nabla\bullet\mathbf{B} = 0$

Accuracy and Spectral Pollution

Because the externally imposed toroidal field in a tokamak is very strong, any plasma instability will slip through this field and not compress it. We need to be able to model this motion very accurately because of the weak forces causing the instability.

In **M3D-C^1**, we express the velocity and magnetic fields as shown: (R, ϕ, Z)

$$\mathbf{V} = R^2 \nabla U \times \nabla \phi + R^2 \omega \nabla \phi + \frac{1}{R^2} \nabla_\perp \chi$$

$$\mathbf{B} = \nabla \psi \times \nabla \phi - \nabla \frac{\partial f}{\partial \phi} + (F_0 + R^2 \nabla^2 f) \nabla \phi$$

big!

Consider now the action of the first term in $\mathbf{V}$ on the external toroidal field:

$$\mathbf{B} = F_0 \nabla \phi \qquad \mathbf{V} = R^2 \nabla U \times \nabla \phi$$

$$\begin{aligned} \nabla \phi \bullet \left[\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V} \times \mathbf{B} \right) \right] &= \nabla \phi \bullet \nabla \times \left[\left(R^2 \nabla U \times \nabla \phi \right) \times F_0 \nabla \phi \right] \\ &= -F_0 \nabla \bullet \left[\nabla U \times \nabla \phi \right] \\ &= 0 \end{aligned}$$

The velocity field U does not compress the external toroidal field!

The unstable mode will mostly consist of the velocity component U.

The plasma motion will be such as to minimize δW

This is the term associated with the fast wave…it is positive definite with a large multiplier. Any unstable plasma motion will make this term small

$$\nabla\bullet\boldsymbol{\xi}_\perp \cong -2\boldsymbol{\xi}_\perp\bullet\boldsymbol{\kappa} \sim 0$$

$$\delta W = \frac{1}{2}\int d\tau\left[\begin{array}{l}\frac{1}{\mu_0}\mathbf{Q}_\perp^2 + \frac{1}{\mu_0}B^2\left[\nabla\bullet\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\bullet\boldsymbol{\kappa}\right]^2 + \frac{5}{3}p\left|\nabla\bullet\boldsymbol{\xi}\right|^2 \\ -2\left(\boldsymbol{\xi}_\perp\bullet\nabla p\right)\left(\boldsymbol{\kappa}\bullet\boldsymbol{\xi}_\perp\right) - \sigma\boldsymbol{\xi}_\perp\times\mathbf{B}\bullet\mathbf{Q}_\perp\end{array}\right]$$

$\mathbf{b} = \mathbf{B}/B$ unit vector in direction of field

$\boldsymbol{\kappa} = \mathbf{b}\bullet\nabla\mathbf{b}$ curvature of magnetic field

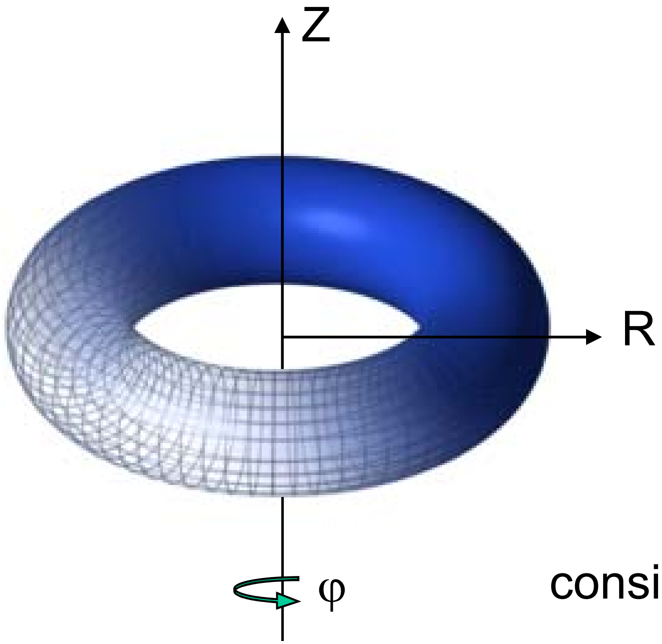
$\sigma = \mathbf{J}\bullet\mathbf{B}/B^2$ parallel current density

$\mathbf{Q} = \nabla\times\left(\boldsymbol{\xi}\times\mathbf{B}\right)$ perturbed magnetic field

$\mathbf{Q}_\perp =$ perpendicular component of $\mathbf{Q}$

$\boldsymbol{\xi}_\perp =$ perpendicular component of $\boldsymbol{\xi}$

$$L\{\mathbf{V}\} = \frac{1}{\mu_0}\{\nabla\times[\nabla\times(\mathbf{V}\times\mathbf{B})]\}\times\mathbf{B} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times[\nabla\times(\mathbf{V}\times\mathbf{B})] + \nabla(\mathbf{V}\bullet\nabla p + \gamma p\nabla\bullet\mathbf{V})$$



This is the ideal MHD operator of Bernstein, Freeman, Kruskal, and Kulsrud (1958)

Define now 2 displacement (velocity) fields:

$$\mathbf{V} = R^2\nabla U\times\nabla\varphi + \omega R^2\nabla\varphi + \frac{1}{R^2}\nabla_\perp\chi$$

$$\tilde{\mathbf{V}} = R^2\nabla\tilde{U}\times\nabla\varphi + \tilde{\omega} R^2\nabla\varphi + \frac{1}{R^2}\nabla_\perp\tilde{\chi}$$

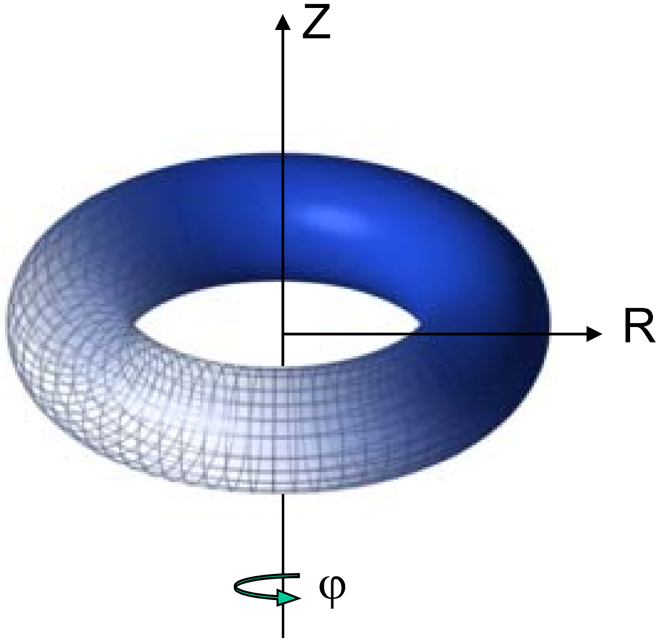
consider the functional:

$$\delta W(\tilde{\mathbf{V}},\mathbf{V}) \equiv \iint d^2R\, \tilde{\mathbf{V}}\bullet L(\mathbf{V})$$

$$\begin{aligned} &= \delta W_{11}(\tilde{U},U) + \delta W_{12}(\tilde{U},\omega) + \delta W_{13}(\tilde{U},\chi) \\ &+ \delta W_{21}(\tilde{\omega},U) + \delta W_{22}(\tilde{\omega},\omega) + \delta W_{23}(\tilde{\omega},\chi) \\ &+ \delta W_{31}(\tilde{\chi},U) + \delta W_{32}(\tilde{\chi},\omega) + \delta W_{33}(\tilde{\chi},\chi) \end{aligned}$$

can be broken up into these 9 parts, each of which is a quadratic functional →

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n+1/2}$$



• To solve this by the finite element method, we need to take projections to get scalar equations, and then to take the weak form of those equations.

$$\mathbf{V}=R^2\nabla U\times\nabla\varphi+\omega R^2\nabla\varphi+\frac{1}{R^2}\nabla_\perp\chi$$

$\nu_i(R,Z)$ is ith finite element trial function

$$-\iint d^2R\,\nu_i\nabla\varphi\bullet\nabla_\perp\times R^2 \qquad \iint d^2R\,R^2\nabla_\perp\nu_i\times\nabla\varphi\bullet$$

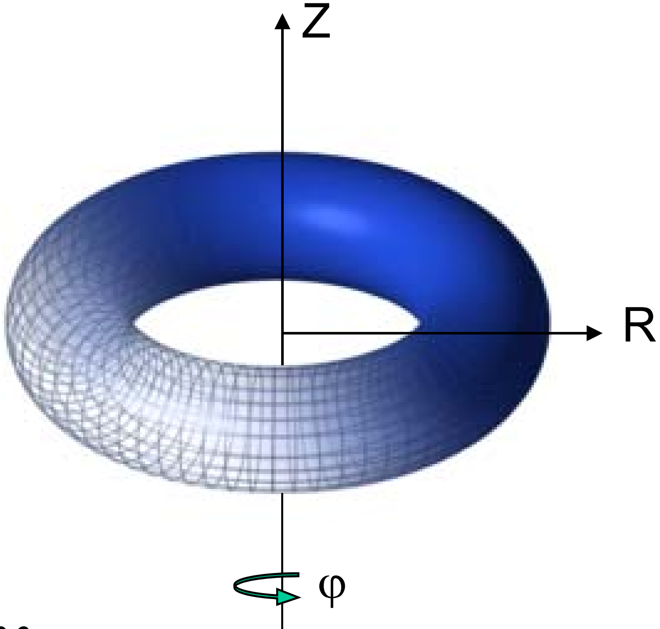
$$\iint d^2R\,\nu_i R^2\nabla\varphi\bullet \quad \xrightarrow{\text{by parts}} \quad \iint d^2R\,\nu_i R^2\nabla\varphi\bullet$$

$$-\iint d^2R\,\nu_i\nabla_\perp\bullet\frac{1}{R^2} \qquad \iint d^2R\,\frac{1}{R^2}\nabla_\perp\nu_i\bullet$$

Projection or annihilation operators:

Same form as velocity!

$$\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n+1}=\left\{\rho-\theta^2(\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}\right\}^{n+1/2}$$



• Consider the effect of these projection operators on the MHD operator

$$L\{\mathbf{V}\}=\frac{1}{\mu_0}\left\{\nabla\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\right\}\times\mathbf{B}$$
$$+\frac{1}{\mu_0}\left(\nabla\times\mathbf{B}\right)\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)$$

$$\mathbf{V}=R^2\nabla U\times\nabla\varphi+\omega R^2\nabla\varphi+\frac{1}{R^2}\nabla_\perp\chi$$

$$\iint d^2R\, R^2\nabla_\perp\nu_i\times\nabla\varphi\bullet L\{\mathbf{V}\} \qquad \delta W_{11}(\nu_i,U)+\delta W_{12}(\nu_i,\omega)+\delta W_{13}(\nu_i,\chi)$$

$$\iint d^2R\, \nu_i R^2\nabla\varphi\bullet L\{\mathbf{V}\} \quad\longrightarrow\quad \delta W_{21}(\nu_i,U)+\delta W_{22}(\nu_i,\omega)+\delta W_{23}(\nu_i,\chi)$$

$$\iint d^2R\, \frac{1}{R^2}\nabla_\perp\nu_i\bullet L\{\mathbf{V}\} \qquad \delta W_{31}(\nu_i,U)+\delta W_{32}(\nu_i,\omega)+\delta W_{33}(\nu_i,\chi)$$

same functions!

these “energy terms” add to mass matrix to make a fully stable implicit system.

The sparse matrix equation to be solved for the velocity variables take the form:

$$\begin{bmatrix} S_{11}^{v} & S_{12}^{v} & S_{13}^{v} \\ S_{21}^{v} & S_{22}^{v} & S_{23}^{v} \\ S_{31}^{v} & S_{32}^{v} & S_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ \omega \\ \chi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{v} & D_{12}^{v} & D_{13}^{v} \\ D_{21}^{v} & D_{22}^{v} & D_{23}^{v} \\ D_{31}^{v} & D_{32}^{v} & D_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ \omega \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{v} & R_{12}^{v} & R_{13}^{v} \\ R_{21}^{v} & R_{22}^{v} & R_{23}^{v} \\ R_{31}^{v} & R_{32}^{v} & R_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ f \\ F \end{bmatrix}^{n+1/2}$$

S^v matrix is self-adjoint!

$$S_{11}^{v} = D_{11}^{v} = \rho\left(\nu_i, U\right) - (\theta\delta t)^2 \delta W_{11}(\nu_i, U)$$

etc.

- Corresponds to projections of the operator equation derived on earlier vg:

$$\left\{\rho - \theta^2(\delta t)^2 L\right\} \mathbf{V}^{n+1} = \left\{\rho - \theta^2(\delta t)^2 L\right\} \mathbf{V}^{n} + \delta t \left\{ -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \right\}^{n+1/2}$$

- Also contains 2 non-trivial sub-systems (reduced MHD) that conserve appropriate “energy” and are numerically stable

$$\left[S_{11}^{v}\right] \bullet [U]^{n+1} = \left[D_{11}^{v}\right] \bullet [U]^{n} + \left[R_{11}^{v}\right] \bullet [\psi]^{n}$$ etc.

First 3 δW_{ij} terms

$$\delta W_{11}(v_i,U) = +\frac{1}{R^2}\left([U,\psi],[\hat{v}_i,\psi]\right) - \frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,[U,\psi]\right] - \frac{2F}{R^2}\left[U,\frac{F}{R^2}\right]\hat{v}_{iZ}$$

$$+\frac{F}{R^4}\left(U',[\hat{v}_i,\psi]\right) - \frac{F}{R^4}\left([U,\psi],\hat{v}_i\right)' - \Delta^*\psi\left(\frac{F}{R^4}[\hat{v}_i,U]\right)' - \frac{F}{R^4}\left(\frac{F}{R^2}(U',\hat{v}_i)\right)'$$

$$\delta W_{12}(v_i,\omega) = -\frac{2F}{R^2}[\omega,\psi]\hat{v}_{iZ} - \frac{\omega'}{R^2}\left(\psi,[\hat{v}_i,\psi]\right) + \frac{\omega'}{R^2}\Delta^*\psi[\hat{v}_i,\psi] + \frac{F}{R^4}\left(\omega'(\psi,\hat{v}_i)\right)'$$

$$\delta W_{13}(v_i,\chi) = -\frac{1}{R^2}\left((\chi,\psi),[\hat{v}_i,\psi]\right) + \frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,(\chi,\psi)\right] + \frac{2F}{R^2}\hat{v}_{iZ}\nabla\bullet\frac{F}{R^2}\nabla_\perp$$

$$-\frac{F}{R^2}\left[[\hat{v}_i,\psi],\chi'\right] + \frac{F}{R^4}\left((\chi,\psi),\hat{v}_i\right)' - \frac{1}{R^2}\Delta^*\psi\left(\frac{F}{R^2}(\chi,\hat{v}_i)\right)' - \frac{F}{R^4}\left(F[\chi,\hat{v}_i]\right)''$$

-present in 2D

-3D only

$$[a,b] \equiv [\nabla a\times\nabla b\bullet\nabla\varphi] = \frac{1}{R}\left(a_Z b_R - a_R b_Z\right)$$

$$(a,b) \equiv \nabla a\bullet\nabla b = a_R b_R + a_Z b_Z$$

$$f' \equiv \partial f/\partial\varphi$$

note: at most second order derivatives on each scalar compatible with C^1 elements →

Magnetic Field

$$\mathbf{A} = R^2 \nabla \varphi \times \nabla f + \psi \nabla \varphi - F_0 \ln R \hat{z}$$

$$\nabla_\perp \bullet \frac{1}{R^2} \mathbf{A} = 0 \quad \text{(gauge condition)}$$

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \nabla \psi \times \nabla \varphi - \nabla_\perp f' + F \nabla \varphi \\ &= \nabla \psi \times \nabla \varphi - \nabla f' + F^* \nabla \varphi \end{aligned}$$

$$F \equiv F_0 + R^2 \nabla \bullet \nabla_\perp f$$

$$F^* \equiv F_0 + R^2 \nabla^2 f = F + f''$$

$$f' \equiv \partial f / \partial \varphi$$

$$\begin{aligned} \mathbf{J} &\equiv \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} \\ &= \nabla F^* \times \nabla \varphi + \frac{1}{R^2} \nabla_\perp \psi' - \Delta^* \psi \nabla \varphi \end{aligned}$$

2 scalar variables and a gauge condition

Magnetic Field Advance Equations

$$\mathbf{A} = R^2 \nabla\varphi \times \nabla f + \psi \nabla\varphi - F_0 \ln R \hat{z}$$

$$\mathbf{B} = \nabla\psi \times \nabla\varphi - \nabla_\perp f' + F \nabla\varphi \qquad F = F_0 + R^2 \nabla_\perp^2 f$$

(1) $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\mathbf{V} \times \mathbf{B} - \eta \mathbf{J} + \cdots \right]$$

$$\iint d^2R\, \nu_i \nabla\varphi \bullet \nabla_\perp \times (1) \quad \rightarrow \quad \iint d^2R\, \nabla_\perp \nu_i \times \nabla\varphi \bullet (1)$$

$$\iint d^2R\, \nu_i \nabla\varphi \bullet (1) \quad \rightarrow \quad \iint d^2R\, \nu_i \nabla\varphi \bullet (1)$$

$$-\iint d^2R\, \nu_i \nabla \bullet (1) \quad \rightarrow \quad \iint d^2R\, \nabla_\perp \nu_i \bullet (1)$$

Not needed!

Energy Conservation:

$$\mathbf{V}\bullet\left[\frac{\partial \mathbf{V}}{\partial t} = (\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p\right] = \mathbf{V}\bullet\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] - \mathbf{V}\bullet\nabla p$$

$$\mathbf{B}\bullet\left[\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B})\right] = -\mathbf{V}\bullet\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] + \nabla\bullet\left[(\mathbf{V}\times\mathbf{B})\times\mathbf{B}\right]$$

$$\frac{1}{\gamma-1}\frac{\partial p}{\partial t} = -\frac{1}{\gamma-1}\left[\mathbf{V}\bullet\nabla p + \gamma p\nabla\bullet\mathbf{V}\right] = -\frac{\gamma}{\gamma-1}\nabla\bullet(p\mathbf{V}) + \mathbf{V}\bullet\nabla p$$

For energy conservation, the like colored terms must cancel exactly. Since this only requires that the projections we take of the momentum equation are equivalent to the dot product with the velocity, we will have energy conservation for each of the 3 velocity fields:

$$\mathbf{V} = R^2\nabla U\times\nabla\varphi$$

$$\mathbf{V} = R^2\nabla U\times\nabla\varphi + \omega R^2\nabla\varphi$$

Reduced MHD

$$\mathbf{V} = R^2\nabla U\times\nabla\varphi + \omega R^2\nabla\varphi + \frac{1}{R^2}\nabla_\perp\chi$$

Full MHD

2-Variable 3D Toroidal subset of full equations….or, (1,1) component

$$\mathbf{A} = \psi \nabla \varphi - F_0 \ln R \hat{Z}$$

$$\mathbf{V} = R^2 \nabla U \times \nabla \varphi$$

$$\frac{1}{R^2}\Delta^* \dot{U} = -\left[\frac{\Delta^* \dot{U}}{R^2}, U\right] + \left[\frac{\Delta^* \psi}{R^2}, \psi\right] + \frac{F_0}{R^4}\Delta^* \psi' + \frac{1}{R^2}\left(\frac{F_0}{R^2}, \psi'\right) + \left[\frac{1}{R^2}\Delta^*\left(\mu \Delta^* U\right) + \nabla \bullet \frac{1}{R^4}\mu \nabla U''\right]$$

$$\frac{1}{R^2}\Delta^* \dot{\psi} = -\frac{1}{R^2}\Delta^*[\psi, U] + \nabla \bullet \frac{F_0}{R^4}\nabla_\perp U' + \nabla \bullet \frac{1}{R^2}\left[\nabla(\eta \Delta^* \psi) + \eta \frac{1}{R^2}\nabla_\perp \psi''\right]$$

These reduced equations have an energy conservation theorem, and the ideal terms have an associated energy principle and variational form

Weak form for the velocity equation:

$$-\frac{1}{R^2}\left(v_i, \dot{U}\right) - (\theta \delta t)^2 \delta W^{2D}\left(\dot{U}, v_i\right) = \frac{\Delta^* U}{R^2}[v_i, U] - \frac{\Delta^* \psi}{R^2}[v_i, \psi] - \frac{F_0}{R^4}(v_i, \psi') + \mu\left[\frac{\Delta^* U}{R^2}\Delta^* v_i - \frac{1}{R^4}\nabla v_i \bullet \nabla U''\right]$$

$$\delta W^{2D}(U,V) = \iint R dR dZ \left\{ \begin{array}{l} \frac{1}{R^2}([U,\psi],[V,\psi]) + \frac{F_0}{R^4}(U',[V,\psi]) - \frac{F_0}{R^4}(V,[U',\psi]) \\ -\frac{F_0^2}{R^6}(U'',V) - \frac{\Delta^* \psi}{R^2}[V,[U,\psi]] - \Delta^* \psi \frac{F_0}{R^4}[V,U'] \end{array} \right\}$$

4-Variable 3D Toroidal subset of full equations….or, 2x2 submatrix

$$\mathbf{A} = R^2 \nabla \varphi \times \nabla f + \psi \nabla \varphi - F_0 \ln R \hat{z}$$
$$\mathbf{V} = R^2 \nabla U \times \nabla \varphi + R^2 \omega \nabla \varphi$$

$$\Delta^* \dot{U} = -R^2 \left[\frac{\Delta^* U}{R^2}, U \right] - (\omega, U') - \omega \Delta^* U' - R^2 \frac{\partial}{\partial z}(\omega^2) + R^2 \left[\frac{\Delta^* \psi}{R^2}, \psi \right] - (\Delta^* \psi) \Delta^* f' - (\Delta^* \psi, f')$$
$$+ \frac{F}{R^2} \Delta^* \psi' + \left(\frac{F}{R^2}, \psi' \right) + R^2 \left[\frac{F}{R^2}, f'' \right] + \frac{\partial}{\partial z} \left[\left(\frac{F}{R} \right)^2 \right] + \left[\Delta^* \mu (\Delta^* U) + R^2 \nabla \bullet \frac{\mu}{R^4} \nabla U'' \right]$$

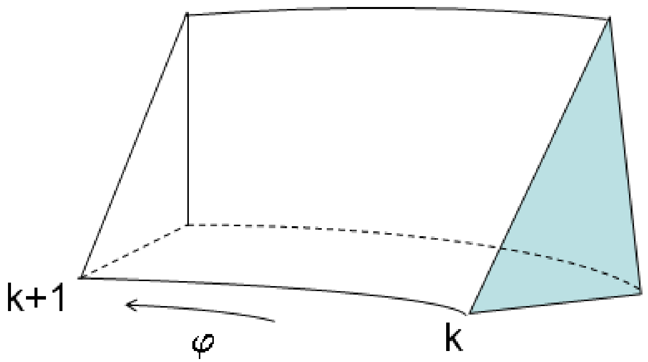
$$R^2 \dot{\omega} = -\left[\omega R^2, U \right] - \omega \omega' R^2 - \frac{1}{R^2}(\psi, \psi') + \left[F^*, \psi \right] + [f', \psi'] - (f', F^*) - p' + \mu \Delta^* R^2 \omega + 2 \mu_C \omega''$$

$$\nabla \bullet \frac{1}{R^2} \nabla_\perp \dot{\psi} = \nabla \bullet \frac{1}{R^2} \nabla_\perp \begin{bmatrix} -[\psi, U] - (U, f') + \eta \Delta^* \psi + [\psi, F^*] \\ +(F^*, f') + \frac{1}{R^2}(\psi, \psi') + [\psi', f'] \end{bmatrix} + \nabla \bullet \frac{1}{R^2} \begin{bmatrix} \frac{F}{R^2} \nabla_\perp U - \omega \nabla_\perp \psi - \omega R^2 \nabla_\perp f' \times \nabla \varphi + \eta \frac{1}{R^2} \nabla_\perp \psi' + \eta \nabla_\perp F^* \times \nabla \varphi \\ -\frac{1}{R^2} \Delta^* \psi \nabla_\perp \psi - \frac{F}{R^2} \nabla_\perp F^* + \left[\frac{F}{R^2} \nabla_\perp \psi' - \Delta^* \psi \nabla_\perp f' \right] \times \nabla \varphi \end{bmatrix}'$$

$$\dot{F} = R^2 \nabla_\perp^2 \dot{f} = R^2 \nabla_\perp \bullet \begin{bmatrix} \frac{F}{R^2} \nabla \varphi \times \nabla_\perp U - \omega \nabla \varphi \times \nabla_\perp \psi - \omega \nabla_\perp f' + \eta \frac{1}{R^2} \nabla_\perp F^* + \frac{\eta}{R^2} \nabla \varphi \times \nabla_\perp \psi' \\ -\frac{1}{R^2} \Delta^* \psi \nabla \varphi \times \nabla \psi - \frac{F}{R^2} \nabla \varphi \times \nabla_\perp F^* + \frac{F}{R^4} \nabla_\perp \psi' - \frac{1}{R^2} \Delta^* \psi \nabla_\perp f' \end{bmatrix}$$

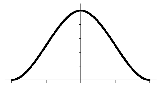
These equations also have an energy theorem and associated energy principle.

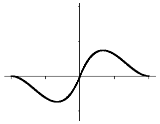
3D C^1 elements by combining Q_{18} triangles in (R,Z) Hermite Cubic representation in the toroidal angle ϕ



Each toroidal plane has two Hermite cubic functions associated with it

$$\Phi_1(x) = (|x|-1)^2 (2|x|+1); \qquad \Phi_2(x) = x(|x|-1)^2$$





Solution for each scalar function is represented in each triangular wedge as the product of Q_{18} and Hermite functions

$$U(R,Z,\varphi) = \sum_{j=1}^{18} \nu_j(R,Z)\Big[U^1_{j,k}\,\Phi_1(\varphi/h) + U^2_{j,k}\Phi_2(\varphi/h) + U^1_{j,k+1}\Phi_1(\varphi/h-1) + U^2_{j,k+1}\Phi_2(\varphi/h-1)\Big]$$

All DOF are still located at nodes: => very efficient representation

3D Nonlinear Solver Strategy

- In 2D, solve efficiently with direct solver up to $(200)^2$ nodes
- In 3D, leads to block triangular structure

$$
\begin{bmatrix}
\mathbf{B}_1 & \mathbf{C}_1 & & & & & & & \mathbf{A}_1 \\
\bullet & \bullet & \bullet & & & & & & \\
 & \bullet & \bullet & \bullet & & & & & \\
 & & \mathbf{A}_j & \mathbf{B}_j & \mathbf{C}_j & & & & \\
 & & & \bullet & \bullet & \bullet & & & \\
 & & & & \bullet & \bullet & \bullet & & \\
 & & & & & \bullet & \bullet & \bullet & \\
\mathbf{C}_N & & & & & & & \mathbf{A}_N & \mathbf{B}_N
\end{bmatrix}
\bullet
\begin{bmatrix}
x_1 \\ . \\ x_{j-1} \\ x_j \\ x_{j+1} \\ . \\ . \\ x_N
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ . \\ y_{j-1} \\ y_j \\ y_{j+1} \\ . \\ . \\ y_N
\end{bmatrix}
$$

$\mathbf{A}_j, \mathbf{B}_j, \mathbf{C}_j$

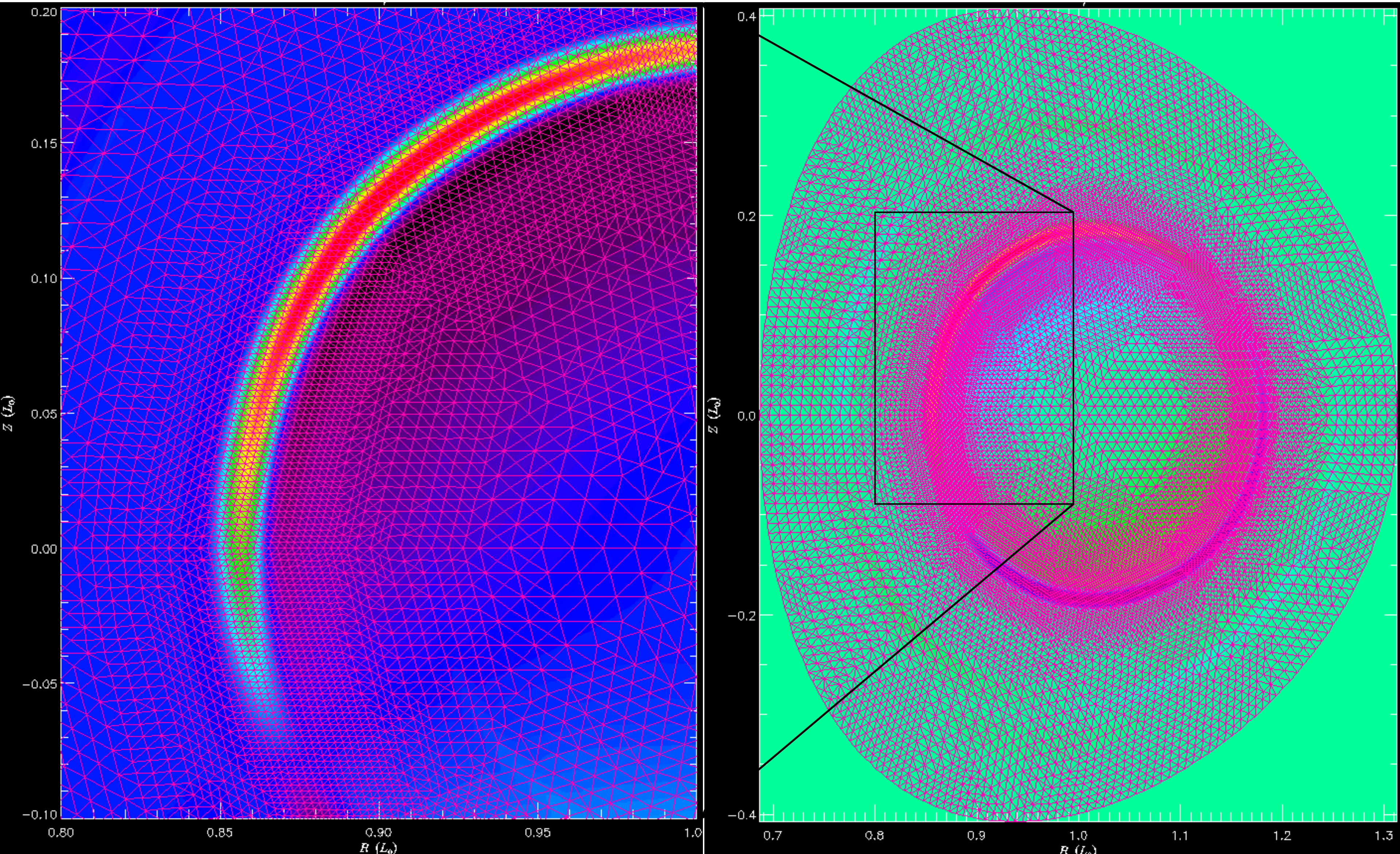
are 2D sparse matrices at plane j

Block Jacobi preconditioner corresponds to multiplying each row by $\mathbf{B}_j^{-1}$

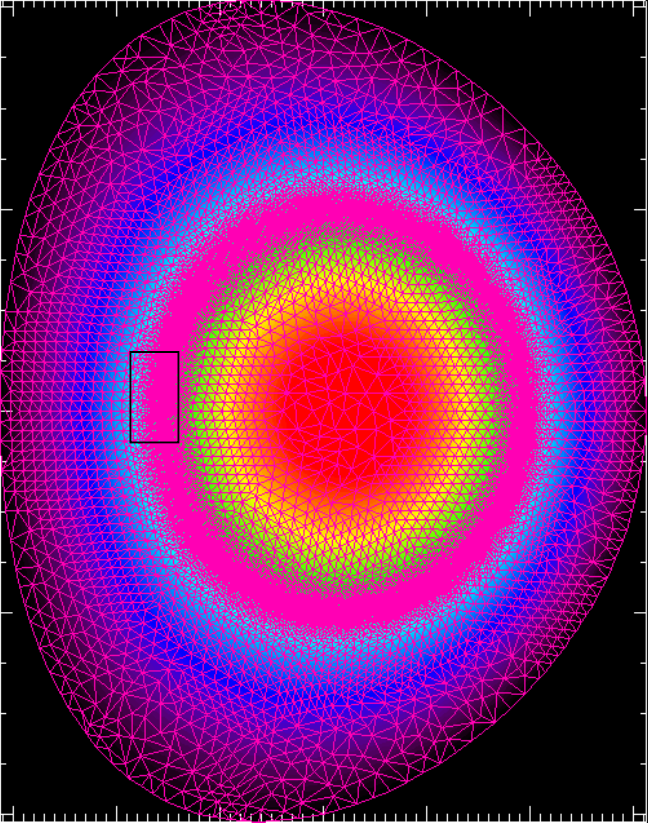
PETSc has the capability of doing this using SuperLU_Dist

N=1 Resistive Internal Kink mode in CMOD with $S=10^7$

Close-up **Perturbed Current with Mesh**

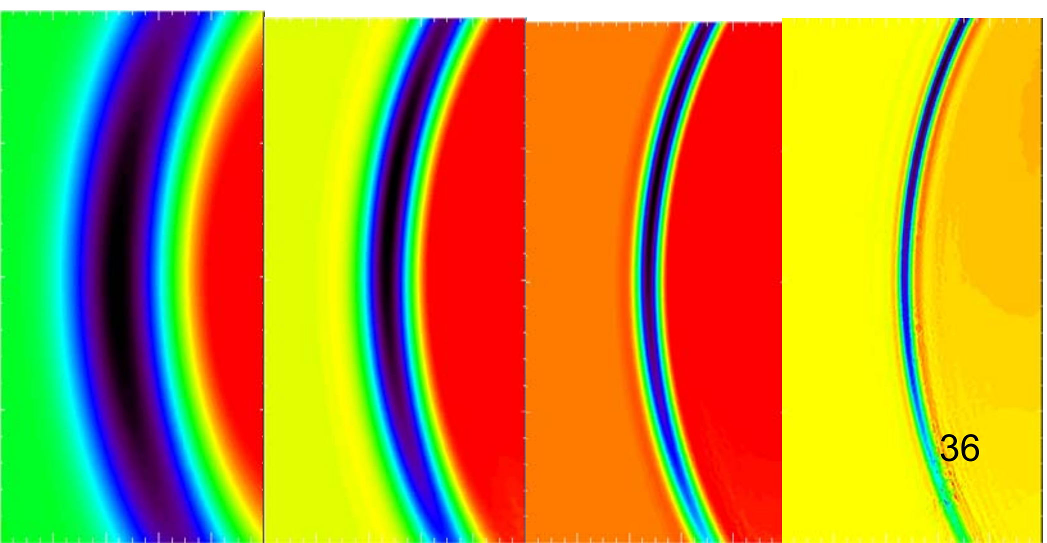


High-S tearing mode studies



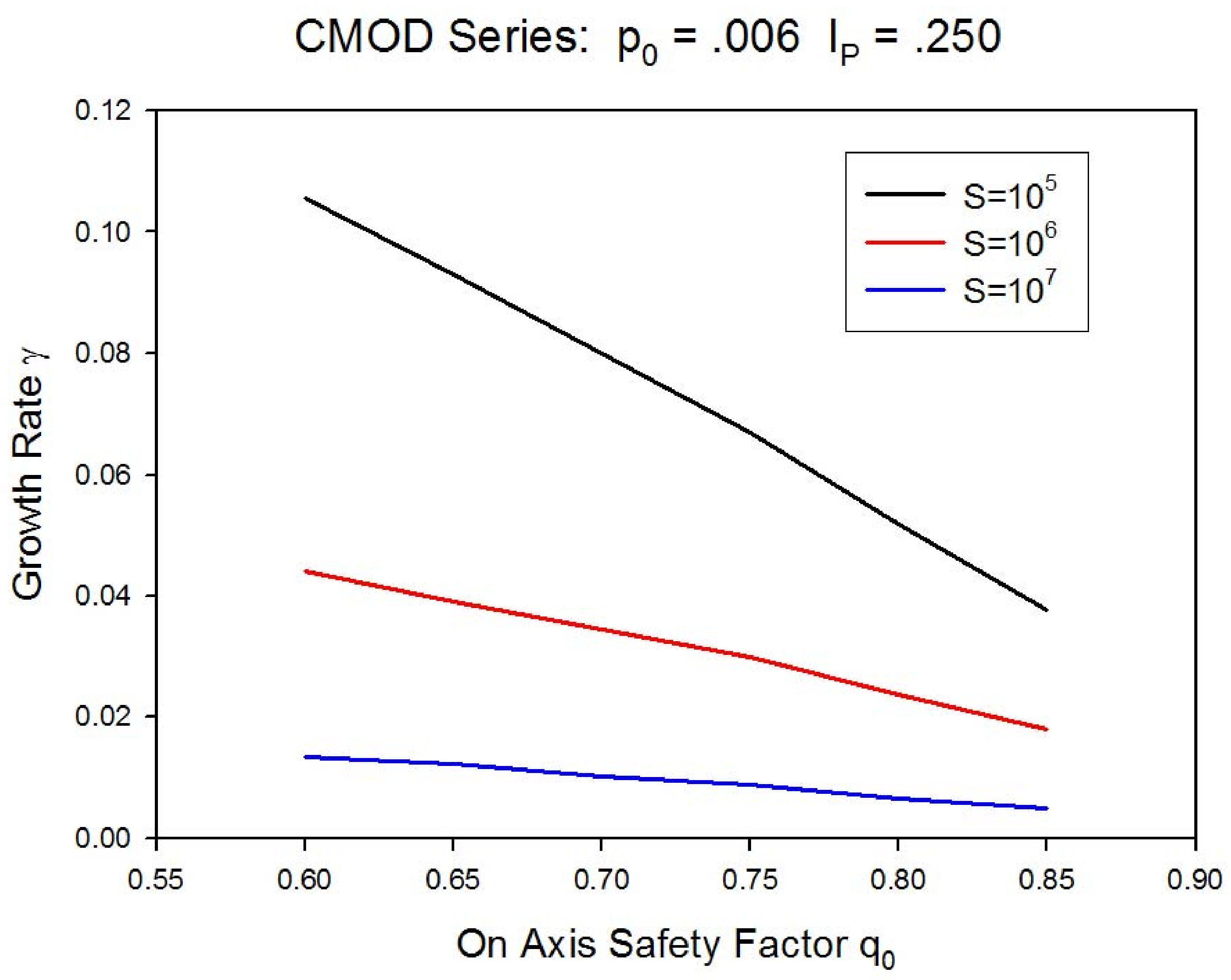
- M3D-C^1, is now being used for linear physics studies in NSTX, CMOD and ITER
- high order C^1 finite elements, adaptive mesh, and fully implicit time advance allow high resolution studies of localized modes
- Now being used to study tearing (and double tearing) modes at realistic S values, including pressure (Glasser) stabilization

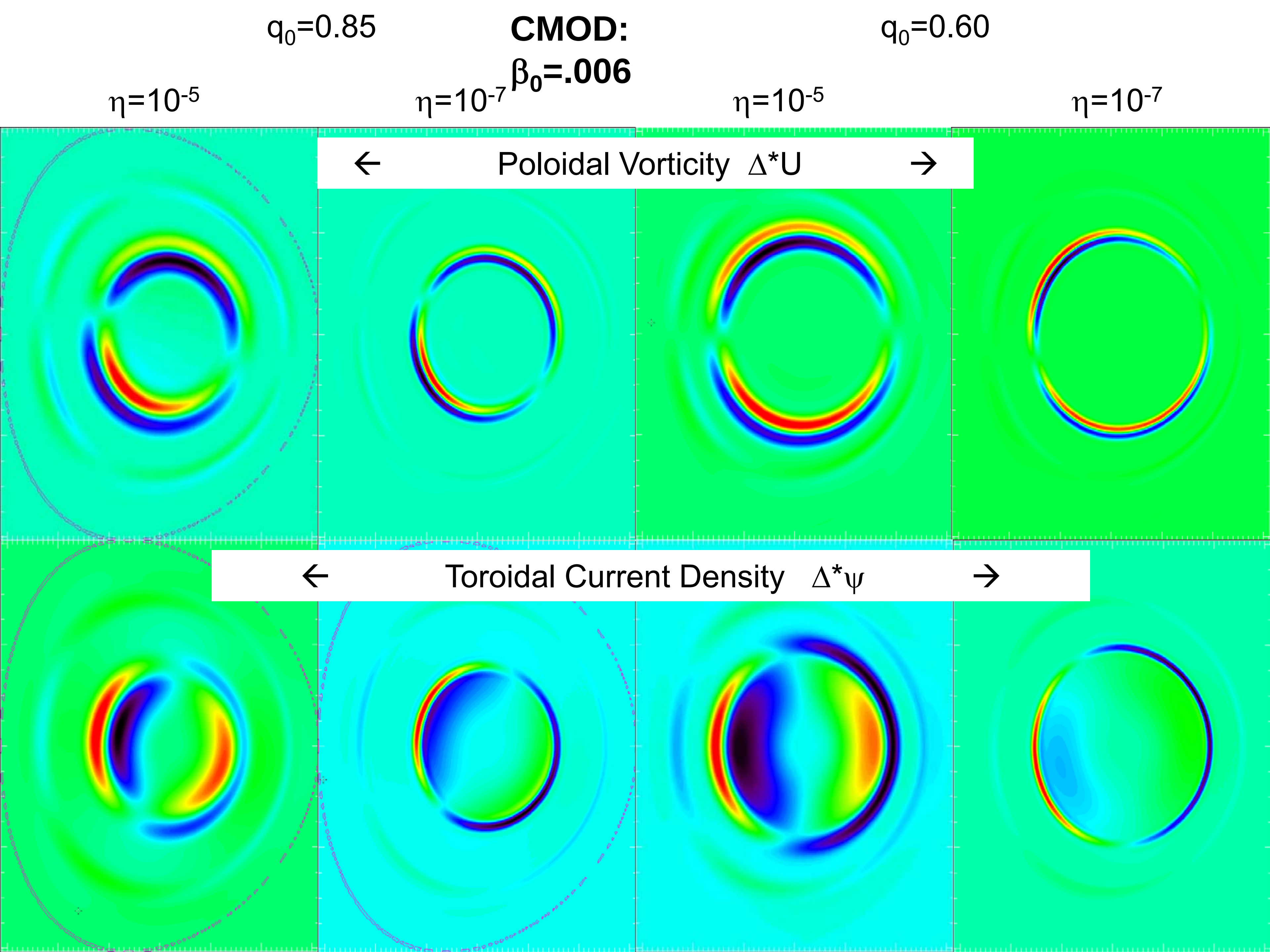
S=10^5 S=10^6 S=10^7 S=10^8



(Top) Equilibrium current density with adaptive mesh superimposed.

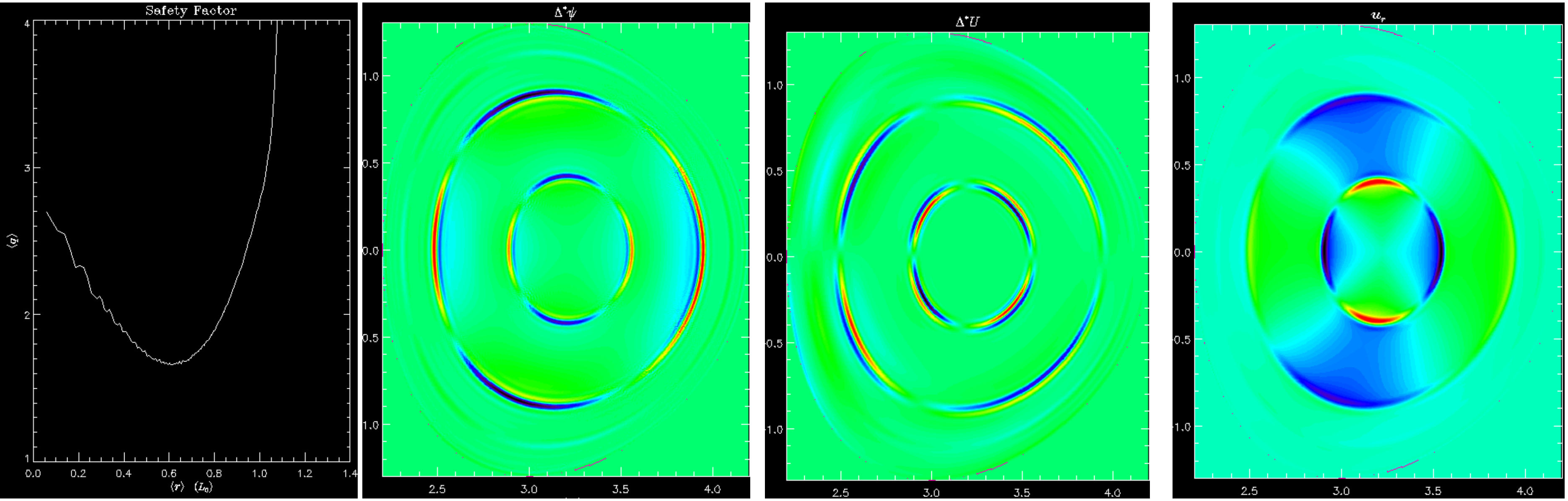
(Left) perturbed current density for (1,1) tearing mode at different S. Rightmost figure corresponds to NSTX parameters


CMOD Series: p0 = .006 IP = .250
Growth Rate γ
On Axis Safety Factor q0
S=10^5
S=10^6
S=10^7
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90

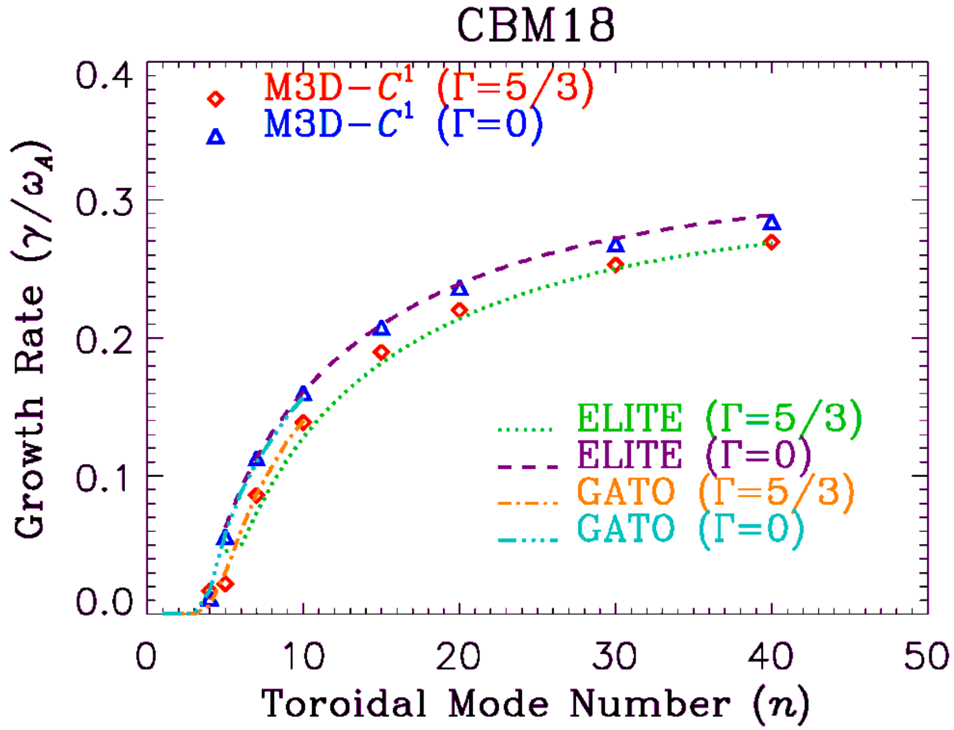

q0=0.85
CMOD:
β0=.006
q0=0.60
η=10-5
η=10-7
η=10-5
η=10-7
← Poloidal Vorticity Δ*U →
← Toroidal Current Density Δ*ψ →

n=1 Double Tearing Mode in NSTX….S = 10^8

q-profile | Toroidal current | Vorticity | Normal displacement



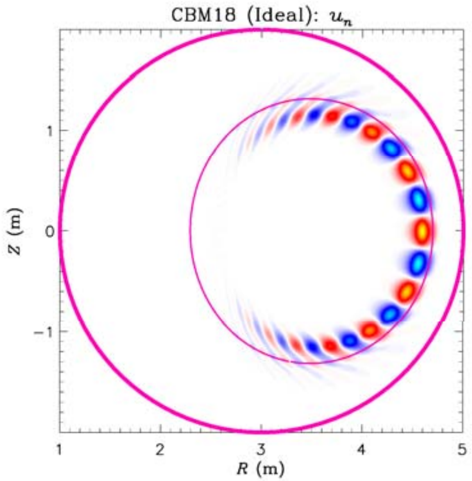
Linear ELM[1]s: Code Verification



[1]Edge Localized Modes

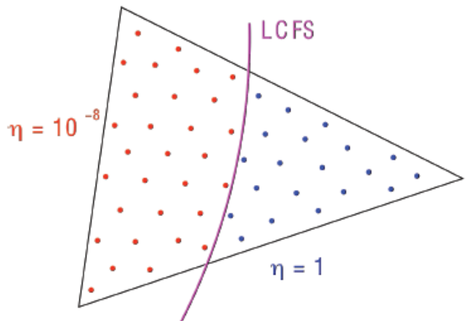
ideal limit:

plasma resistivity → 0.
vacuum region resistivity → ∞,
vacuum region density → 0



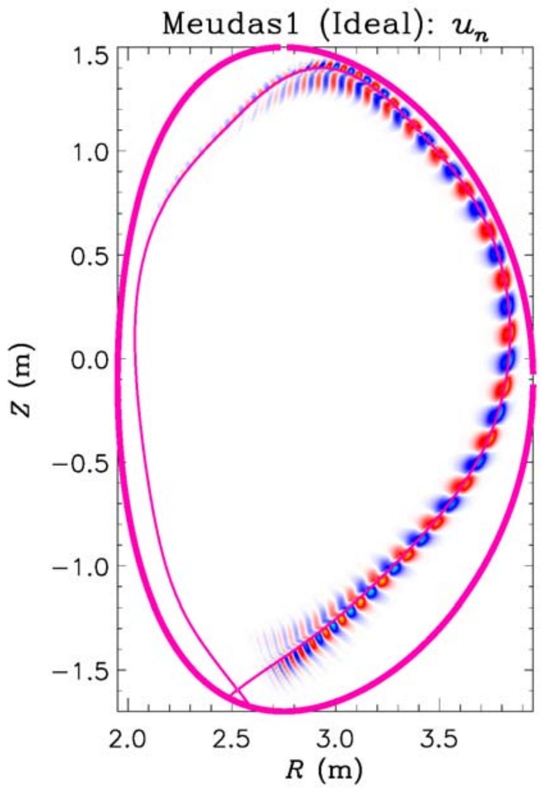
We have performed detailed benchmarking for ELM unstable equilibrium in the **ideal limit** between **M3D-*C*[1]** and GATO and ELITE up to n=40

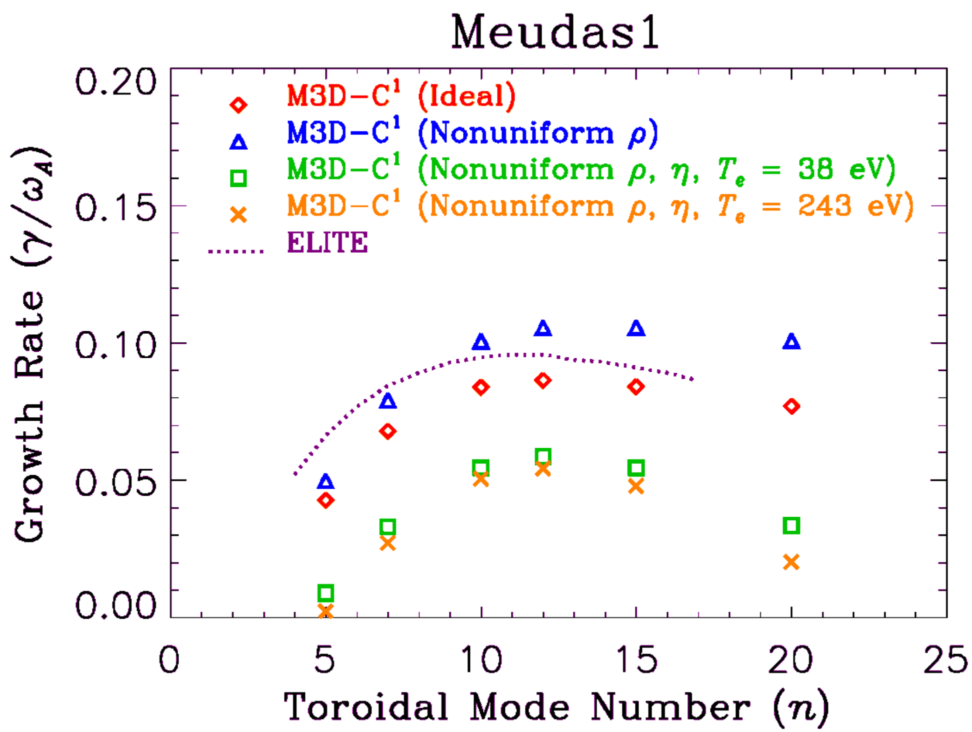
- required discontinuous η and ρ profiles with jump of 10^8

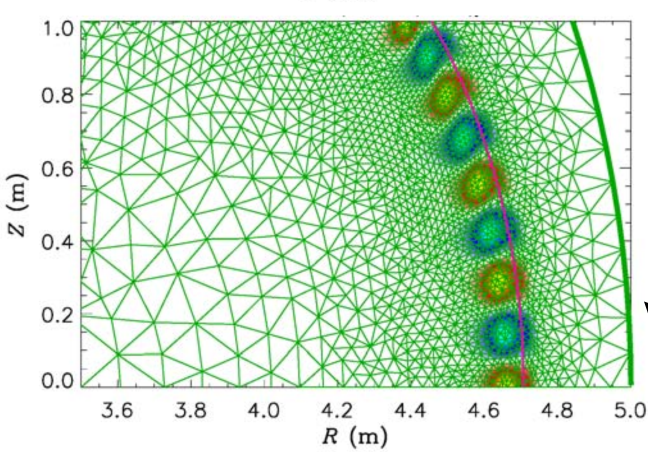


Ferraro

Linear ELMs: Code Verification-2







Close-up showing M3D-*C¹* triangular adaptive mesh

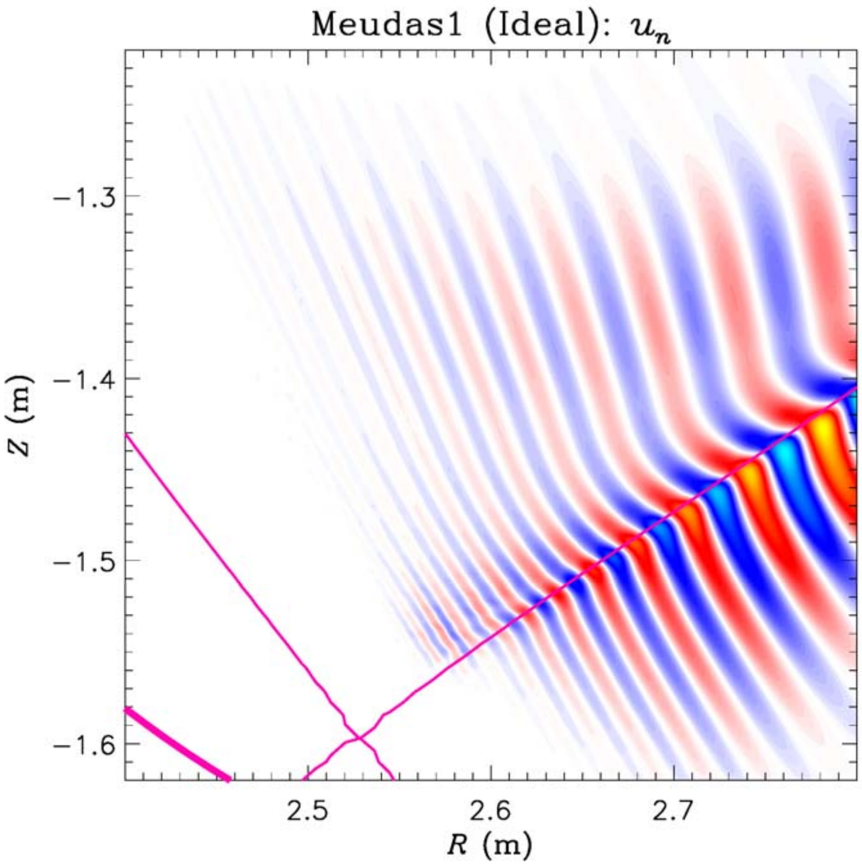
Studies have been extended to:

- diverted equilibrium (JT-60)
- finite resistivity in the plasma and SOL
- realistic density profiles

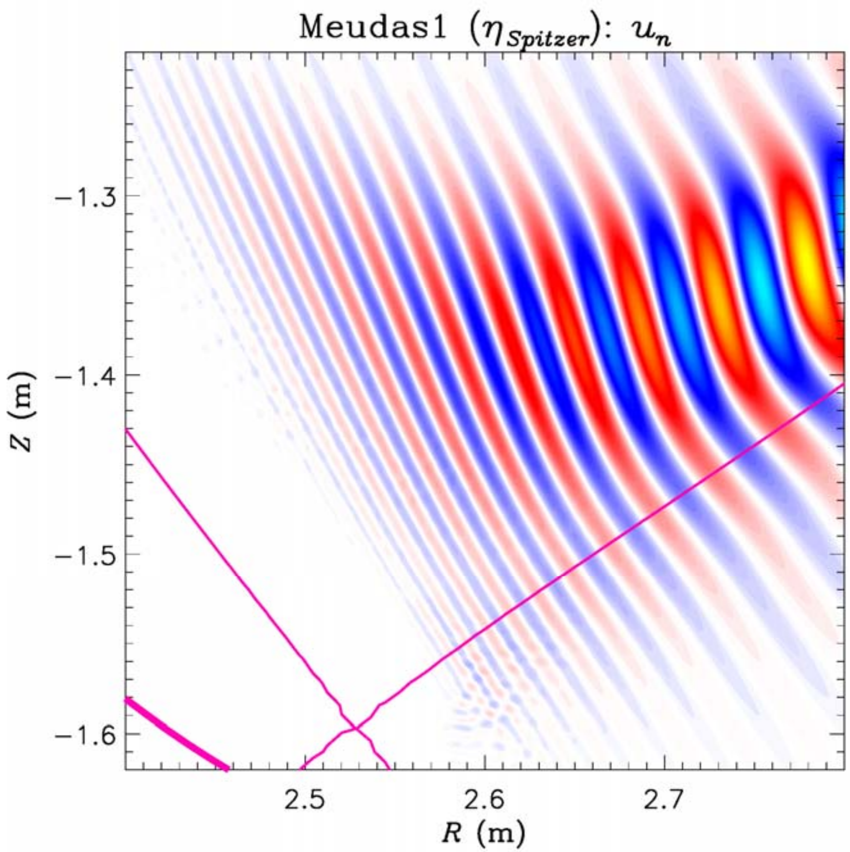
Ferraro

Linear ELMs: Code Verification-3

Comparison of eigenfunctions of normal plasma displacement for "ideal limit" and more realistic Spitzer resistivity with SOL with M3D-*C*[1]



Ideal MHD limit

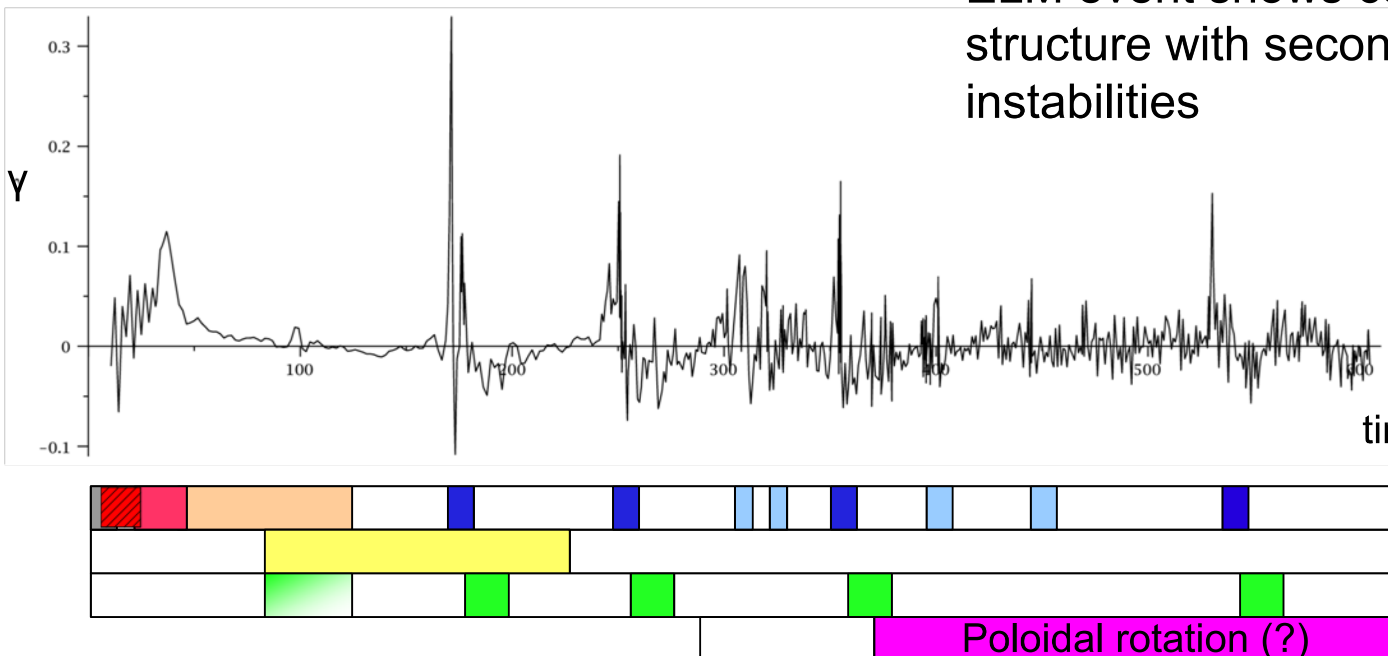


Sptizer resistivity with SOL

Non-linear ELM simulation with M3D

Multi-stage ELM – DIII-D 119690

Full simulation of nonlinear ELM event shows complex structure with secondary instabilities

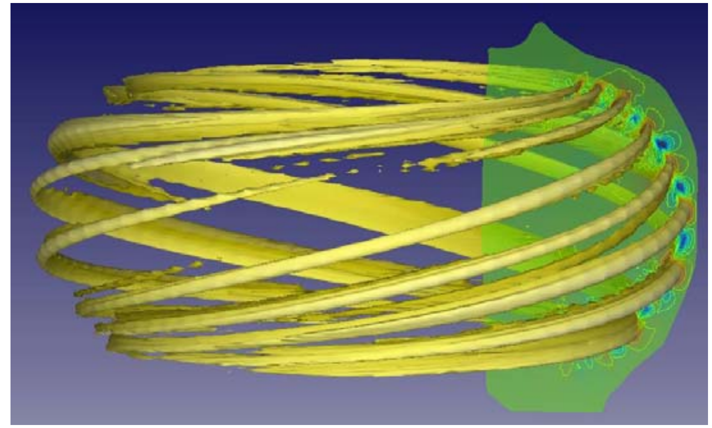


- Linear mode growth and mode consolidation
- Plasma burst outboard, midplane n reduced
- Density to outboard divertor
- Inboard edge instability
- Density to inboard divertor

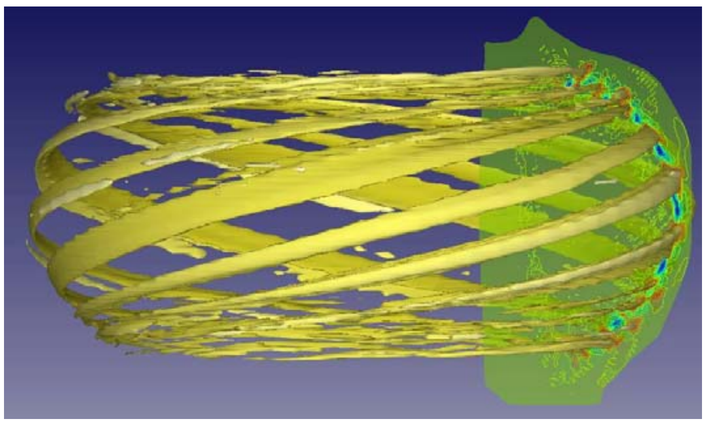
First 50 τ_A: linear mode growth
Nonlinear harmonic consolidation

Initially many unstable linear modes.
These rapidly consolidate into lower-n field-aligned mode ``filaments''
(n=6-10 at t=43)

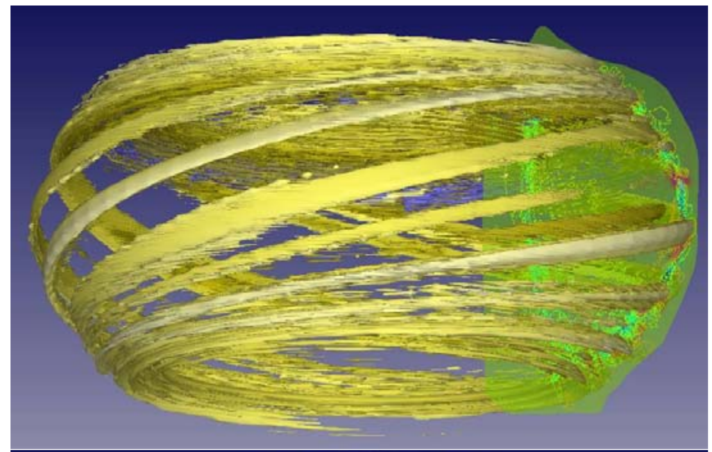
Similar to what is seen experimentally.



ψ-pert

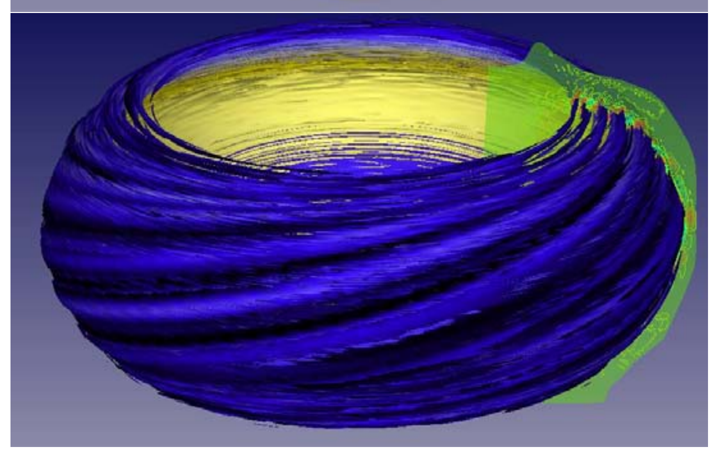


n
pert



u

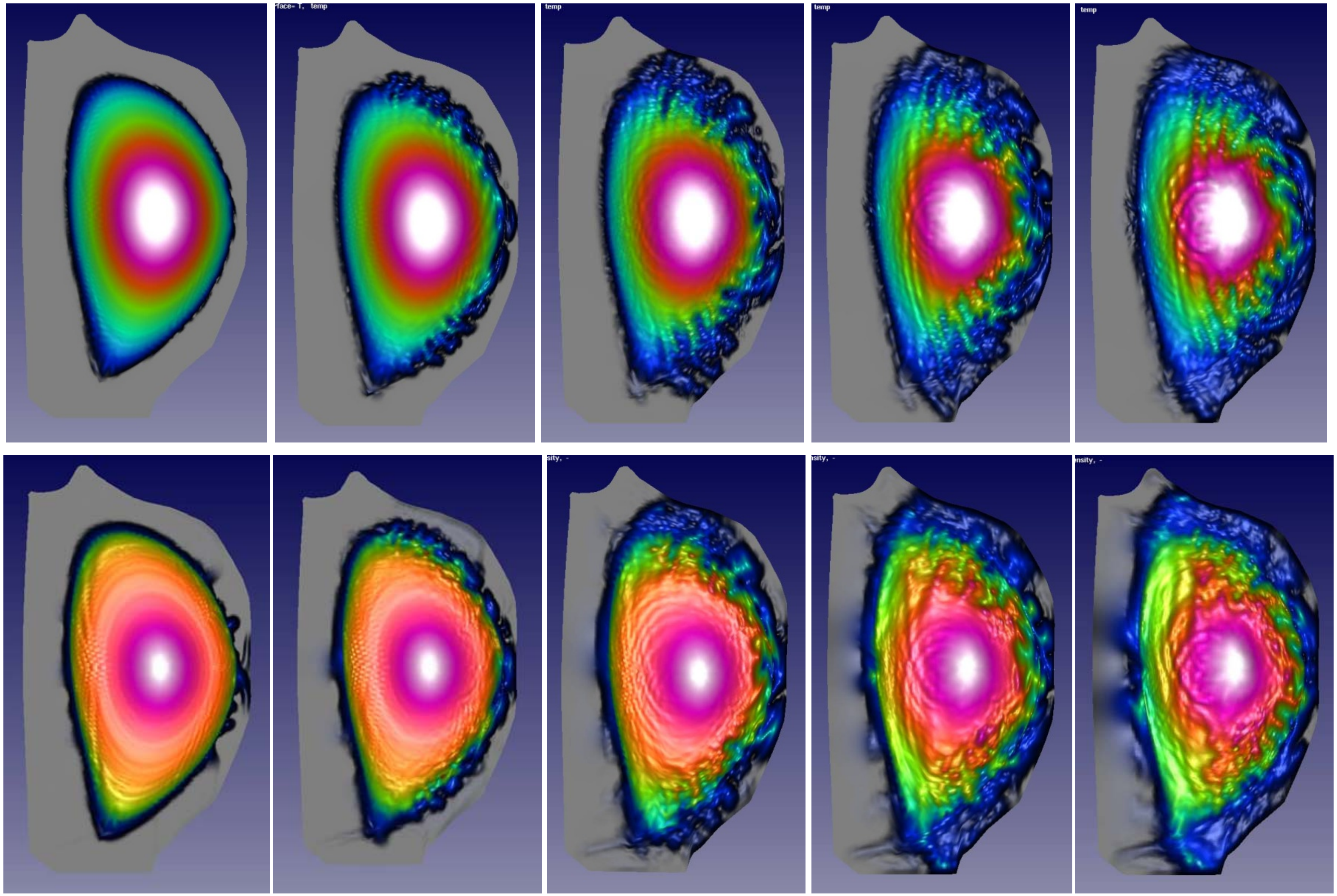
n



RJ_φ

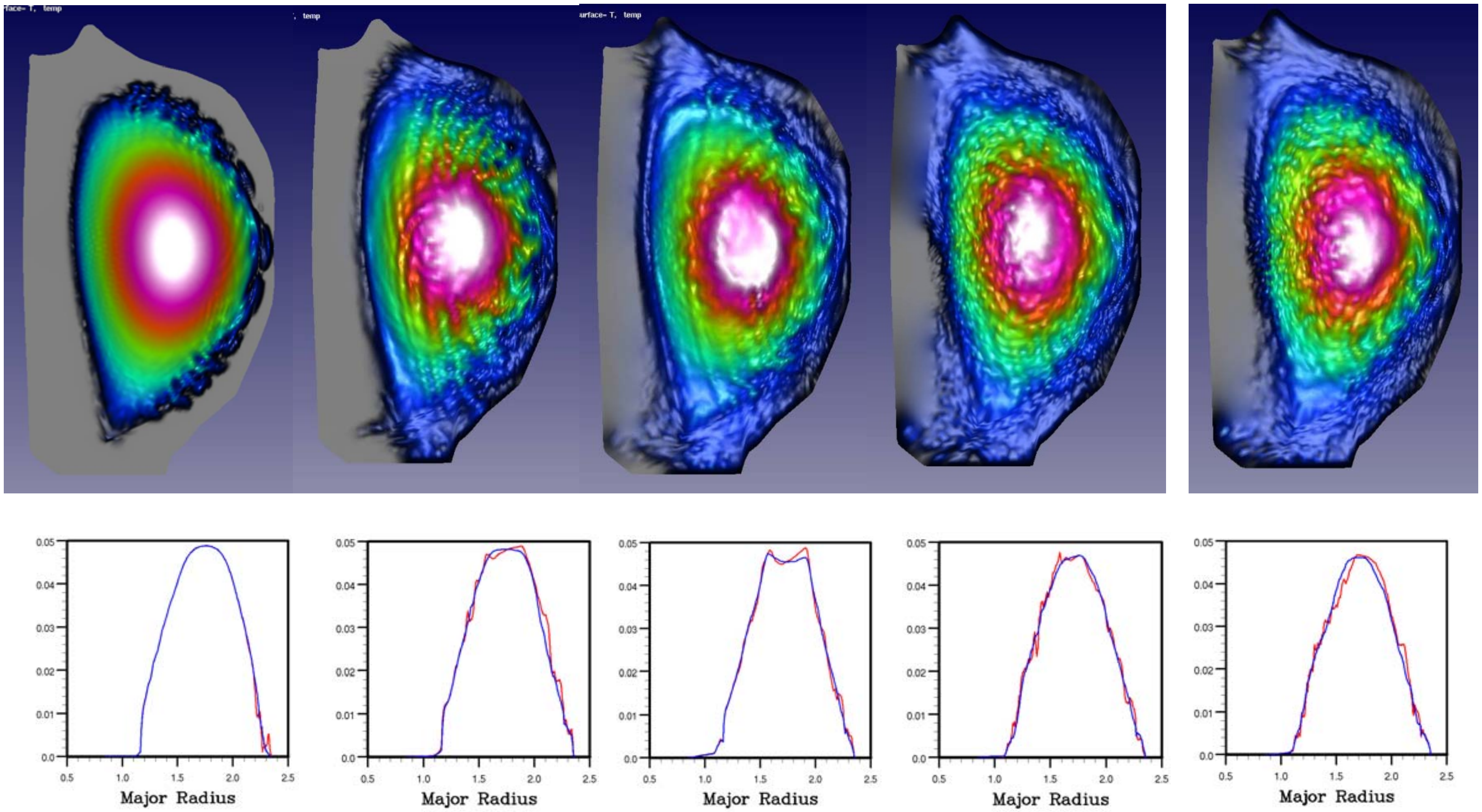
Sugiyama

Early time: T and n ballooning in rapid burst



Sugiyama

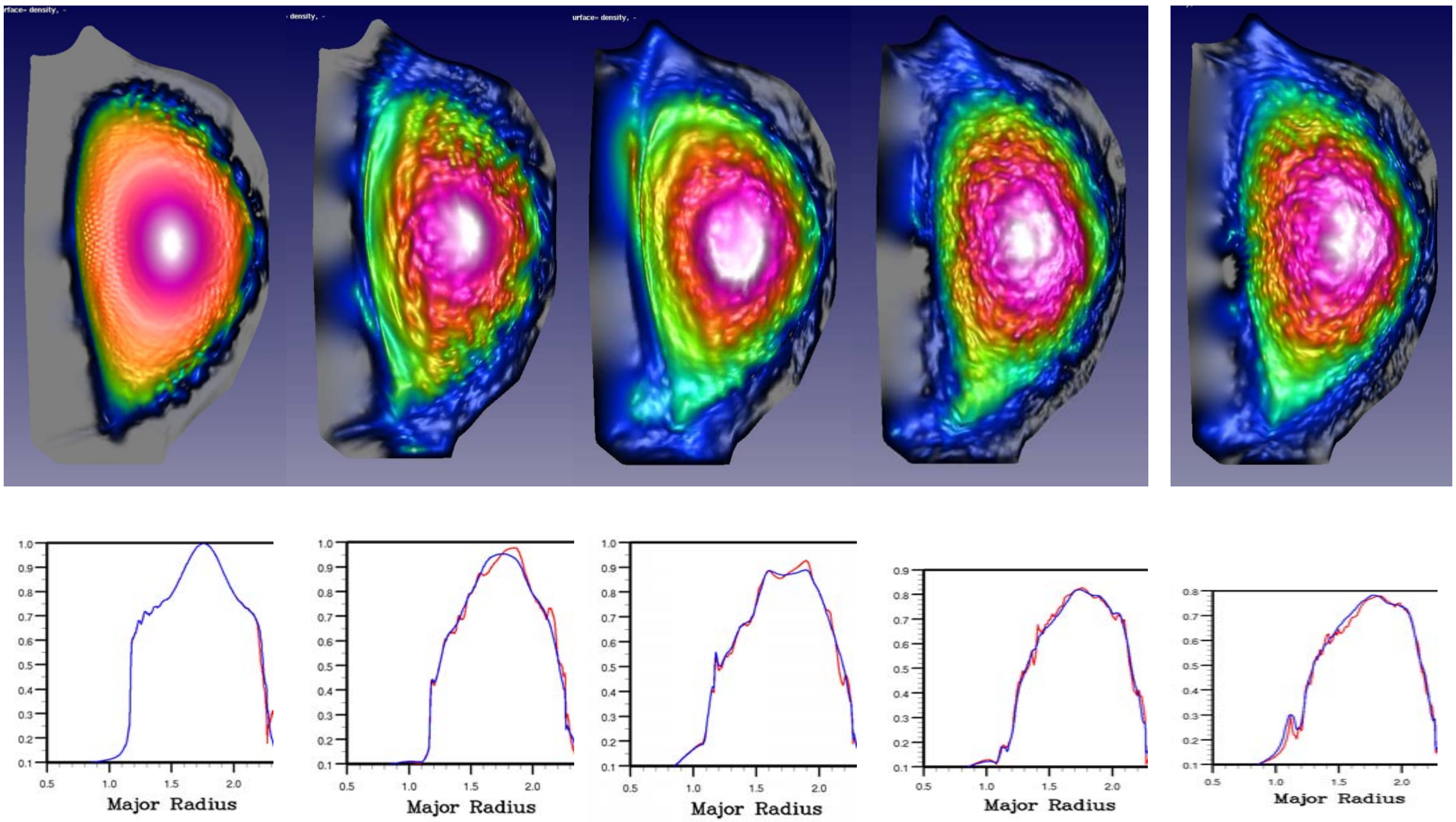
Longer time: T



t=43 126 227 461 529

Sugiyama

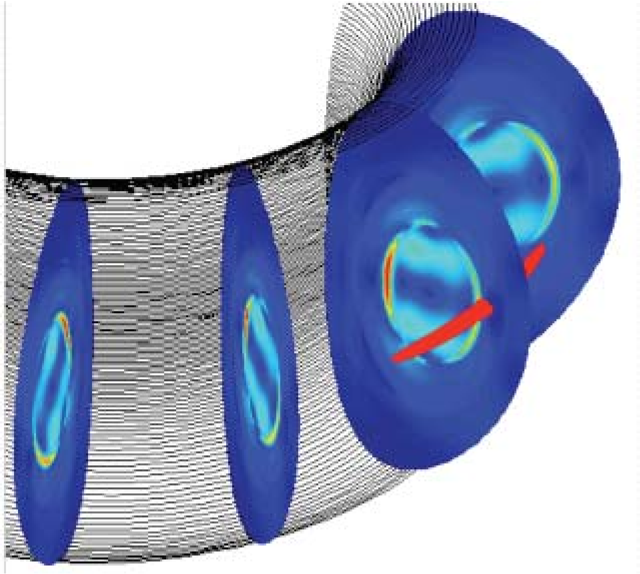
Longer time: n

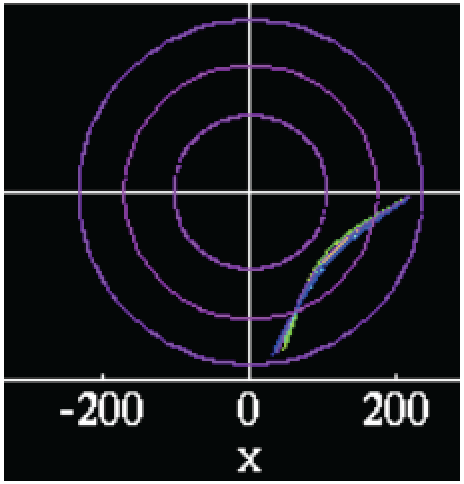


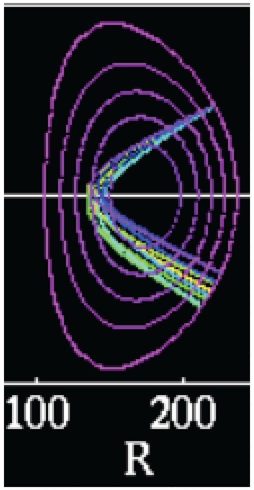
t=43 126 227 461 529

Sugiyama

ECCD Stabilization of NTM





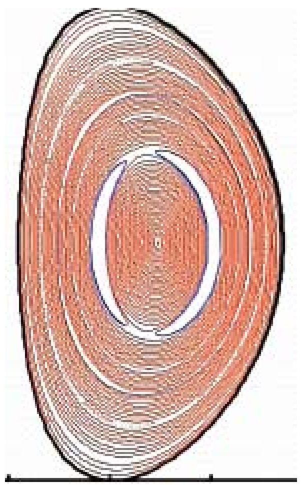


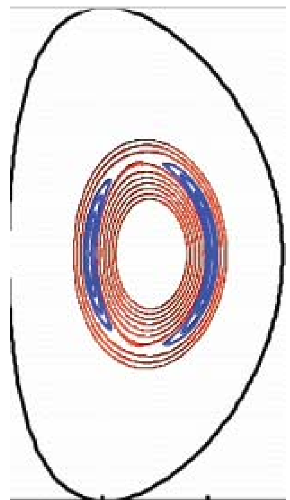
NIMROD code calculates the MHD growth of NTM c

GENRAY code computes wave induced ECCD current drive term

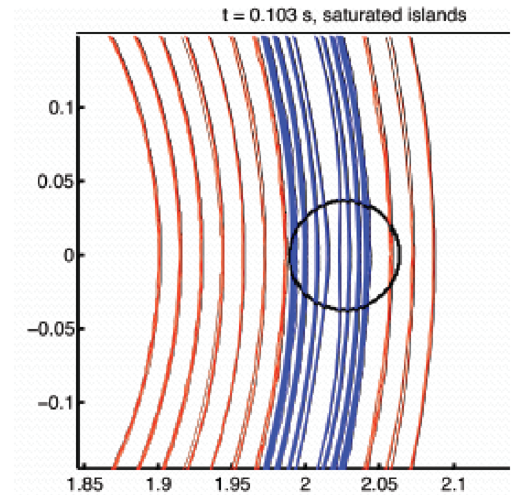
Code coupling provided by SWIM framework

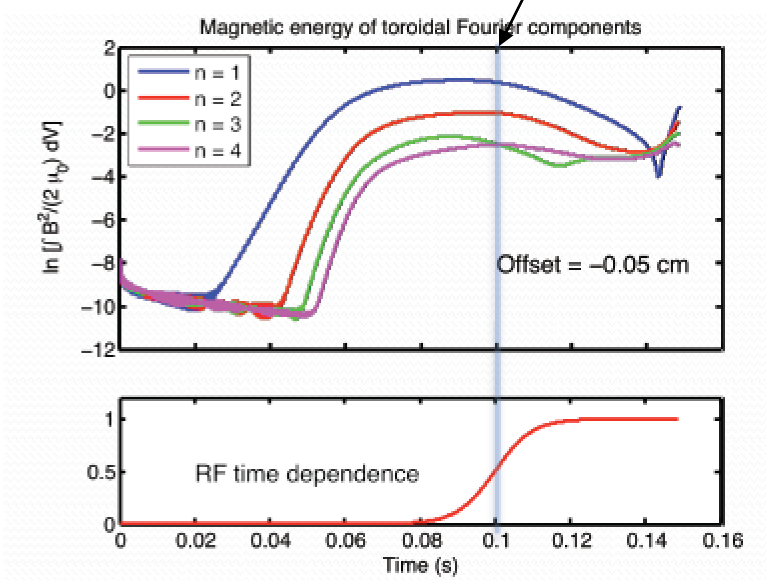
Results to date are for an equilibrium that is tearing unstable and using a model toroidally localized CD term





Close up





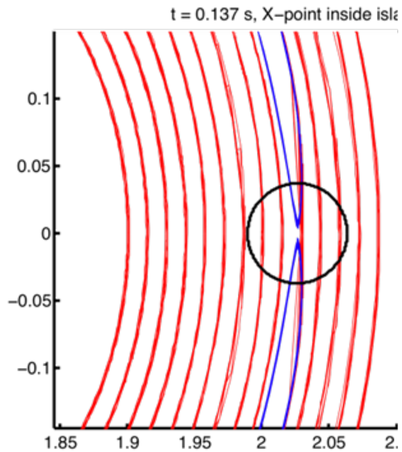
Model current drive source applied to original O-point in 1 toroidal location.

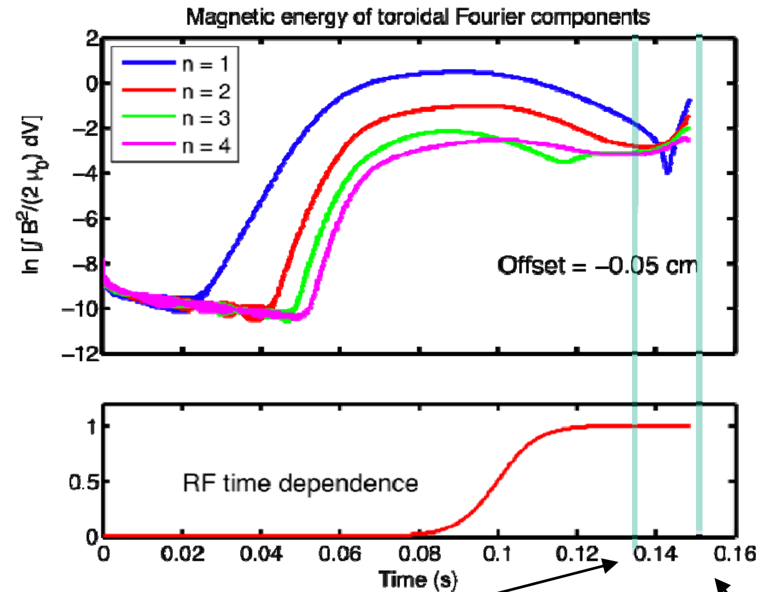
(2,1) island shrinks, becomes (4,2)

(4,2) island shrinks, (2,1) grows

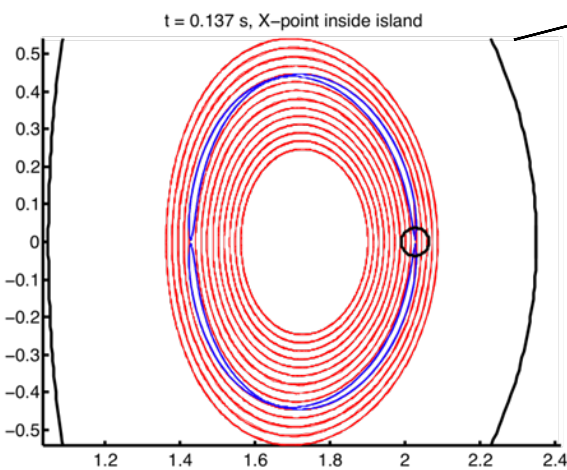
New (2,1) 90^0 our of phase with old

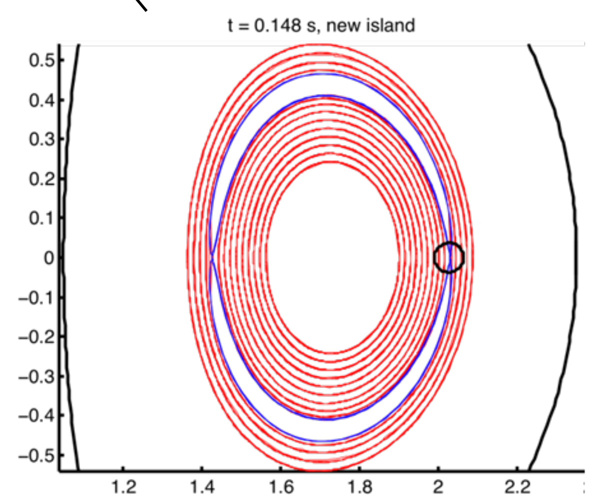
As RF suppresses the original islands, new islands arise





(2,1) islands grow up again after (4,2) island has been suppressed; new islands are 90° out of phase (poloidally) from the old ones.

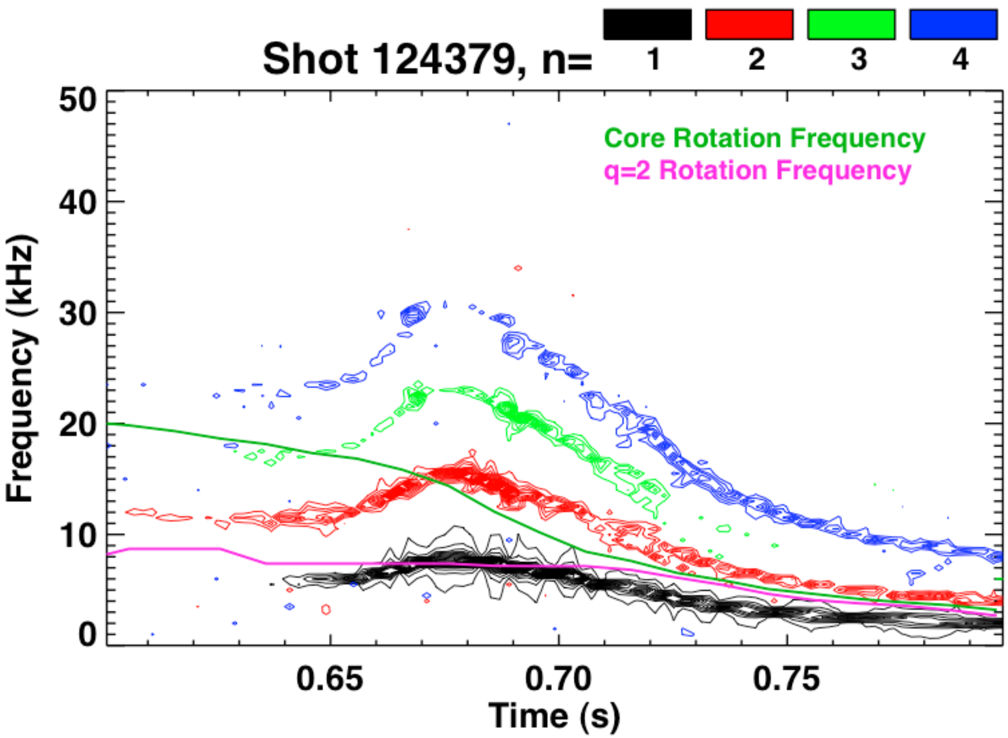


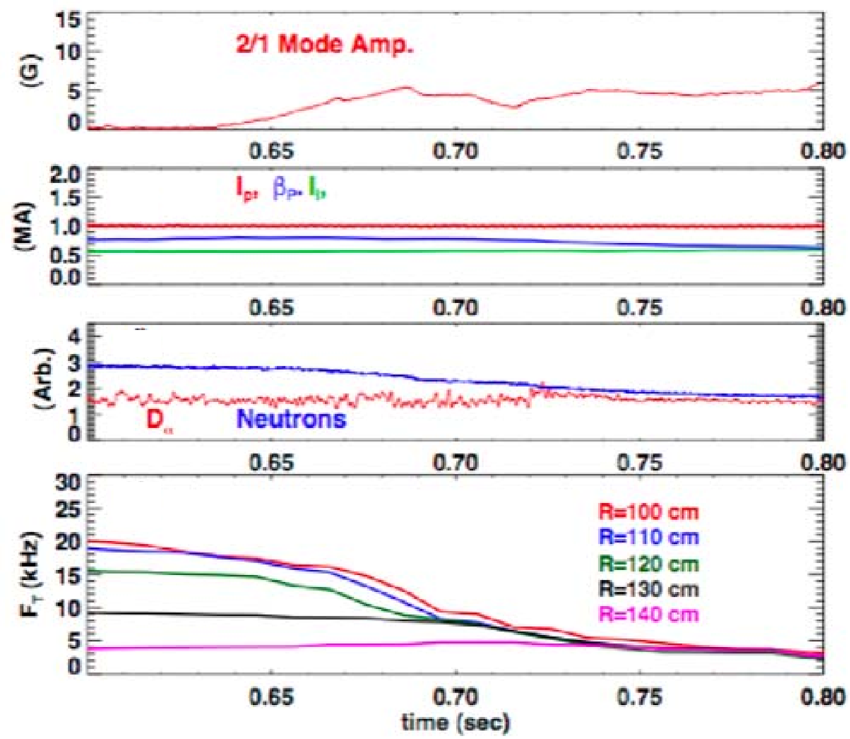


Jenkins

Study of saturated mode in NSTX-Motivation

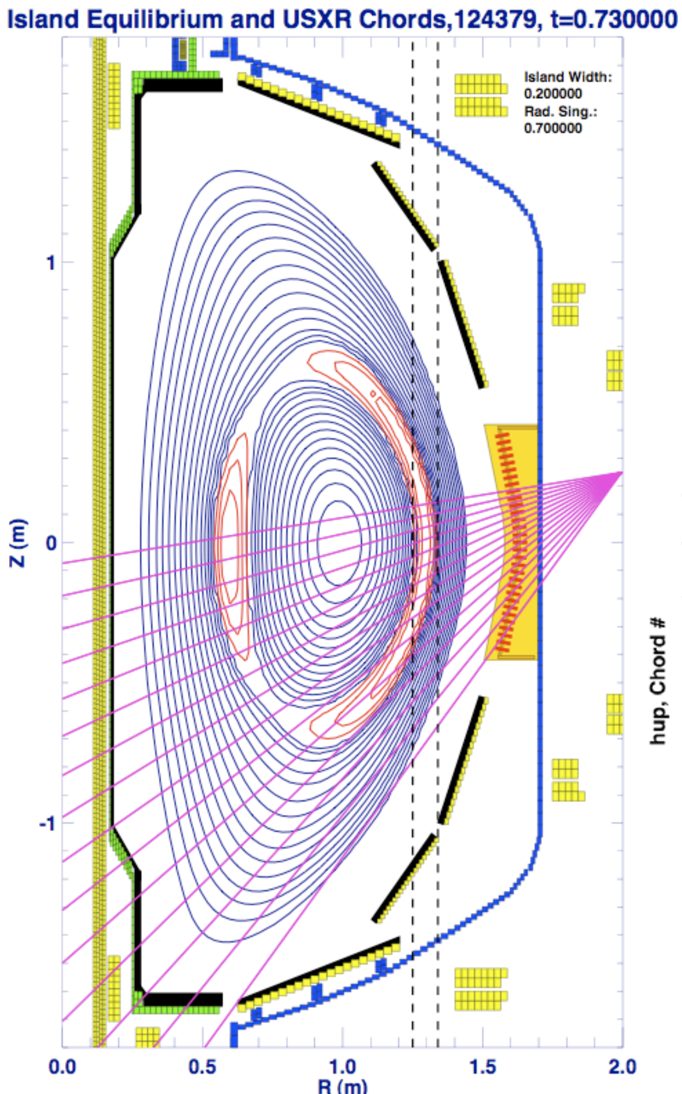
NSTX shot 124379 has a steadily growing 2,1 mode with no apparent trigger seen by the USXR, D_{α}, or neutron diagnostics.

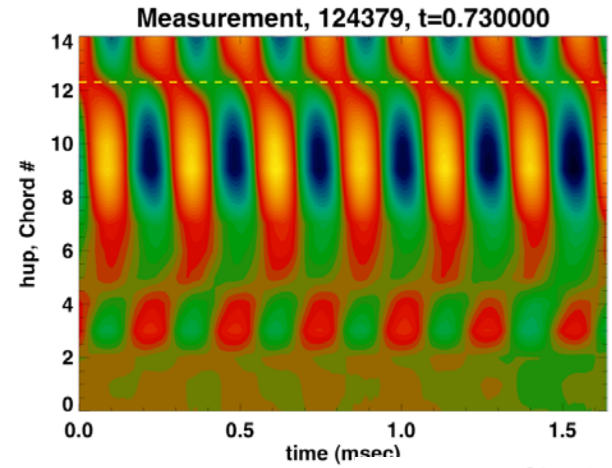


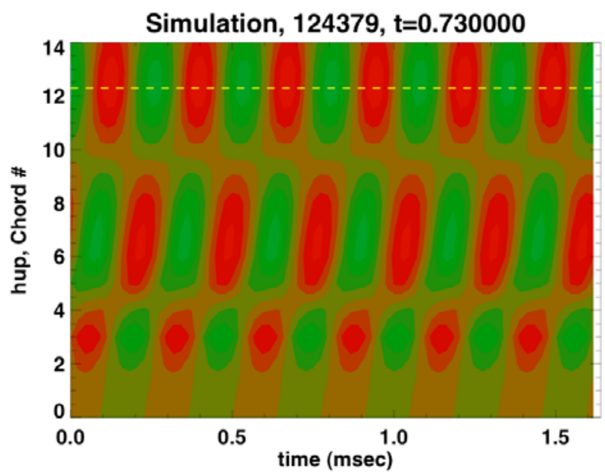


Gerhardt

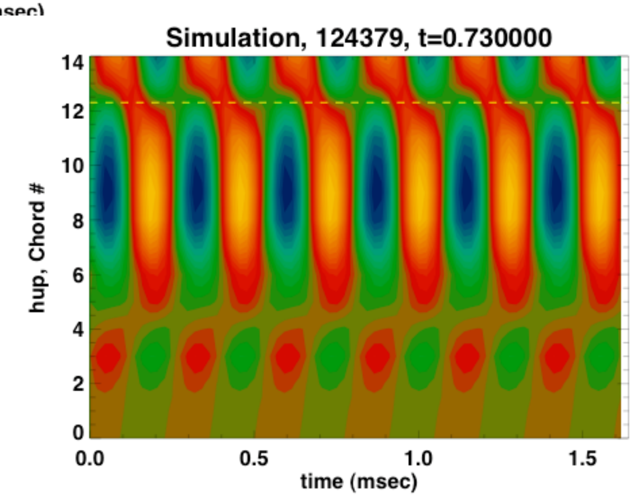
Eigenfunction Analysis of Multichord Data Suggests Coupling to 1,1 Ideal Kink







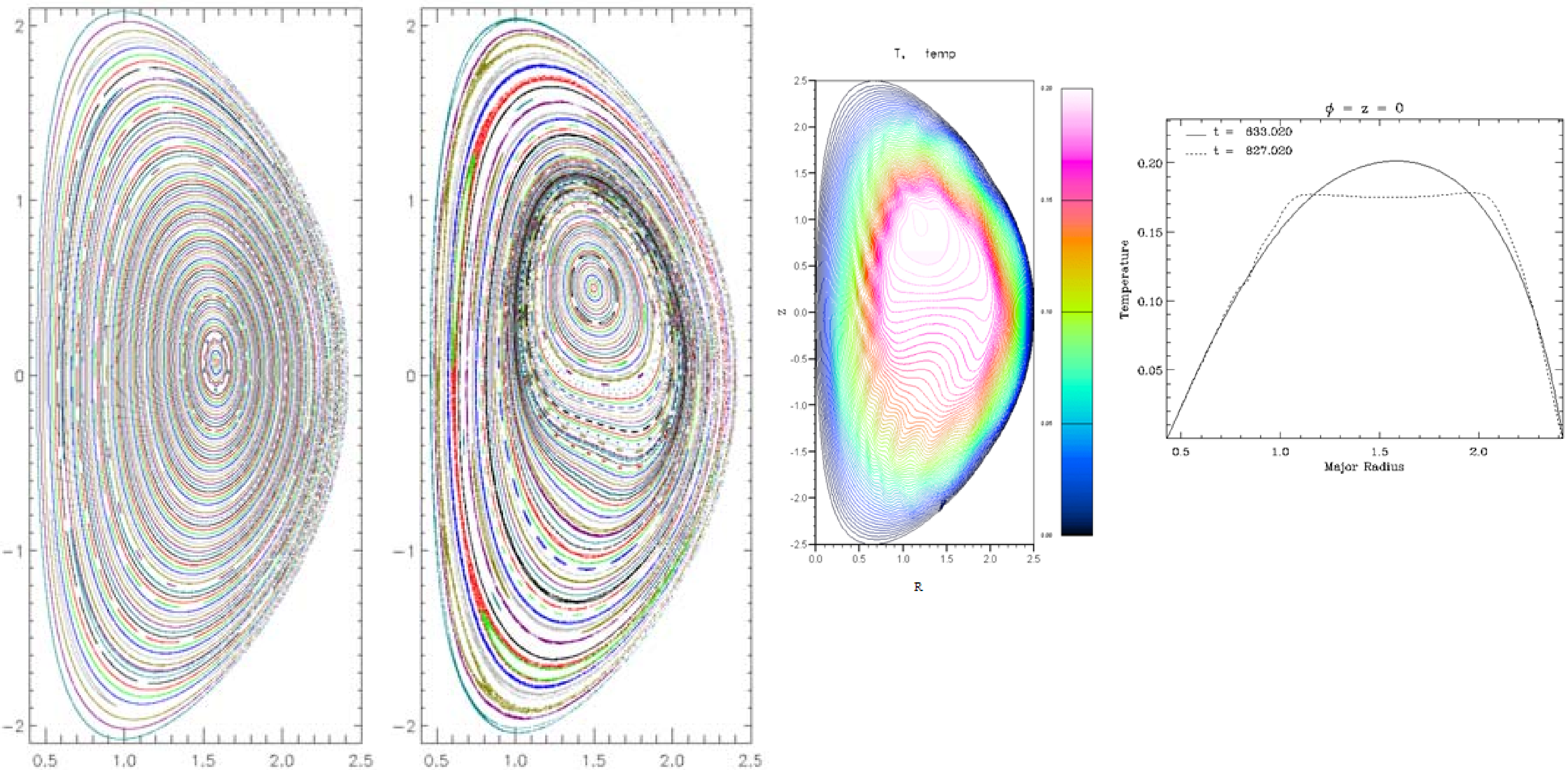
2,1 only



2,1 + 1,1 pert

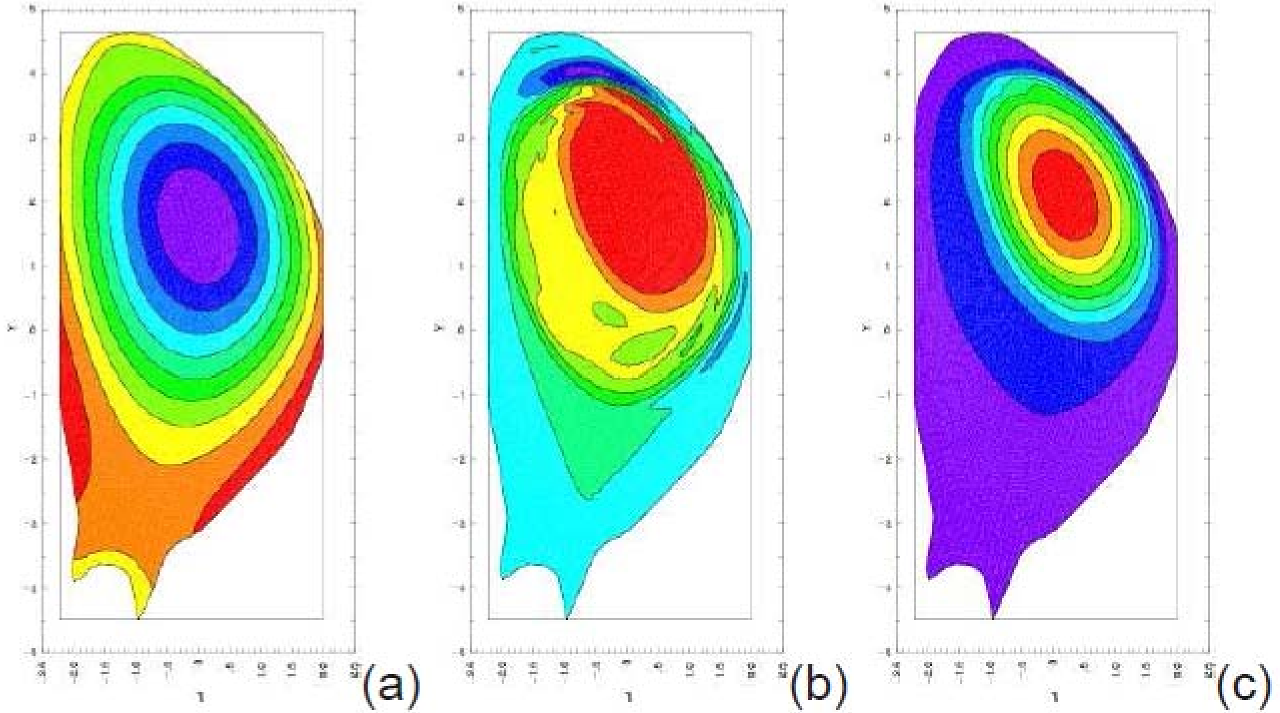
Gerhardt

M3D simulation of saturated mode in NSTX when $q_0 > 1$



Saturated n=1 mode can set develop when q_0 slightly > 1, as seen in Poincare plot on left. Can flatten temperature (right) and also drive m=2 islands.
Breslau, et al. IAEA 2010

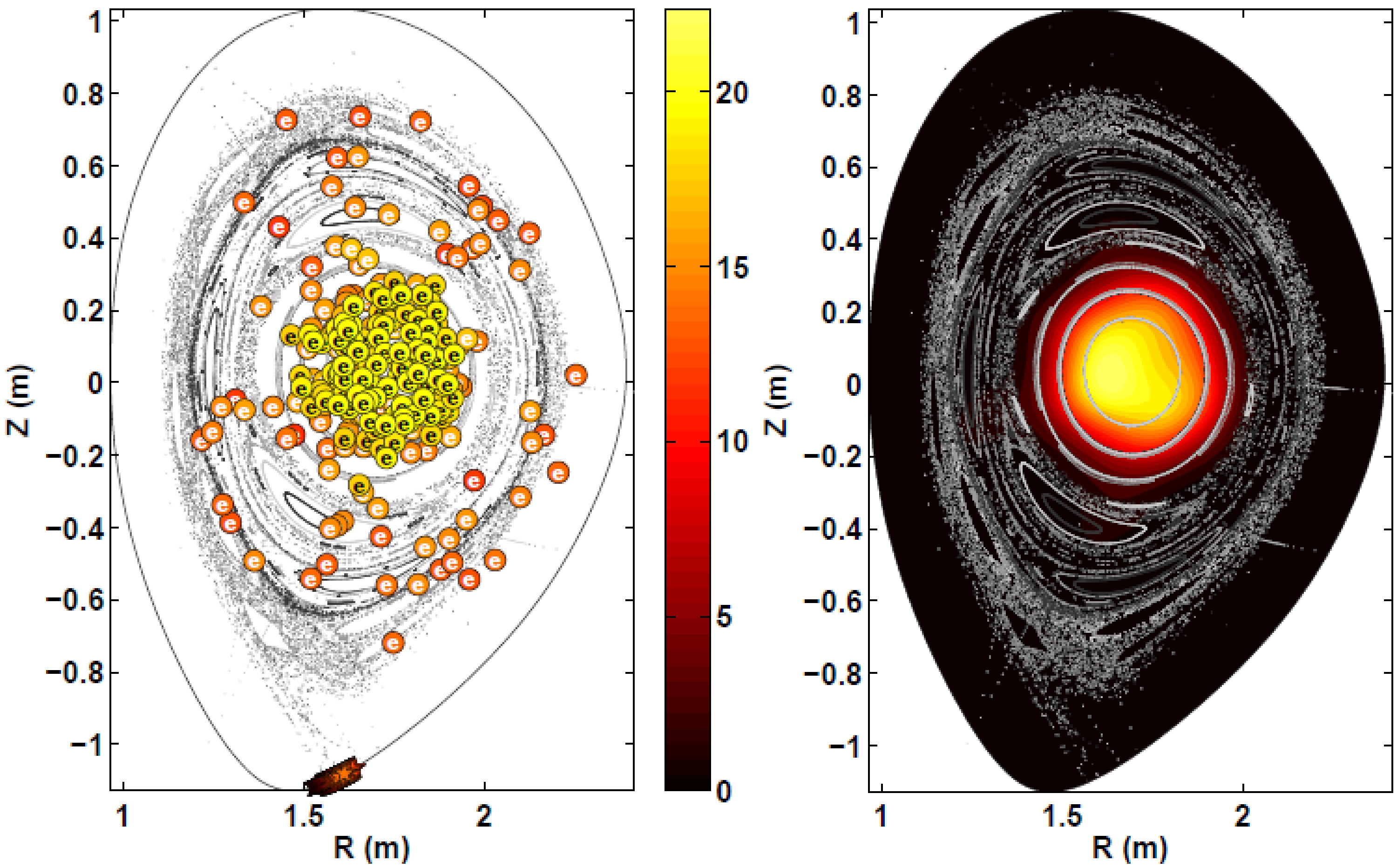
VDE[1] and Plasma Disruption simulations in ITER



(a) Poloidal flux, (b) toroidal current, and (c) temperature during a vertical displacement event. A VDE brings the plasma to the upper wall where a (m,n) = (1,1) kink mode grows. Forces on the vacuum vessel are calculated.

[1]Vertical Displacement Event

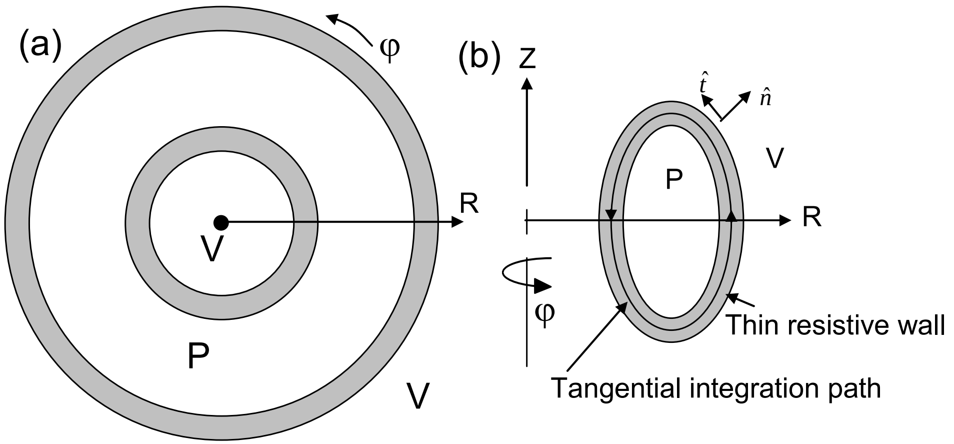
Runaway electron evolution in disrupting plasma is computed.



Simulation of DIII-D Ar pellet experiments. Runaway electrons of different energy shown. Synchrotron emission on right.

Izzo

Thin Resistive Wall Boundary Conditions



New instabilities can be present if the plasma is surrounded by a thin resistive wall. This can be modeled by modifying the boundary conditions.

All the boundary conditions follow from imposing that the normal component of **B** is continuous across the wall, and the tangential components can have a jump.

Follows from: $\nabla \bullet \mathbf{B} = 0$

Thin Resistive Wall Boundary Conditions-2

$$\hat{n}\cdot\mathbf{B}_V = \hat{n}\cdot\mathbf{B}_P$$

$$\mathbf{B}_V = \nabla\phi_V \qquad \nabla^2\phi_V = 0$$

$$\frac{\partial}{\partial t}\hat{n}\cdot\mathbf{B} = \frac{\partial}{\partial l}\frac{\eta_W}{\delta}\left[\mathbf{B}_P\cdot\hat{l} - \mathbf{B}_V\cdot\hat{l}\right] + \frac{1}{R}\frac{\partial}{\partial\varphi}\frac{\eta_W}{\delta}\left[\mathbf{B}_P\cdot\hat{\varphi} - \mathbf{B}_V\cdot\hat{\varphi}\right]$$

$\mathbf{B}_V$ = magnetic field on vacuum side of wall

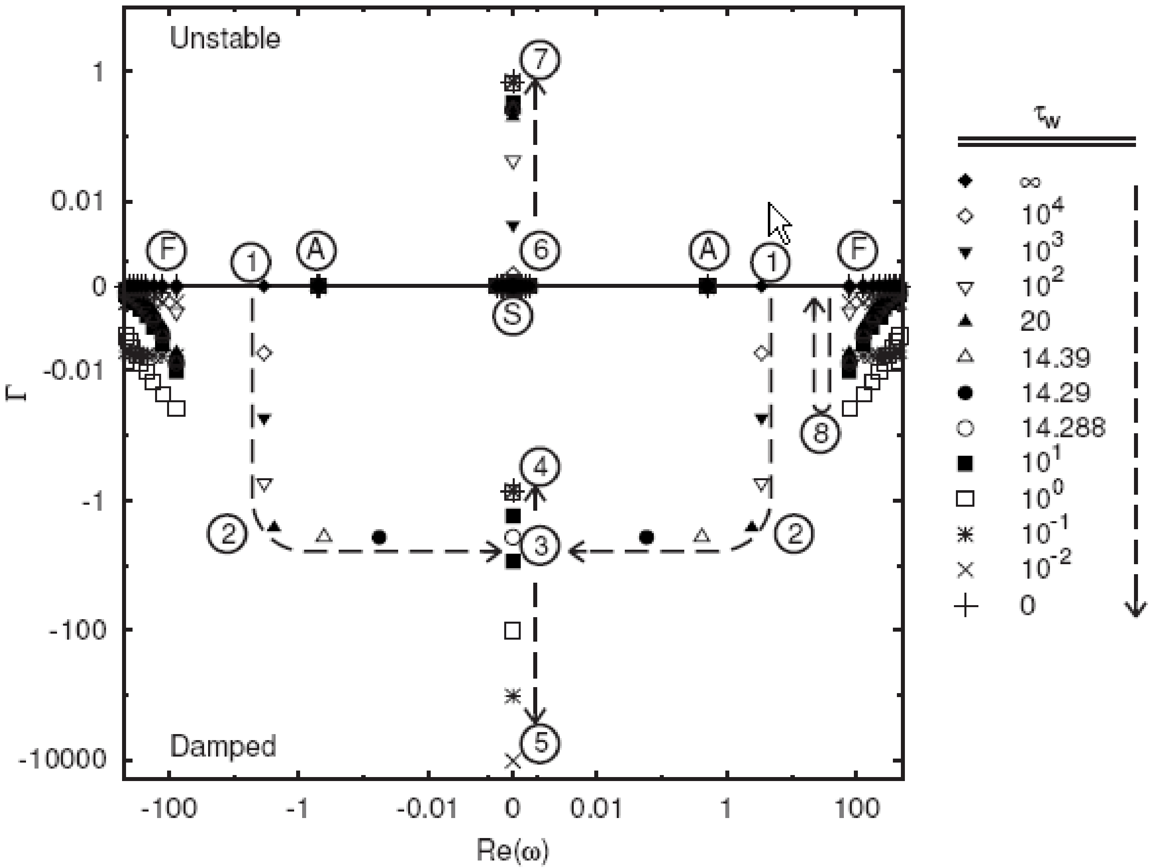
$\mathbf{B}_P$ = magnetic field on plasma side of wall

ϕ_V = magnetic scalar potential in wall

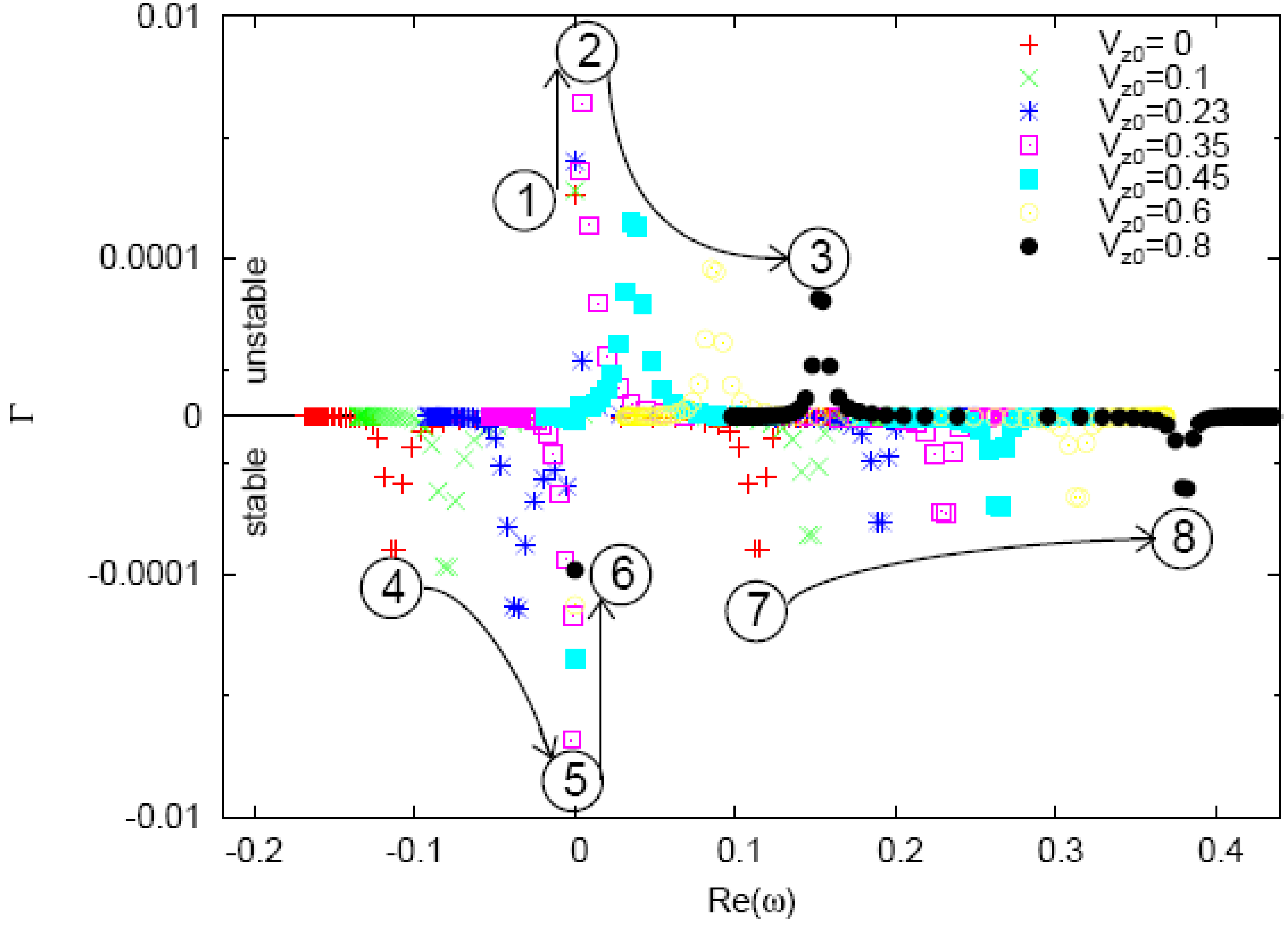
η_W = resistivity of wall

δ = thickness of wall

Without the presence of flow in the plasma, a wall stabilized plasma will become unstable as wall resistance is increased

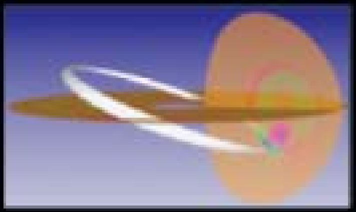


With sufficient plasma flow, the unstable mode will stabilize due to doppler shifted resonance with sound continua



Summary

- Implicit Methods
  - Use differential approximation to reduce matrix size and improve condition number
- Highly magnetized plasma
  - Stream-function/potential representation of velocity and magnetic fields
- Momentum equation projections
  - Gives energy conservation for full and reduced equation sets
- Finite Elements
  - High-order C^1 elements needed for high-order equations.
- Solver Strategy
  - Block Jacobi preconditioner
- Recent Results
  - Demanding ELM ideal MHD benchmarking studies give excellent results
  - Lundquist numbers up to 10^8 or higher are possible



Jardin

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