MHD Simulations for Fusion Applications

Lecture 2

Diffusion and Transport in Axisymmetric Geometry

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CEMRACS '10

Marseille, France

July 20, 2010

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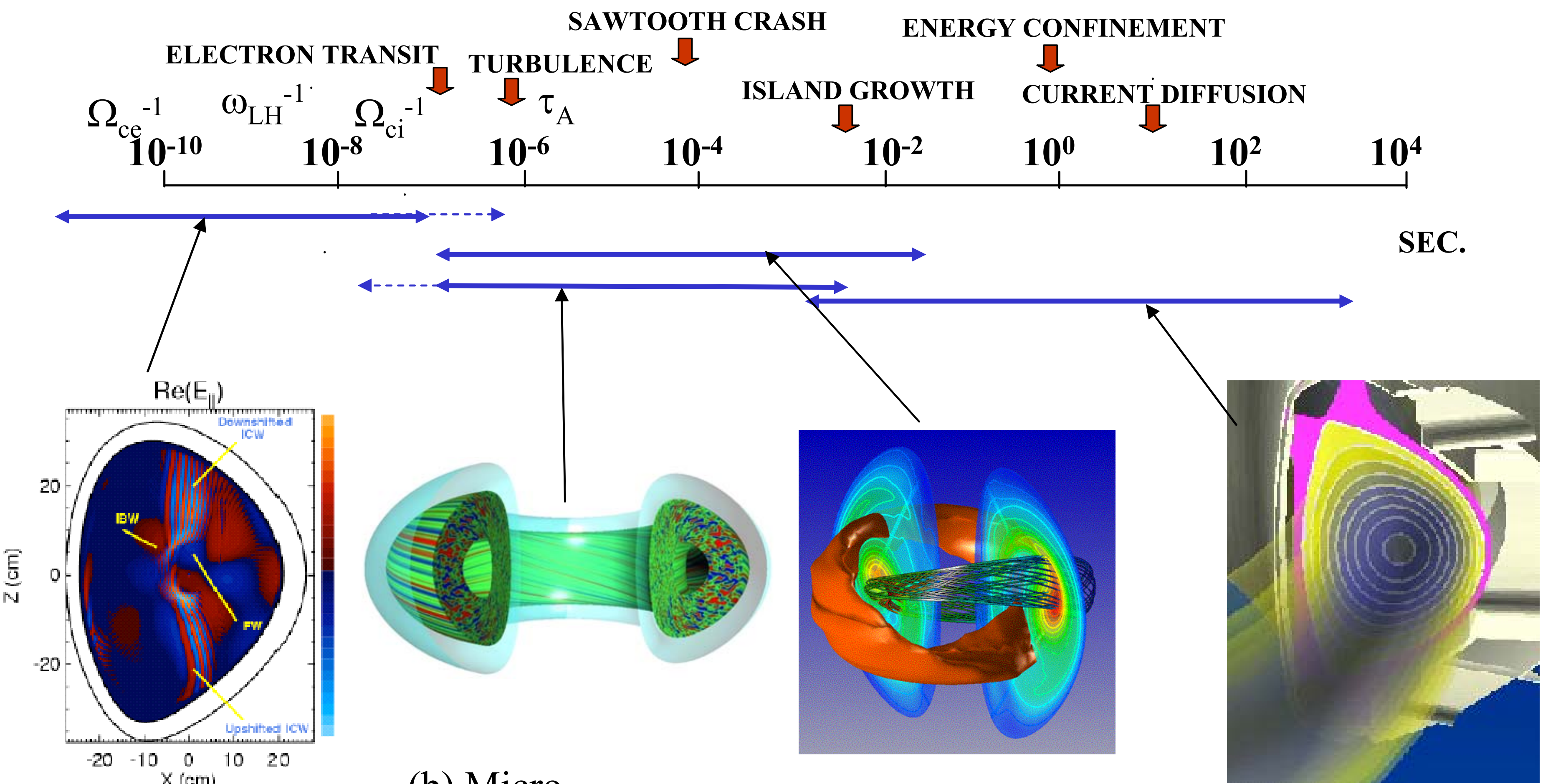
These 4 areas address different timescales and are normally studied using different codes



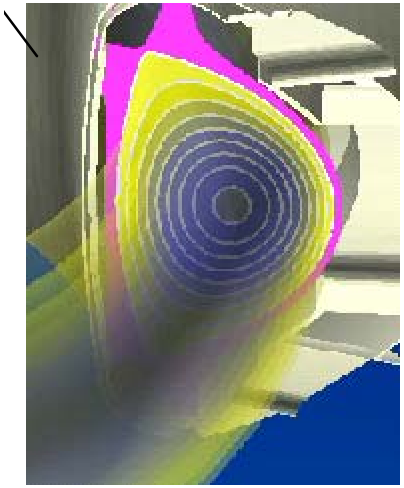
(a) RF codes

(b) Micro-turbulence codes

(c) Extended-MHD codes

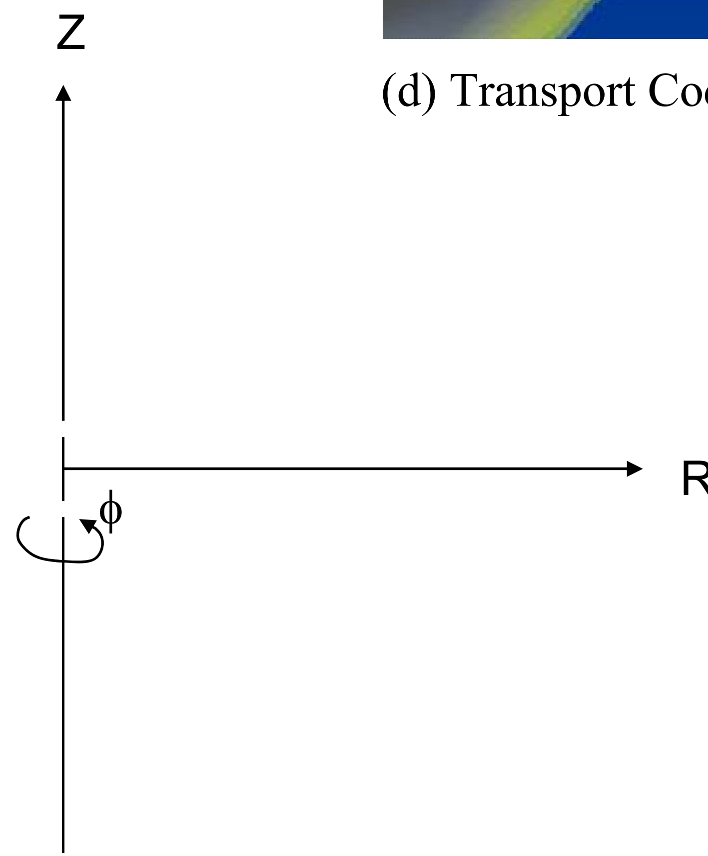
(d) Transport Codes

Transport codes solve the same equations as the Extended MHD codes, but in 2D rather than 3D



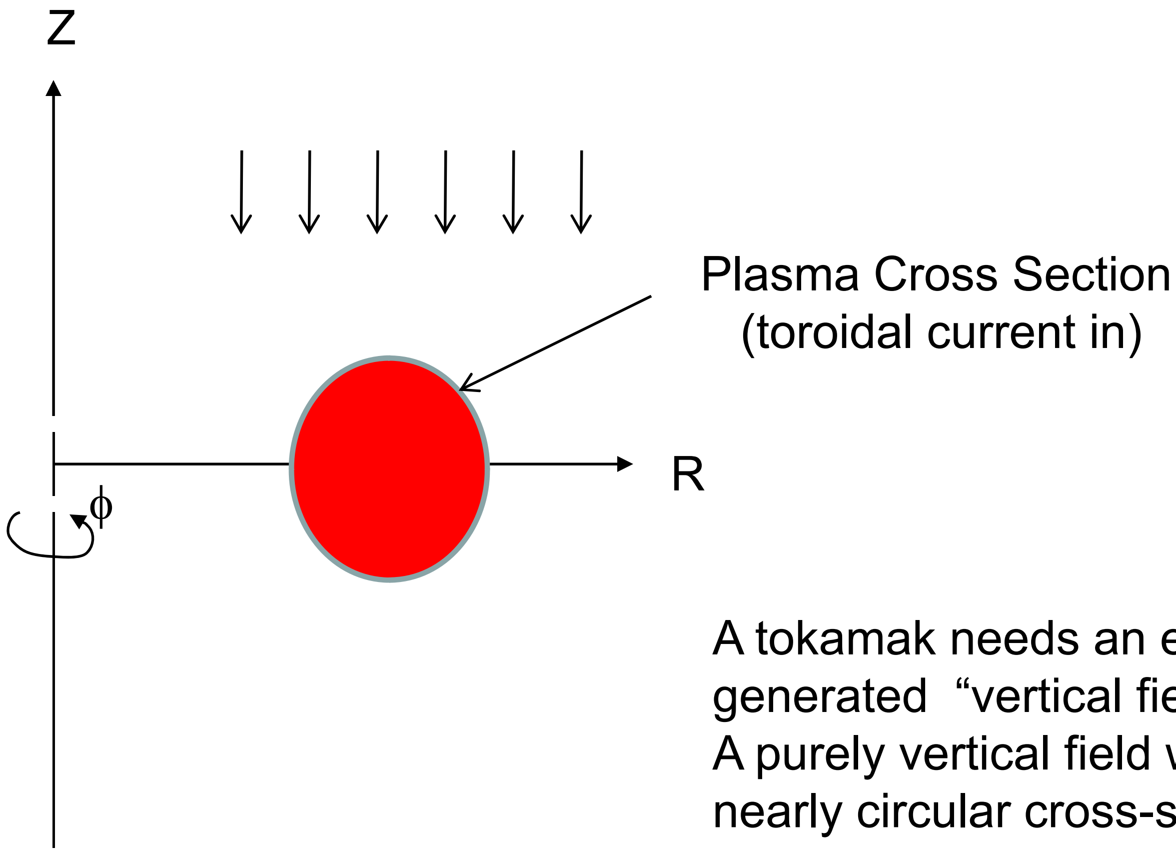
(d) Transport Codes

$$\frac{\partial}{\partial \phi} = 0$$ ➔ axisymmetry



First we will consider a model problem to better understand the timescales in 2D, then will proceed to a general approach

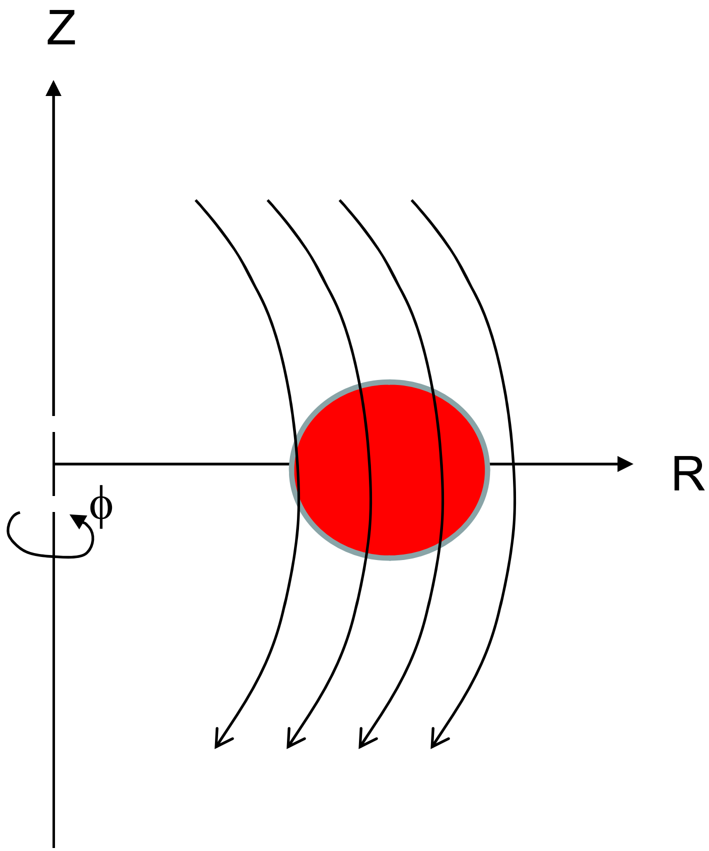
Tokamak Equilibrium Basics: Need for a vertical field

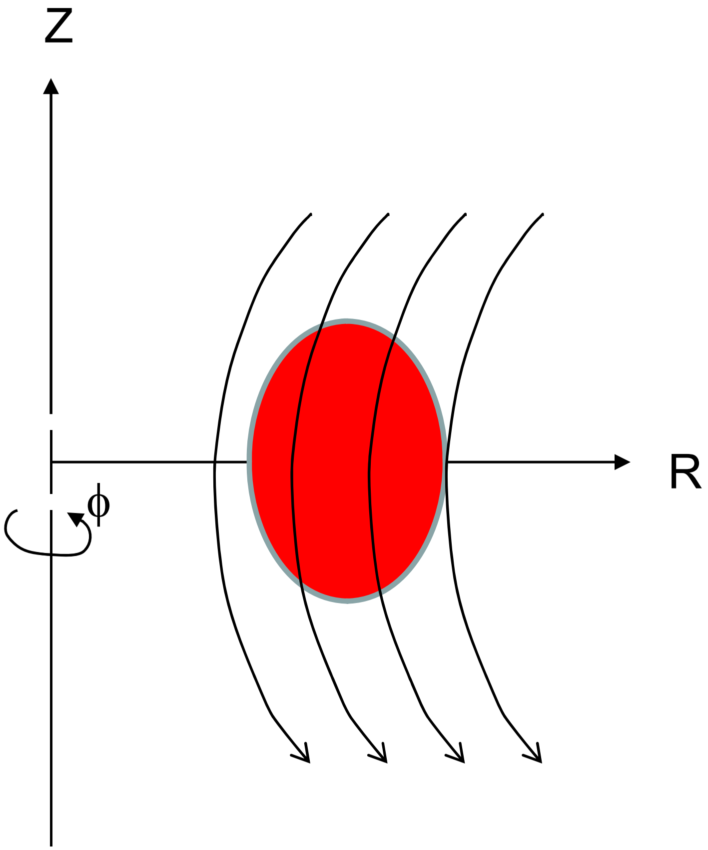


A tokamak needs an externally generated “vertical field” for equilibrium. A purely vertical field will produce a nearly circular cross-section plasma.

We first consider a very crude “rigid plasma” model to better understand where the slow resistive time scale comes from

In actual tokamak experiments, external poloidal field is not purely straight but has some curvature to it





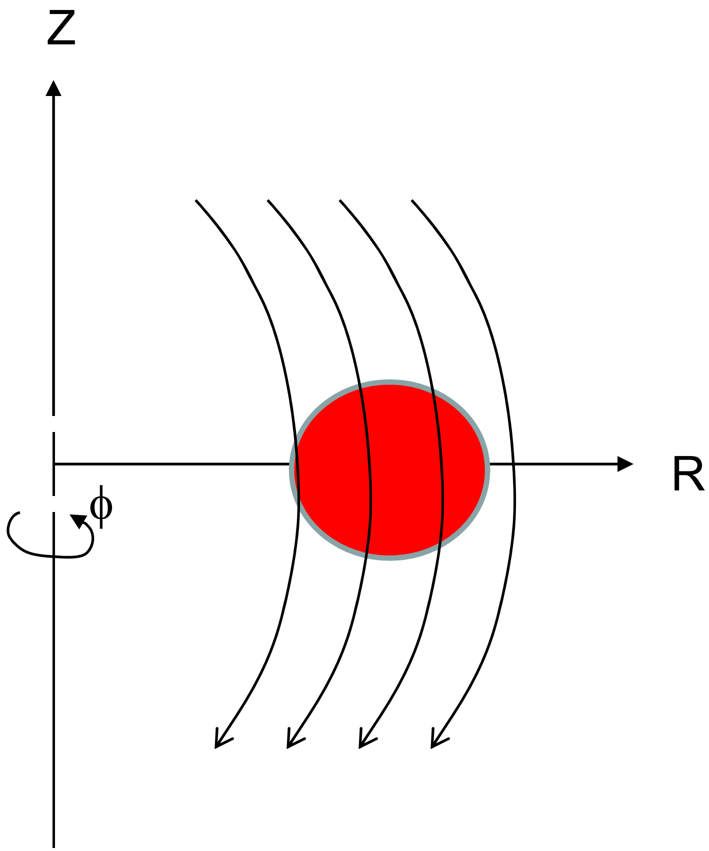
<u>Good Curvature</u>

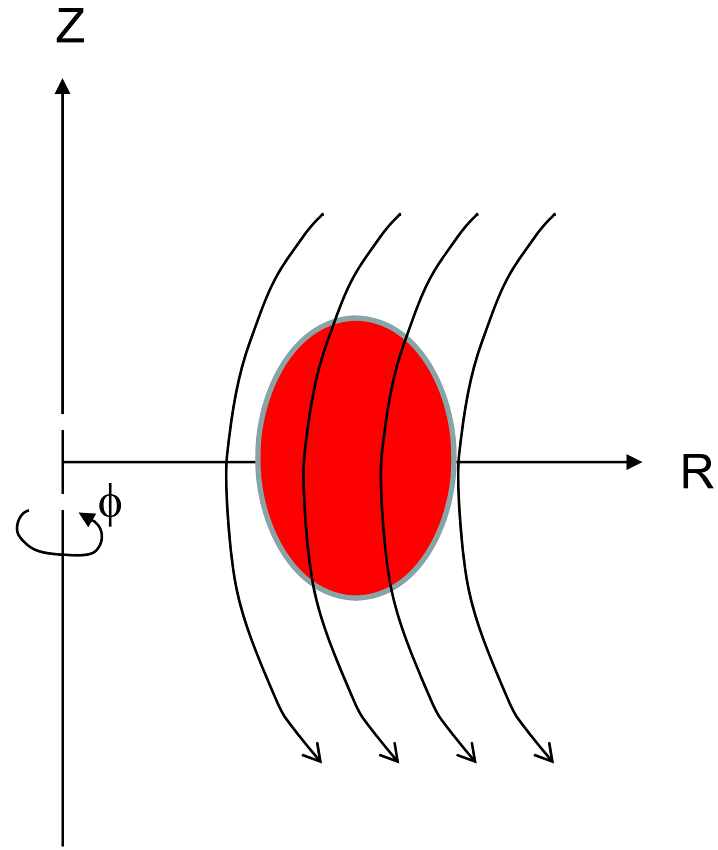
- Stable to vertical mode
- Oblate plasma
- low beta limits

<u>Bad Curvature</u>

- Unstable to vertical mode
- Elongated plasma
- higher beta limits

In actual tokamak experiments, external field is not purely straight but has some curvature to it

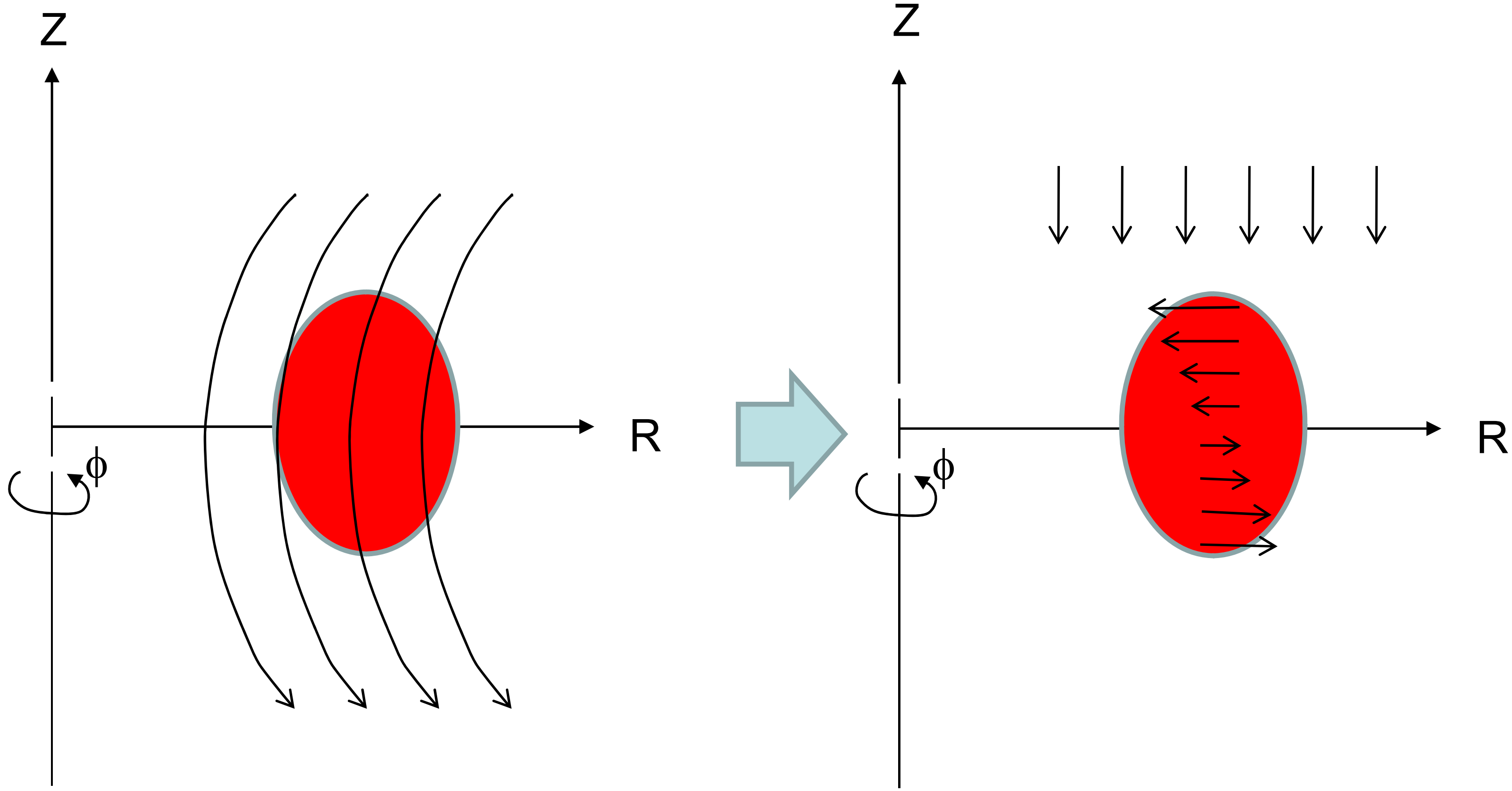




<u>Good Curvature</u>
- Stable to vertical mode
- Oblate plasma
- low beta limits

<u>Bad Curvature</u>
- Unstable to vertical mode
- Elongated plasma
- higher beta limits

External poloidal field with curvature can be thought of as a superposition of vertical and radial field.



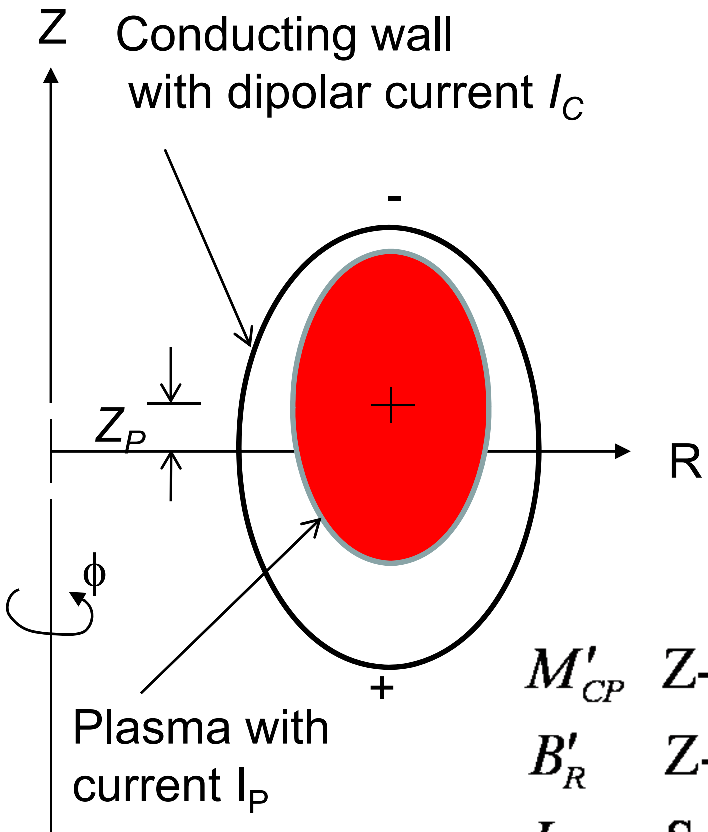
If plasma column is displaced upward, the force
$J \times B = I_P \times B_{Rext}$
will accelerate it further upward. Same for downward.

Vertical Instability

Alfven wave time scale: very fast!

A nearby conductor will produce eddy currents which act to stabilize

Describe the plasma as a rigid body of mass m with Z position Z_P. Assume time dependence $e^{i\omega t}$



Equation of motion:

$$-m\omega^2 Z_P = I_P M'_{CP} I_C + I_P B'_R Z_P$$

inertia   conductor   external field

Circuit equation for wall:

$$i\omega L I_C + R I_C + i\omega M'_{CP} I_P Z_P = 0$$

inductance   resistance   plasma coupling

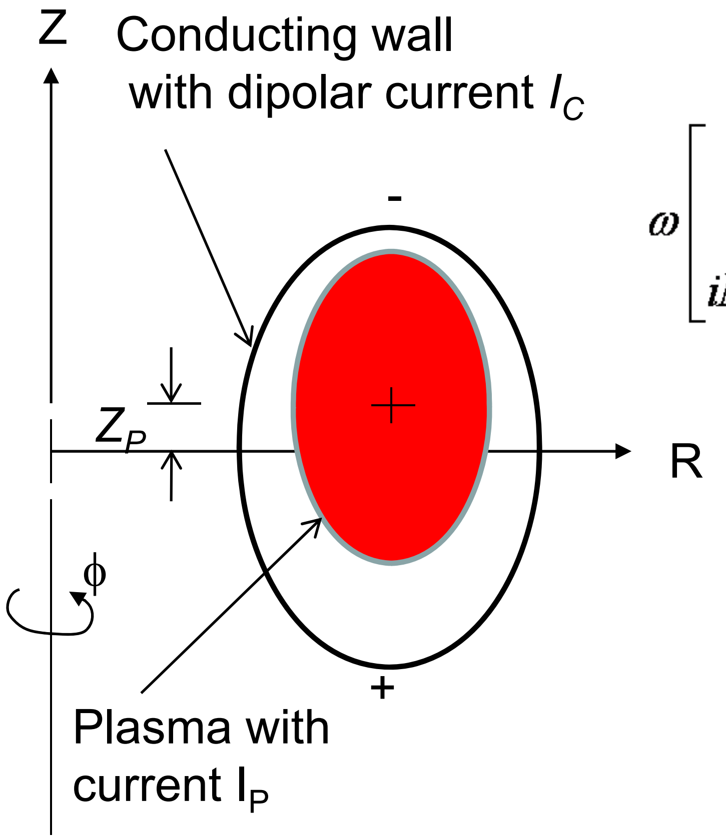
M'_{CP} Z-derivative of plasma/wall mutual inductance

B'_R Z-derivative of external radial magnetic field

L Self-inductance of wall

R Resistance of wall

Introduce plasma velocity $V_P = i\,\omega Z_P$ to get a 3x3 matrix eigenvalue equation for ω of standard form



$$\omega\begin{bmatrix} i & 0 & 0 \\ 0 & im & 0 \\ iM'_{CP}I_P & 0 & iL \end{bmatrix}\bullet\begin{bmatrix} Z_P \\ V_P \\ I_C \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ I_P B'_R & 0 & I_P M'_{CP} \\ 0 & 0 & -R \end{bmatrix}\bullet\begin{bmatrix} Z_P \\ V_P \\ I_C \end{bmatrix}$$

<u>Three roots:</u>

$$\omega_{1,2} = \pm\left[\omega_S^2-\omega_0^2\right]^{1/2} + i\,\frac{\gamma_R}{2}\,\frac{\omega_S^2}{\left(\omega_S^2-\omega_0^2\right)}$$

$$\omega_3 = -i\,\frac{\gamma_R\omega_0^2}{\left(\omega_S^2-\omega_0^2\right)}$$

	$\gamma_R \equiv R/L$	natural resistive decay rate of wall
$\gamma_R^2 \ll \omega_0^2, \omega_s^2$	$\omega_0^2 \equiv I_P B'_R / m$	destabilizing force
	$\omega_s^2 \equiv \left(I_P M'_{CP}\right)^2 / mL$	stabilizing force from wall

With only passive conductor, still an unstable root but much smaller.
Not on Alfven wave time scale but on L/R timescale of conductor.

$$\omega\begin{bmatrix} i & 0 & 0 \\ 0 & im & 0 \\ iM'_{CP}I_P & 0 & iL \end{bmatrix}\bullet\begin{bmatrix} Z_P \\ V_P \\ I_C \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ I_P B'_R & 0 & I_P M'_{CP} \\ 0 & 0 & -R \end{bmatrix}\bullet\begin{bmatrix} Z_P \\ V_P \\ I_C \end{bmatrix}$$

These are high frequency ($\sim 10^{-7}$ sec) stable oscillations that are slowly damped by the wall resistivity

This is the unstable mode. Very slow ($\sim 10^{-1}$ sec), and independent of plasma mass.

Three roots:

$$\omega_{1,2}=\pm\left[\omega_S^2-\omega_0^2\right]^{1/2}+i\frac{\gamma_R}{2}\frac{\omega_S^2}{\left(\omega_S^2-\omega_0^2\right)}$$

$$\omega_3=-i\frac{\gamma_R\omega_0^2}{\left(\omega_S^2-\omega_0^2\right)}$$

$$\gamma_R^2 \ll \omega_0^2,\omega_s^2$$

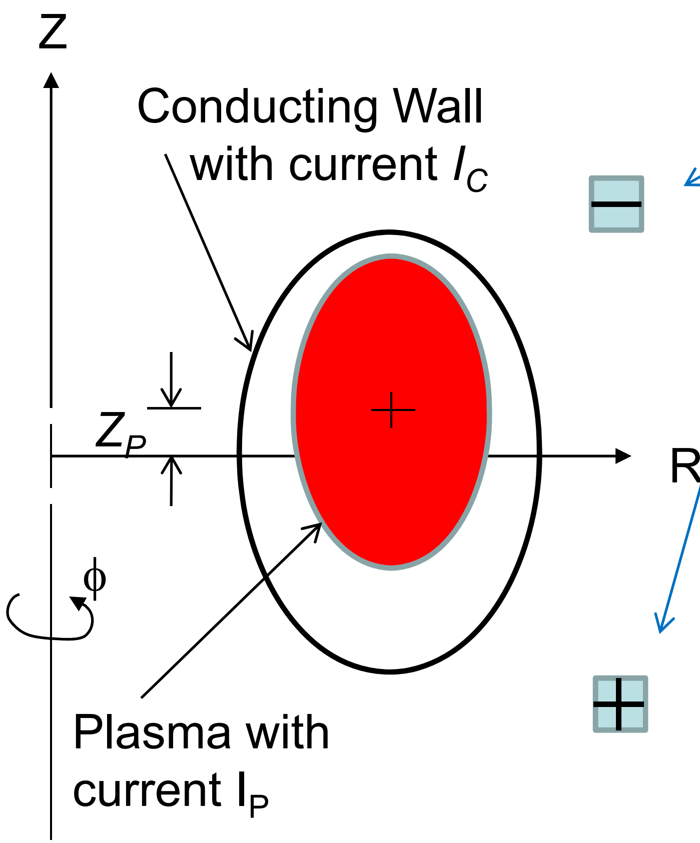
$\gamma_R \equiv R/L$ natural resistive decay rate of wall

$\omega_0^2 \equiv I_P B'_R/m$ destabilizing force

$$\omega_s^2 > \omega_0^2$$

$\omega_s^2 \equiv \left(I_P M'_{CP}\right)^2/mL$ stabilizing force from wall

Total stability is obtained by adding an active feedback system which only needs to act on this slower timescale.



This "rigid" mode is easily stabilized by adding a pair of feedback coils of opposite sign, and applying a voltage proportional to the plasma displacement -or its time integral or time derivative (PID)

Three roots:

$$\omega_{1,2} = \pm\left[\omega_S^2 - \omega_0^2\right]^{1/2} + i\frac{\gamma_R}{2}\frac{\omega_S^2}{\left(\omega_S^2 - \omega_0^2\right)}$$

$$\omega_3 = -i\frac{\gamma_R \omega_0^2}{\left(\omega_S^2 - \omega_0^2\right)}$$

$$\gamma_R^2 \ll \omega_0^2, \omega_s^2$$

$\gamma_R \equiv R/L$ natural resistive decay rate of wall

$\omega_0^2 \equiv I_P B_R' / m$ destabilizing force

$\omega_s^2 \equiv \left(I_P M_{CP}'\right)^2 / mL$ stabilizing force from wall

To model this "vertical instability" in realistic geometry, and take the non-rigid motion of the plasma into account, we take advantage of the fact that the unstable mode does not depend on the plasma mass (or inertia), and the stable modes are very high frequency and very low amplitude.

We start with the basic MHD + circuit equations and apply a "resistive timescale ordering"

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$nM_i\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

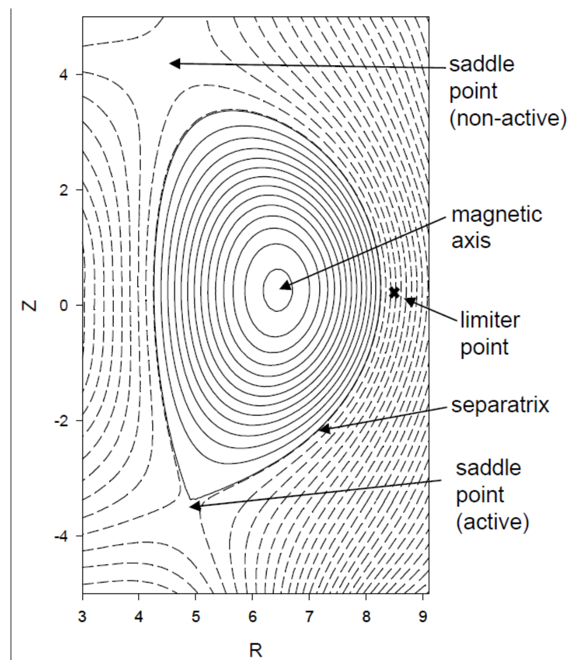
$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\mathbf{q} + \frac{3}{2} p\mathbf{V}\right) = -p\nabla \bullet \mathbf{V} + \eta J^2$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{d}{dt}\left[L_i I_i + \sum_{i \neq j} M_{ij} I_j + \int_P J_\phi G(R_i, R)\, dR\right] + R_i I_i = V_i$$

Introduce small parameter $\varepsilon \ll 1$

$$\frac{\partial}{\partial t} \sim \mathbf{V} \sim \mathbf{E} \sim V_i \sim \eta \sim R \sim \varepsilon$$



To model this “vertical instability” in realistic geometry, and taking the non-rigid motion of the plasma into account, we take advantage of the fact that the unstable mode does not depend on the plasma mass (or inertia), and the stable modes are very high frequency and low amplitude.

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\varepsilon^2 n M_i \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\mathbf{q} + \frac{3}{2} p\mathbf{V}\right) = -p\nabla \bullet \mathbf{V} + \eta J^2$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{d}{dt}\left[L_i I_i + \sum_{i \neq j} M_{ij} I_j + \int_P J_\phi G(R_i, R) dR\right] + R_i I_i = V_i$$

We start with the basic MHD + circuit equations and apply a “resistive timescale ordering”

Introduce small parameter $\varepsilon \ll 1$

$$\frac{\partial}{\partial t} \sim \mathbf{V} \sim \mathbf{E} \sim V_i \sim \eta \sim R \sim \varepsilon$$

All equations pick up a factor of ε , in all terms, which cancels out, except in the momentum equation, where the inertial terms are multiplied by ε^2 .

To model this "vertical instability" in realistic geometry, and taking the non-rigid motion of the plasma into account, we take advantage of the fact that the unstable mode does not depend on the plasma mass (or inertia), and the stable modes are very high frequency and low amplitude.

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\cancelto{0}{\varepsilon^2 nM_i\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right)} + \nabla p = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\mathbf{q} + \frac{3}{2} p\mathbf{V}\right) = -p\nabla \bullet \mathbf{V} + \eta J^2$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{d}{dt}\left[L_i I_i + \sum_{i \neq j} M_{ij} I_j + \int_P J_\phi G(R_i, R)\, dR\right] + R_i I_i = V_i$$

We start with the basic MHD + circuit equations and apply a "resistive timescale ordering"

Introduce small parameter $\varepsilon \ll 1$

$$\frac{\partial}{\partial t} \sim \mathbf{V} \sim \mathbf{E} \sim V_i \sim \eta \sim R \sim \varepsilon$$

This allows us to drop the inertial terms in the momentum equation, and replace it with the equilibrium equation.

Huge simplification….
removes Alfven timescale

To model this “vertical instability” in realistic geometry, and taking the non-rigid motion of the plasma into account, we take advantage of the fact that the unstable mode does not depend on the plasma mass (or inertia), and the stable modes are very high frequency and low amplitude.

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\nabla p = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\mathbf{q} + \frac{3}{2} p\mathbf{V} \right) = -p\nabla \bullet \mathbf{V} + \eta J^2$$

$$\mathbf{J} = \nabla \times \mathbf{B}$$

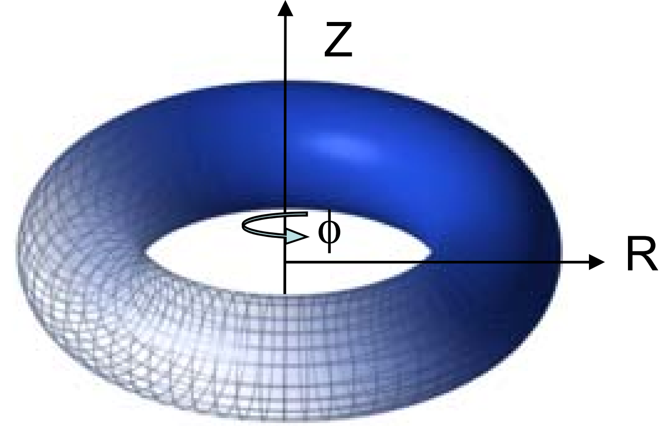
This is the set of equations we solve to simulate control of the plasma position and shape.

There are 3 production codes that solve these nonlinear equations in 2D and are used to design and test control strategies.

- TSC (PPPL)
- DINA (Russia)
- CORSICA (LLNL)

$$\frac{d}{dt}\left[L_i I_i + \sum_{i \neq j} M_{ij} I_j + \int_P J_\phi G(R_i, R) dR \right] + R_i I_i = V_i$$

Consider first the vector magnetic field equation



$$(1) \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$(2) \qquad \mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

The most general form for an equilibrium axisymmetric magnetic field is:

$$(3) \qquad \mathbf{B} = \nabla\phi \times \nabla\Psi + g(\Psi)\nabla\phi$$

Ψ is "flux function"

g is toroidal field function

$\nabla \bullet \mathbf{B} = 0$

$$\frac{\partial}{\partial\phi} = 0, \qquad |\nabla\phi|^2 = \frac{1}{R^2}$$

Substitute (3) into (1):

$$(4) \qquad \nabla\phi \times \nabla\frac{\partial\Psi}{\partial t} + \frac{\partial g}{\partial t}\nabla\phi = -\nabla \times \mathbf{E}$$

Take dot product of (4) with $\nabla\phi$

$$(5) \qquad \frac{1}{R^2}\frac{\partial g}{\partial t} = -\nabla\phi \bullet \nabla \times \mathbf{E} = \nabla \bullet [\nabla\phi \times \mathbf{E}]$$

Noting that $\nabla\phi \times \nabla\frac{\partial\Psi}{\partial t} = -\nabla \times \frac{\partial\Psi}{\partial t}\nabla\phi$, the remaining part of (4) becomes

$$(6) \qquad \frac{\partial\Psi}{\partial t} = R^2\mathbf{E} \bullet \nabla\phi + \cancel{C}(t)$$

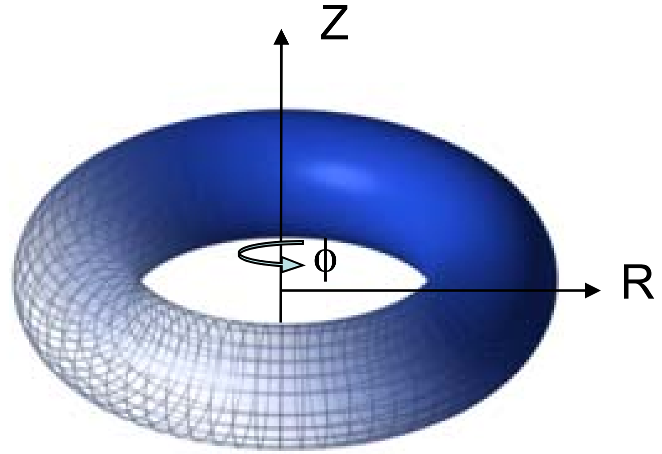
Constant can be taken to vanish to match boundary condition that Ψ=0 at R=0

Recall:

$$(2) \qquad \mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

$$(5) \qquad \frac{1}{R^2}\frac{\partial g}{\partial t} = \nabla \bullet \left[\nabla \phi \times \mathbf{E} \right]$$

$$(6) \qquad \frac{\partial \Psi}{\partial t} = R^2 \mathbf{E} \bullet \nabla \phi$$



$$\mathbf{B} = \nabla \phi \times \nabla \Psi + g(\Psi) \nabla \phi$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

$$= \Delta^* \Psi \nabla \phi + \nabla g \times \nabla \phi$$

Use (2) to eliminate $\mathbf{E}$ from (5) and (6):

$$\frac{1}{R^2}\frac{\partial g}{\partial t} = \nabla \bullet \left[-\nabla \phi \times (\mathbf{V} \times \mathbf{B}) + \nabla \phi \times \eta \mathbf{J} \right]$$

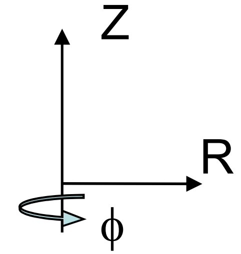
$$(7) \qquad \frac{\partial g}{\partial t} = R^2 \nabla \bullet \left[-\frac{g}{R^2}\mathbf{V} + (\nabla \phi \bullet \mathbf{V}) \nabla \phi \times \nabla \Psi + \frac{1}{R^2}\frac{\eta}{\mu_0} \nabla g \right]$$

$$\frac{\partial \Psi}{\partial t} = -R^2 (\mathbf{V} \times \mathbf{B}) \bullet \nabla \phi + R^2 \eta \mathbf{J} \bullet \nabla \phi$$

$$(8) \qquad \frac{\partial \Psi}{\partial t} = -\mathbf{V} \bullet \nabla \Psi + \frac{\eta}{\mu_0} \Delta^* \Psi$$

$$\Delta^* \Psi \equiv R^2 \nabla \bullet R^{-2} \nabla \Psi$$

Summary of scalar equations:



$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\mathbf{q} + \frac{3}{2} p\mathbf{V} \right) + p\nabla \bullet \mathbf{V} = \eta \mathbf{J}^2$$

$$\frac{\partial \Psi}{\partial t} + \mathbf{V} \bullet \nabla \Psi = \frac{\eta}{\mu_0} \Delta^* \Psi$$

$$\frac{\partial g}{\partial t} + R^2 \nabla \bullet \left[\frac{g}{R^2} \mathbf{V} - \left(\nabla \phi \bullet \mathbf{V} \right) \nabla \phi \times \nabla \Psi - \frac{1}{R^2} \frac{\eta}{\mu_0} \nabla g \right] = 0$$

$$\mathbf{B} = \nabla \phi \times \nabla \Psi + g\left(\Psi\right) \nabla \phi$$

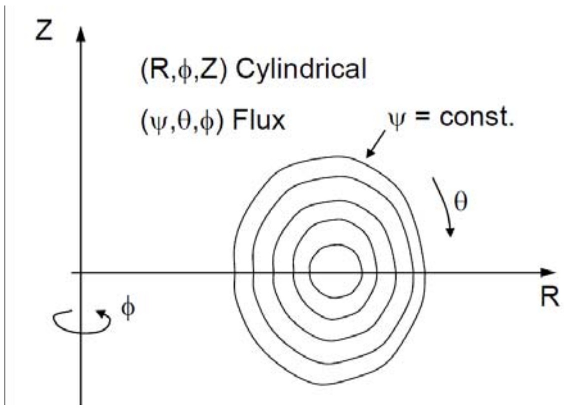
$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

And, the equilibrium constraint: $\nabla p = \mathbf{J} \times \mathbf{B}$

Note:

- The pressure and magnetic field variables obey separate time advancement equations, yet they must always satisfy the equilibrium constraint
- Each of the equations contains the velocity variable $\mathbf{V}$, yet there is no equation to advance $\mathbf{V}$.
- The heat flux vector $\mathbf{q}$ is very anisotropic, much larger parallel to the magnetic field than perpendicular to it.

Because of the anisotropy of the heat conduction, we want to transform to a moving coordinate system aligned with the magnetic flux surfaces.

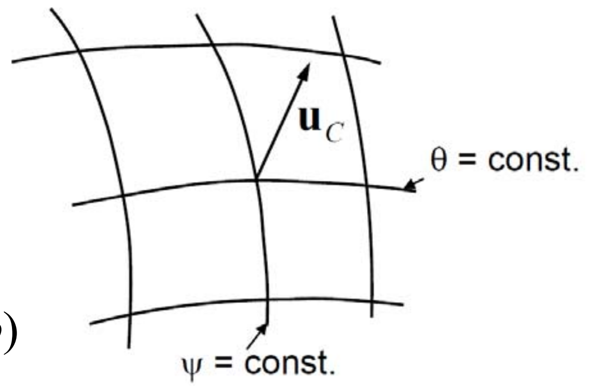


At any given time, we will define the non-orthogonal flux coordinate system:

$$\left(\psi(\mathbf{x}),\theta(\mathbf{x}),\phi(\mathbf{x})\right) \qquad \psi=\psi\left(\Psi\right) \quad \mathbf{B}\bullet\nabla\psi=0$$

This has the associated volume element: and Jacobian:

$$d\tau=\frac{d\psi\, d\theta\, d\phi}{\nabla\psi\times\nabla\theta\bullet\nabla\phi}\equiv J d\psi\, d\theta\, d\phi \qquad J\equiv\left[\nabla\psi\times\nabla\theta\bullet\nabla\phi\right]^{-1}$$



We also have the inverse representation: $\mathbf{x}(\psi,\theta,\phi,t)$

We next define the *coordinate velocity* at a particular (ψ,θ,ϕ) location as:

$$\mathbf{u}_C=\left.\frac{\partial\mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi}$$

For any scalar function α:

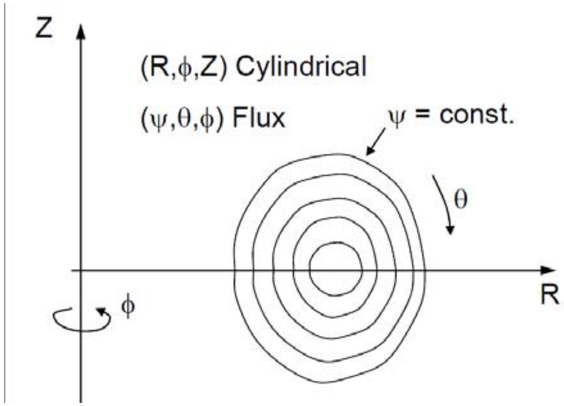
$$\left.\frac{\partial\alpha}{\partial t}\right|_{\psi,\theta,\phi}=\left.\frac{\partial\alpha}{\partial t}\right|_{\mathbf{x}}+\frac{\partial\alpha}{\partial\mathbf{x}}\bullet\left.\frac{\partial\mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi} \Rightarrow \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}}=\left.\frac{\partial}{\partial t}\right|_{\psi,\theta,\phi}-\mathbf{u}_C\bullet\nabla$$

Also, one can verify the relation for the time derivative of the Jacobian:

$$\left.\frac{\partial J}{\partial t}\right|_{\psi,\theta,\phi}=J\nabla\bullet\mathbf{u}_C$$

Time dependent coordinate transformation

$$\psi(\mathbf{x},t),\theta(\mathbf{x},t),\phi$$



$$\mathbf{u}_C = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi} \qquad \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\psi,\theta,\phi} - \mathbf{u}_C \bullet \nabla$$

$$J \equiv \left[\nabla\psi\times\nabla\theta\bullet\nabla\phi\right]^{-1} \qquad \left.\frac{\partial J}{\partial t}\right|_{\psi,\theta,\phi} = J\nabla\bullet\mathbf{u}_C$$

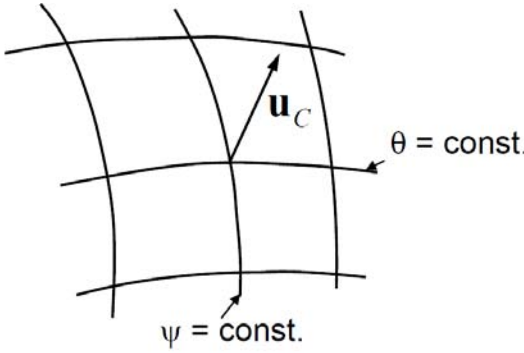
The fluid velocity that appears in the MHD equations is divided into two parts:

$$\mathbf{V} = \mathbf{u}_C + \mathbf{u}_R$$

actual fluid velocity

Velocity of the coordinates

Velocity relative to the moving coordinates



Transform the continuity equation to the moving frame:

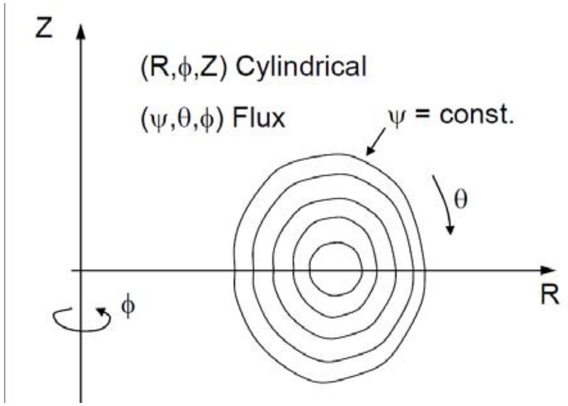
$$\left.\frac{\partial n}{\partial t}\right|_{\mathbf{x}} + \nabla\bullet(n\mathbf{V}) = 0$$

$$\left.\frac{\partial n}{\partial t}\right|_{\psi,\theta,\phi} - \mathbf{u}_C\bullet\nabla n + \left(\mathbf{u}_C + \mathbf{u}_R\right)\bullet\nabla n + n\nabla\bullet\left(\mathbf{u}_C + \mathbf{u}_R\right) = 0$$

$$\left.\frac{\partial (nJ)}{\partial t}\right|_{\psi,\theta,\phi} + J\nabla\bullet(n\mathbf{u}_R) = 0$$

$$\mathbf{u}_C = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\psi,\theta,\phi} \qquad \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\psi,\theta,\phi} - \mathbf{u}_C \bullet \nabla \qquad \psi(\mathbf{x},t),\theta(\mathbf{x},t),\phi$$

$$J \equiv \left[\nabla\psi \times \nabla\theta \bullet \nabla\phi\right]^{-1} \qquad \left.\frac{\partial J}{\partial t}\right|_{\psi,\theta,\phi} = J\nabla \bullet \mathbf{u}_C$$



Apply the same technique to all the time-advance equations:

$$\left.\frac{\partial}{\partial t}\right|_{\psi,\theta,\phi} \rightarrow \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t}(nJ) + J\nabla \bullet (n\mathbf{u}_R) = 0$$

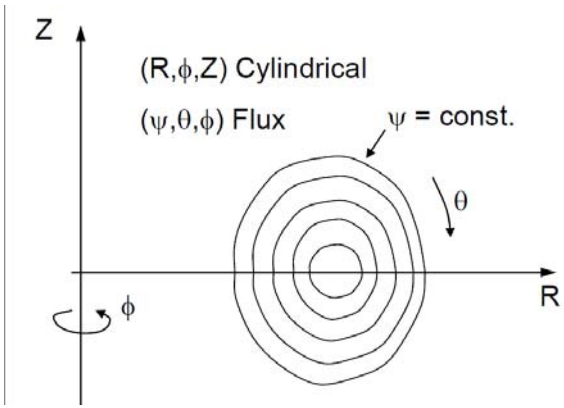
$$\frac{\partial}{\partial t}\left(p^{3/5}J\right) + J\nabla \bullet \left(p^{3/5}\mathbf{u}_R\right) + \frac{2}{5}Jp^{-2/5}\left[\nabla \bullet \mathbf{q} - \eta \mathbf{J}^2\right] = 0$$

$$\frac{\partial \Psi}{\partial t} + \mathbf{u}_R \bullet \nabla\Psi = \frac{\eta}{\mu_0}\Delta^*\Psi$$

$$\frac{\partial}{\partial t}\left(g\frac{J}{R^2}\right) + J\nabla \bullet \left[\frac{g}{R^2}\mathbf{u}_R - (\nabla\phi \bullet \mathbf{u}_R)\nabla\phi \times \nabla\Psi - \frac{1}{R^2}\frac{\eta}{\mu_0}\nabla g\right] = 0$$

These equations are now in the moving frame. Because these are all conservation equations, only the relative velocity appears in the equations!

$$\psi(\mathbf{x},t),\theta(\mathbf{x},t),\phi$$



Next, we integrate the equations in the poloidal angle around the flux surface.

Use the property, that for any vector $\mathbf{A}$

$$J\nabla\bullet\mathbf{A}=\frac{\partial}{\partial\psi}\left(J\mathbf{A}\bullet\nabla\psi\right)+\frac{\partial}{\partial\theta}\left(J\mathbf{A}\bullet\nabla\theta\right)$$

$$2\pi\int_0^{2\pi}J\nabla\bullet\mathbf{A}\,d\theta=2\pi\int_0^{2\pi}\frac{\partial}{\partial\psi}\left(J\mathbf{A}\bullet\nabla\psi\right)d\theta+\cancelto{0}{2\pi\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(J\mathbf{A}\bullet\nabla\theta\right)}$$

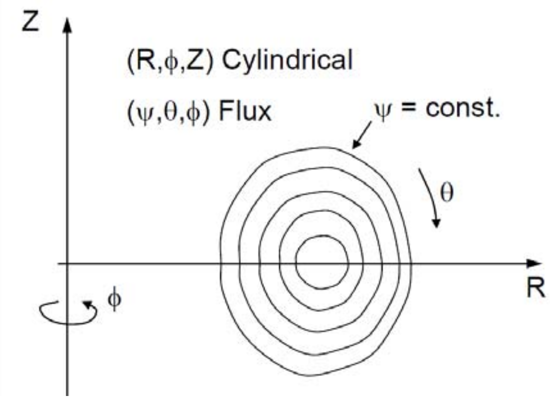
$$=\frac{\partial}{\partial\psi}\left[V'\left\langle\mathbf{A}\bullet\nabla\psi\right\rangle\right]$$

Here, we have defined the *differential volume* and *surface average*:

$$V'(\psi)\equiv\frac{dV}{d\psi}=2\pi\int_0^{2\pi}J\,d\theta \qquad \left\langle a\right\rangle=\frac{2\pi}{V'}\int_0^{2\pi}Ja\,d\theta$$

Apply to continuity equation: $\frac{\partial}{\partial t}(nJ)+J\nabla\bullet(n\mathbf{u}_R)=0 \;\rightarrow\; \frac{\partial}{\partial t}(nV')+\frac{\partial}{\partial\psi}\left[nV'\left\langle\nabla\psi\bullet\mathbf{u}_R\right\rangle\right]=0$

$$\psi(\mathbf{x},t),\theta(\mathbf{x},t),\phi$$



After integrating all the equations over a flux surface:

$$\frac{\partial}{\partial t}(nV')+\frac{\partial}{\partial\psi}\left[nV'\langle\nabla\psi\bullet\mathbf{u}_R\rangle\right]=0$$

$$\frac{\partial}{\partial t}\left(p^{3/5}V'\right)+\frac{\partial}{\partial\psi}\left[p^{3/5}V'\langle\nabla\psi\bullet\mathbf{u}_R\rangle\right]+\frac{2}{5}p^{-2/5}\left[\frac{\partial}{\partial\psi}\left(V'\langle\mathbf{q}\bullet\nabla\psi\rangle\right)-\eta V'\mathbf{J}^2\right]=0$$

$$\frac{\partial\Psi}{\partial t}+\mathbf{u}_R\bullet\nabla\Psi=\frac{\eta}{\mu_0}\Delta^*\Psi$$

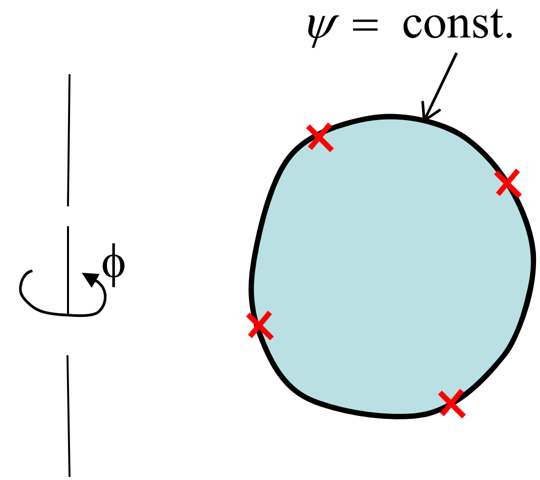
$$\frac{\partial}{\partial t}\left(gV'\langle R^{-2}\rangle\right)+\frac{\partial}{\partial\psi}\left[gV'\langle R^{-2}\nabla\psi\bullet\mathbf{u}_R\rangle-V'\frac{\eta}{\mu_0}\left\langle\frac{1}{R^2}\nabla g\bullet\nabla\psi\right\rangle\right]=0 \qquad (*)$$

These are now 1-dimensional equations for the surface averages. We can use one of these equations to eliminate the relative velocity from the others.

Note that Equation (*) is for the derivative of the toroidal magnetic flux inside a flux surface

$$\frac{\partial}{\partial t}\left(gV'\left\langle R^{-2}\right\rangle\right)+\frac{\partial}{\partial \psi}\left[gV'\left\langle R^{-2}\nabla\psi\bullet\mathbf{u}_R\right\rangle - V'\frac{\eta}{\mu_0}\left\langle\frac{1}{R^2}\nabla g\bullet\nabla\psi\right\rangle\right]=0 \qquad (*)$$

Equation (*) is for the derivative of the toroidal magnetic flux within a flux surface. If we choose the relative velocity $\mathbf{u}_R$ so as to make the time derivative vanish, we can identify the flux coordinate as the toroidal flux. $\psi \rightarrow \Phi$



Toroidal flux is the integral of the toroidal magnetic field over the area inside a surface

$$\mathbf{B}=\nabla\phi\times\nabla\Psi+g\left(\Psi\right)\nabla\phi$$

$$\begin{aligned}\Phi\left(\psi\right)&=\frac{1}{2\pi}\int\mathbf{B}\bullet\nabla\phi\, d\tau\\&=\frac{1}{2\pi}\int\nabla\bullet\left[\mathbf{B}\phi\right]d\tau\\&=\frac{1}{2\pi}\int\phi\mathbf{B}\bullet d\mathbf{S}\Big|_{\phi=0}^{\phi=2\pi}\\&=\int\mathbf{B}\bullet d\mathbf{S}\end{aligned}$$

$$\begin{aligned}\Phi\left(\psi\right)=\frac{1}{2\pi}\int\mathbf{B}\bullet\nabla\phi\, d\tau&=\int_0^{2\pi}d\theta\int_0^{\psi}d\psi\, JgR^{-2}\\&=\int_0^{\psi}d\psi\left(gV'\left\langle R^{-2}\right\rangle\right)\end{aligned}$$

There is a constraint on the relative velocity in that the coordinate ψ must remain a flux coordinate as time involves

$$\nabla\phi\times\nabla\psi\bullet\frac{\partial\Psi}{\partial t}=0$$

$$\nabla\phi\times\nabla\psi\bullet\left[\mathbf{u}_R\bullet\nabla\Psi-\frac{\eta}{\mu_0}\Delta^*\Psi\right]=0$$

$$\mathbf{u}_R\bullet\nabla\Psi-\frac{\eta}{\mu_0}\Delta^*\Psi=f(\psi)$$

Presently undetermined

We now determine this function by requiring that the flux coordinate ψ be the toroidal flux inside a surface

$$gV'\left\langle R^{-2}\nabla\psi\bullet\mathbf{u}_R\right\rangle-V'\frac{\eta}{\mu_0}\left\langle\frac{1}{R^2}\nabla g\bullet\nabla\psi\right\rangle=0$$

$$gV'\left\langle R^{-2}\right\rangle f(\psi)+gV'\frac{\eta}{\mu_0}\left\langle R^{-2}\Delta^*\Psi\right\rangle-V'\frac{\eta}{\mu_0}\left\langle\frac{1}{R^2}\nabla g\bullet\nabla\psi\right\rangle=0$$

$$f(\psi)=-\frac{gV'\frac{\eta}{\mu_0}\left\langle R^{-2}\Delta^*\Psi\right\rangle-V'\frac{\eta}{\mu_0}\left\langle\frac{1}{R^2}\nabla g\bullet\nabla\psi\right\rangle}{gV'\left\langle R^{-2}\right\rangle}=-\frac{\eta}{\mu_0}\frac{\left\langle g\nabla\bullet\frac{1}{gR^2}\nabla\Psi\right\rangle}{\left\langle R^{-2}\right\rangle}$$

Recall:

$$\mathbf{u}_R \bullet \nabla \Psi - \frac{\eta}{\mu_0} \Delta^* \Psi = f(\psi)$$

$$f(\psi) = -\frac{\eta}{\mu_0} \frac{\left\langle g \nabla \bullet \frac{1}{gR^2} \nabla \Psi \right\rangle}{\left\langle R^{-2} \right\rangle}$$

Put these together:

$$(**) \Rightarrow \mathbf{u}_R \bullet \nabla \Psi = \frac{\eta}{\mu_0} \left[\Delta^* \Psi - \frac{\left\langle g \nabla \bullet \frac{1}{gR^2} \nabla \Psi \right\rangle}{\left\langle R^{-2} \right\rangle} \right]$$

define:

$$\Gamma(\psi) \equiv \left\langle \mathbf{u}_R \bullet \nabla \Psi \right\rangle$$

$$\frac{1}{2\pi} V_L(\psi) \equiv \frac{\eta}{\mu_0} \frac{\left\langle g \nabla \bullet \frac{1}{gR^2} \nabla \Psi \right\rangle}{\left\langle R^{-2} \right\rangle}$$

Relative velocity is determined by the equations themselves once we identify the physical meaning of the flux coordinate.

This says that if the toroidal magnetic flux within a magnetic surface is used as the flux coordinate, this Equation (**) gives the fluid velocity relative to the flux surface velocity.

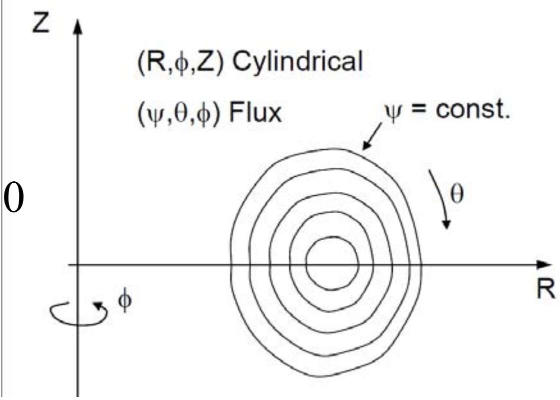
$$\psi \rightarrow \Phi$$

We can therefore eliminate this velocity from each of the previous equations for the density, pressure, poloidal flux.

$$\frac{\partial}{\partial t}(nV') + \frac{\partial}{\partial \psi}\left[nV'\langle \nabla\psi \cdot \mathbf{u}_R \rangle\right] = 0$$

$$\frac{\partial}{\partial t}\left(p^{3/5}V'\right) + \frac{\partial}{\partial \psi}\left[p^{3/5}V'\langle \nabla\psi \cdot \mathbf{u}_R \rangle\right] + \frac{2}{5}p^{-2/5}\left[\frac{\partial}{\partial \psi}\left(V'\langle \mathbf{q}\cdot\nabla\psi\rangle\right) - \eta V'\mathbf{J}^2\right] = 0$$

$$\frac{\partial \Psi}{\partial t} + \mathbf{u}_R \cdot \nabla\Psi = \frac{\eta}{\mu_0}\Delta^*\Psi$$



$$\frac{\partial}{\partial t}\left(gV'\langle R^{-2}\rangle\right) + \frac{\partial}{\partial \psi}\left[gV'\langle R^{-2}\nabla\psi \cdot \mathbf{u}_R\rangle - V'\frac{\eta}{\mu_0}\left\langle \frac{1}{R^2}\nabla g \cdot \nabla\psi \right\rangle\right] = 0$$

$$\psi \to \Phi$$

$$\Rightarrow \mathbf{u}_R \cdot \nabla\Psi = \frac{\eta}{\mu_0}\left[\Delta^*\Psi - \frac{\left\langle g\nabla \cdot \frac{1}{gR^2}\nabla\Psi \right\rangle}{\langle R^{-2}\rangle}\right]$$

$$\Gamma(\psi) \equiv \langle \mathbf{u}_R \cdot \nabla\Psi \rangle$$

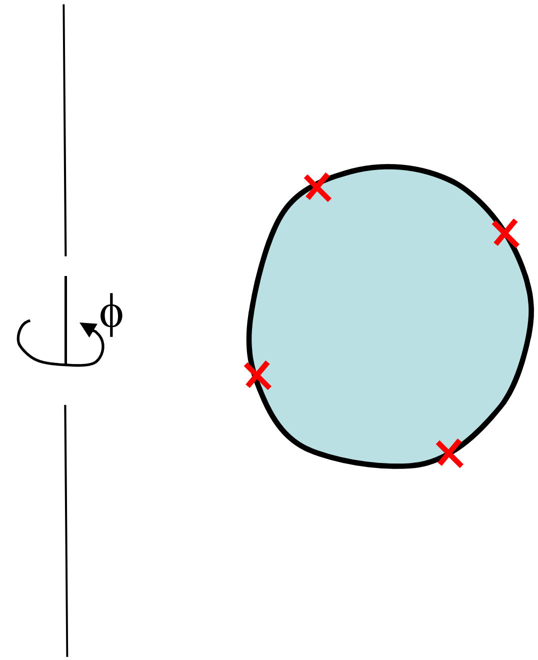
$$\frac{1}{2\pi}V_L(\psi) \equiv \frac{\eta}{\mu_0}\frac{\left\langle g\nabla \cdot \frac{1}{gR^2}\nabla\Psi \right\rangle}{\langle R^{-2}\rangle}$$

$$\frac{\partial}{\partial t}(nV') + \frac{\partial}{\partial \Phi}\left[nV'\Gamma\right] = 0$$

$$\frac{\partial}{\partial t}\left(p^{3/5}V'\right) + \frac{\partial}{\partial \Phi}\left[p^{3/5}V'\Gamma\right] + \frac{2}{5}p^{-2/5}\left[\frac{\partial}{\partial \Phi}\left(V'\langle \mathbf{q}\cdot\nabla\psi\rangle\right) - \eta V'\mathbf{J}^2\right] = 0$$

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2\pi}V_L$$

Just as the toroidal flux is the integral of the toroidal magnetic field over the area inside a surface, we can define the poloidal magnetic flux in a similar way.



Recall:

$$\Phi(\psi) = \frac{1}{2\pi}\int \mathbf{B}\bullet\nabla\phi\, d\tau = \int \mathbf{B}\bullet d\mathbf{S}$$

Similarly:

$$\Psi_{PF}(\psi) = \frac{1}{2\pi}\int \mathbf{B}\bullet\nabla\theta\, d\tau$$

$$= \frac{1}{2\pi}\int \nabla\phi\times\nabla\Psi\bullet\nabla\theta J\, d\psi\, d\theta\, d\phi$$

$$= 2\pi\Psi$$

Rotational Transform:

$$\iota(\psi) \equiv \frac{d\Psi_{PF}}{d\Phi} = \frac{1}{q(\psi)}$$

$$\boxed{\begin{gathered}\frac{\partial}{\partial t}(N') + \frac{\partial}{\partial \Phi}\left[N'\Gamma\right] = 0 \\ \frac{3}{2}(V')^{-2/3}\frac{\partial \sigma}{\partial t} + \frac{\partial Q}{\partial \Phi} = V_L\frac{\partial K}{\partial \Phi} \\ \frac{\partial \iota}{\partial t} = \frac{dV_L}{d\Phi}\end{gathered}}$$

These are the evolution equations for the 1D Adiabatic Variables. Note: time derivatives are zero if there is no dissipation : $(\eta = \mathbf{q} = 0)$

$$N' \equiv nV'$$

$$\sigma \equiv pV'^{5/3}$$

$$\iota(\psi) \equiv \frac{d\Psi_{PF}}{d\Phi}$$

$$\Rightarrow \Gamma(\Phi) = \frac{\eta}{\mu_0}\left[\left\langle \Delta^*\Psi\right\rangle - \frac{\left\langle g\nabla\bullet\frac{1}{gR^2}\nabla\Psi\right\rangle}{\left\langle R^{-2}\right\rangle}\right]$$

$$V_L(\psi) \equiv \frac{2\pi\eta}{\mu_0}\frac{\left\langle g\nabla\bullet\frac{1}{gR^2}\nabla\Psi\right\rangle}{\left\langle R^{-2}\right\rangle}$$

$$K \equiv \frac{V'}{(2\pi)^2\mu_0 q}\left\langle\frac{\left|\nabla\Phi\right|^2}{R^2}\right\rangle$$

$$Q \equiv V'\left[\left\langle q\bullet\nabla\Phi\right\rangle + \frac{5}{2}p\Gamma\right]$$

Just needs heat fluxes to close equations!

Also, $$\nabla p = \mathbf{J}\times\mathbf{B}$$

This is solved in a way that the adiabatic variables stay fixed during solution.

Grad-Hogan Method

H. Grad and J. Hogan, PRL, 24 1337 (1970)

$$\nabla p = \mathbf{J} \times \mathbf{B}$$

This must be solved in a way that the adiabatic variables stay fixed during solution.

$$\mathbf{B} = \nabla \phi \times \nabla \Psi + g(\Psi) \nabla \phi$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \Delta^* \Psi \nabla \phi + \nabla g \times \nabla \phi$$

$$\Delta^* \Psi + R^2 \frac{dp}{d\Psi} + g \frac{dg}{d\Psi} = 0$$

Grad-Shafranov Equation

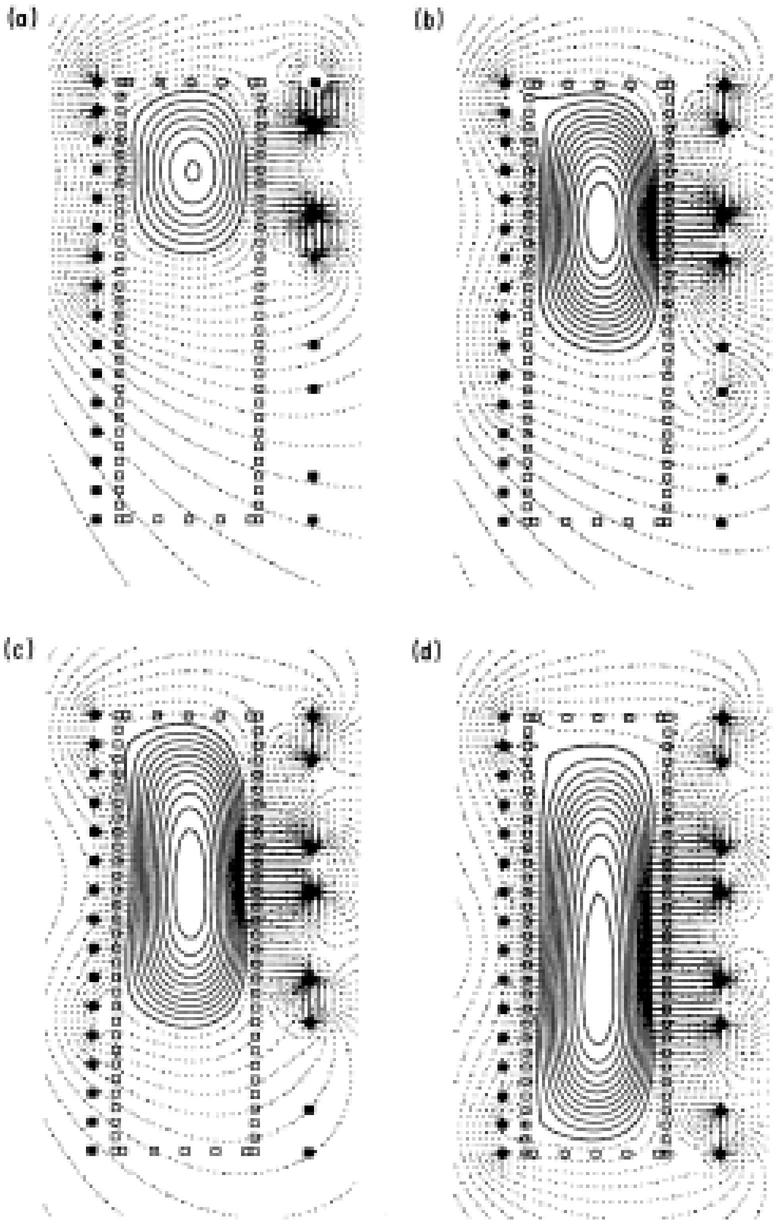
Must express p and g in terms of adiabatic variables:

$$g(\Psi) = \frac{q(\Psi)}{dV / d\Psi \left\langle R^{-2} \right\rangle} \qquad p(\Psi) = \sigma V'^{-5/3}$$

Alternate advancing the adiabatic variables in time, and then solving the equilibrium equation holding the adiabatic variables fixed.

Grad-Hogan Method

H. Grad and J. Hogan, PRL, 24 1337 (1970)



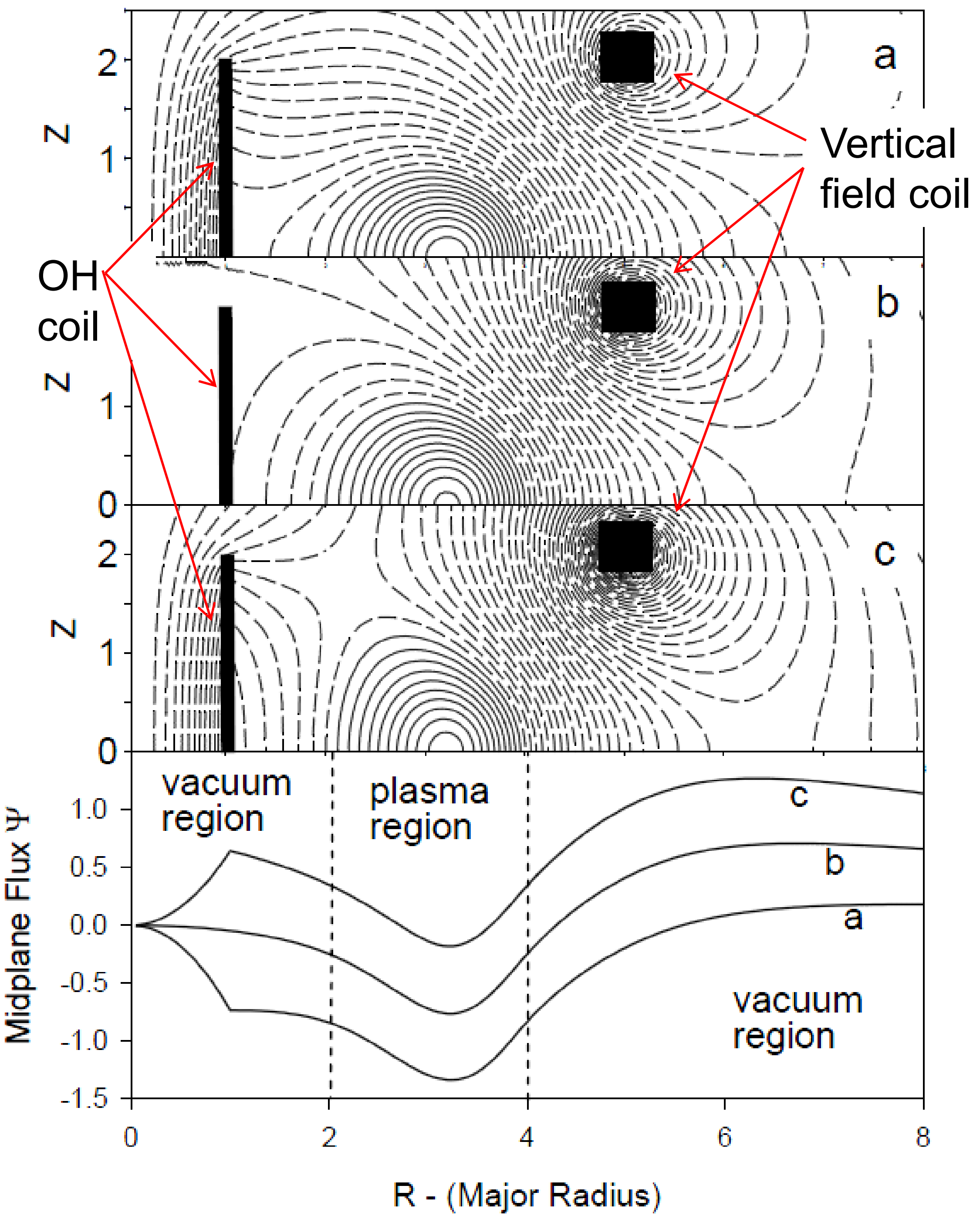
Time sequence of using the TSC code to model the evolution of a highly elongated plasma in the TCV tokamak.

At each instant of time, the vacuum vessel is providing stabilization on the fast (ideal MHD) time scale. The external coils are both feedback stabilizing the plasma and providing shaping fields as they slowly elongate it to fill the entire vessel.

In this case, there were 4 PID feedback systems corresponding to:

- Vertical position
- Radial position
- Elongation
- Squareness

Marcus, Jardin, and Hofmann, PRL, **55** 2289 (1985)



Codes can also accurately model the current drive action of the OH coils.

Simulation of flattop phase of a basic tokamak discharge.

(a) At start of flattop, OH coil has current in same direction as plasma current

(b) Flux in plasma uniformly increases due to resistive dissipation. OH and Vertical field coils adjust boundary values so flux gradient in plasma remains almost unchanged.

$$\mathbf{B_P} = \nabla \phi \times \nabla \Psi$$

(c) At end of flattop, OH coil has current in opposite direction as plasma current.

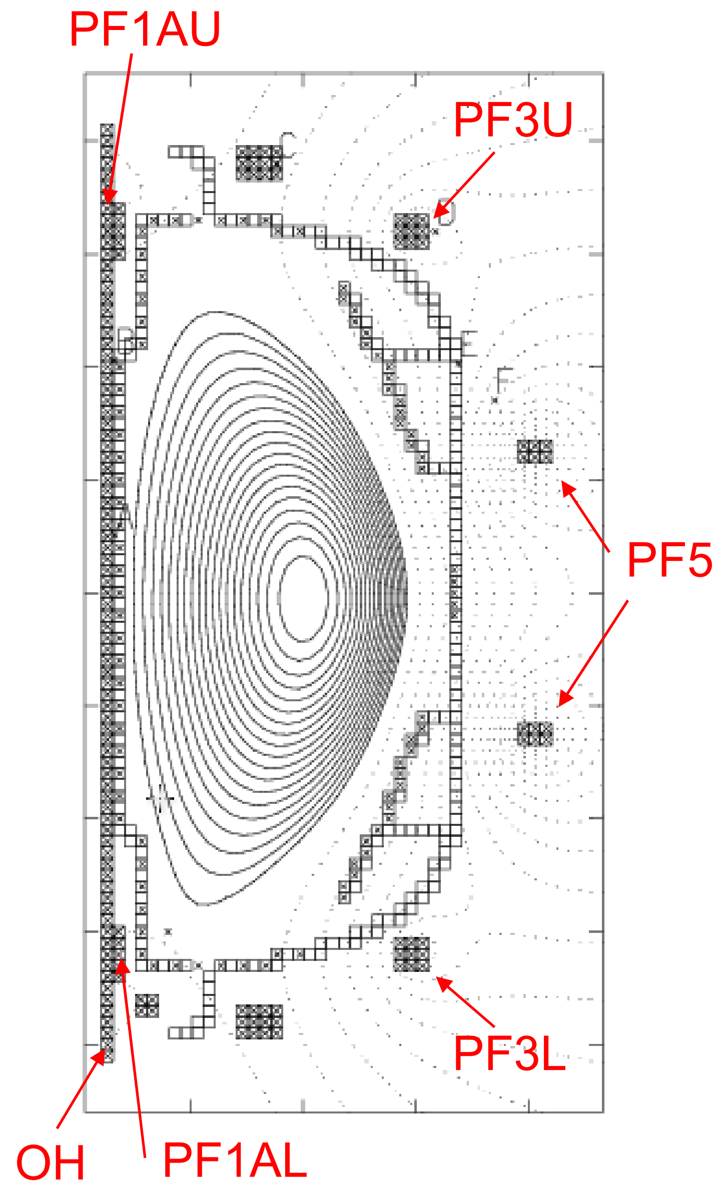
Simulation of NSTX discharge evolution

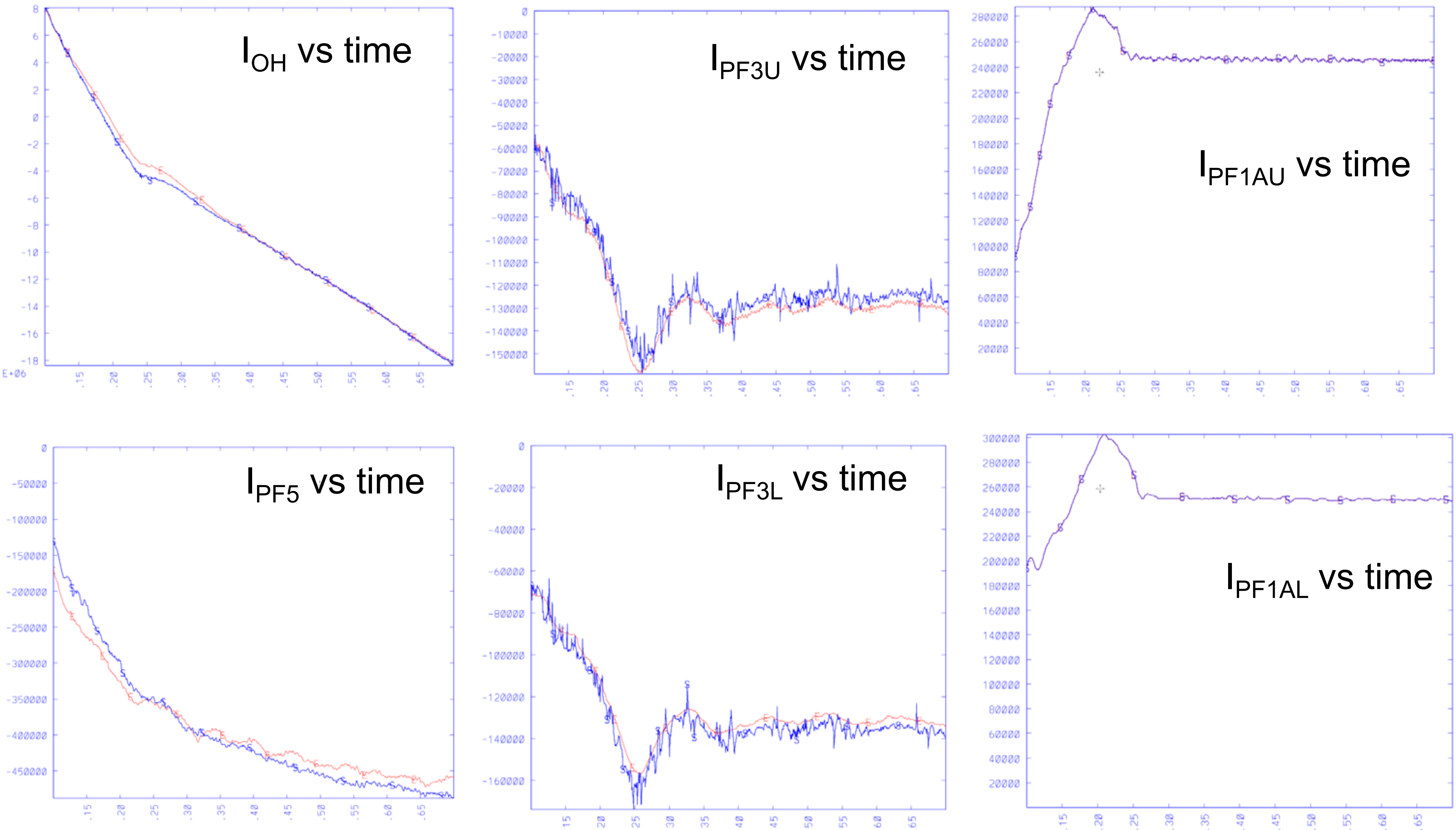
As a validation exercise, we have simulated the evolution of a NSTX discharge using the experimental values of the coil currents as the preprogrammed currents.

$$I_i(t) = I_{PP}(t) + I_{FB}(t)$$

To control the plasma in the simulation, several feedback systems need to be added to the coil groups. The "goodness" of the simulation is measured by how small the current in these feedback systems is to still match other measured quantities (such as the flux in flux loops).

In general, we find that if we can match the plasma density and temperature evolution, then we can predict the plasma current evolution very accurately.





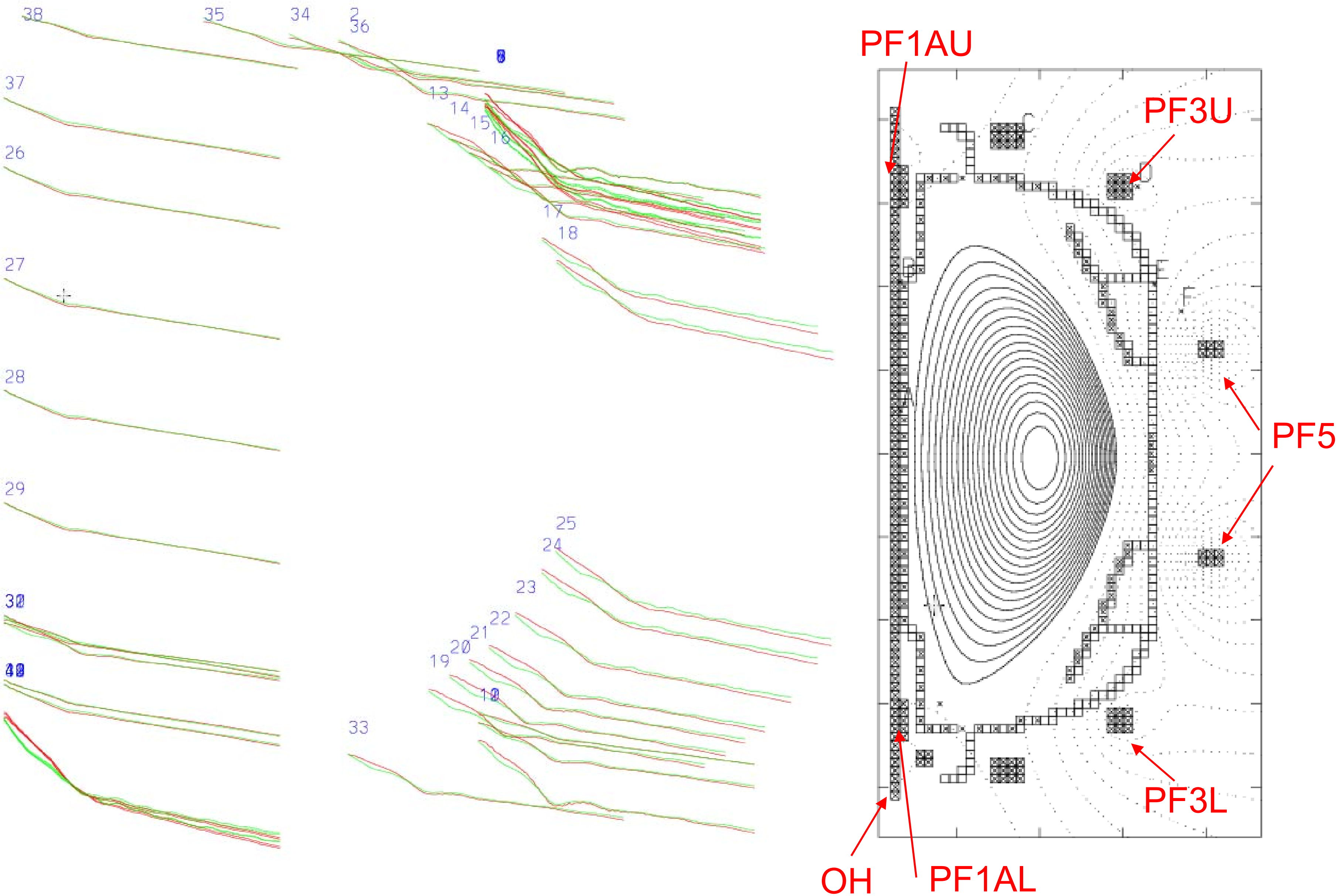
Simulation I_{OH} has feedback added to match experimental plasma current I_P

Simulation I_{PF3U} and I_{PF3L} have vertical stability feedback added

Simulation I_{PF5} have radial feedback system added

experiment

simulation



Red are simulation flux loop data and green are experimental data. Origin of each curve is approximate position of flux loop around machine.

Excellent agreement!

Summary

- Resistive time-scale dynamics arise both from dissipation in plasma (resistivity, thermal conductivity) and from resistance in nearby conductors
- On this timescale, plasma inertia can be ignored
- Equations can be written in a moving flux-coordinate system and averaged over the flux surfaces to give a set of 1D equations + time
- Dissipation coefficients only give relative motion of particles, energies, poloidal, and toroidal fluxes, so one can be chosen as reference
- Choosing the amount of toroidal flux within a surface as the reference flux coordinate is the most natural for tokamaks
- These equations evolve the 1D adiabatic variables in time. We also must solve the equilibrium equation in a way that the adiabatic variables stay fixed during the solution….. Grad-Hogan method

Jardin

Computational Methods in Plasma Physics

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K10512
ISBN: 978-1-4398-1021-7
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