MHD Simulations for Fusion Applications

Lecture 1

Tokamak Fusion Basics and the MHD Equations

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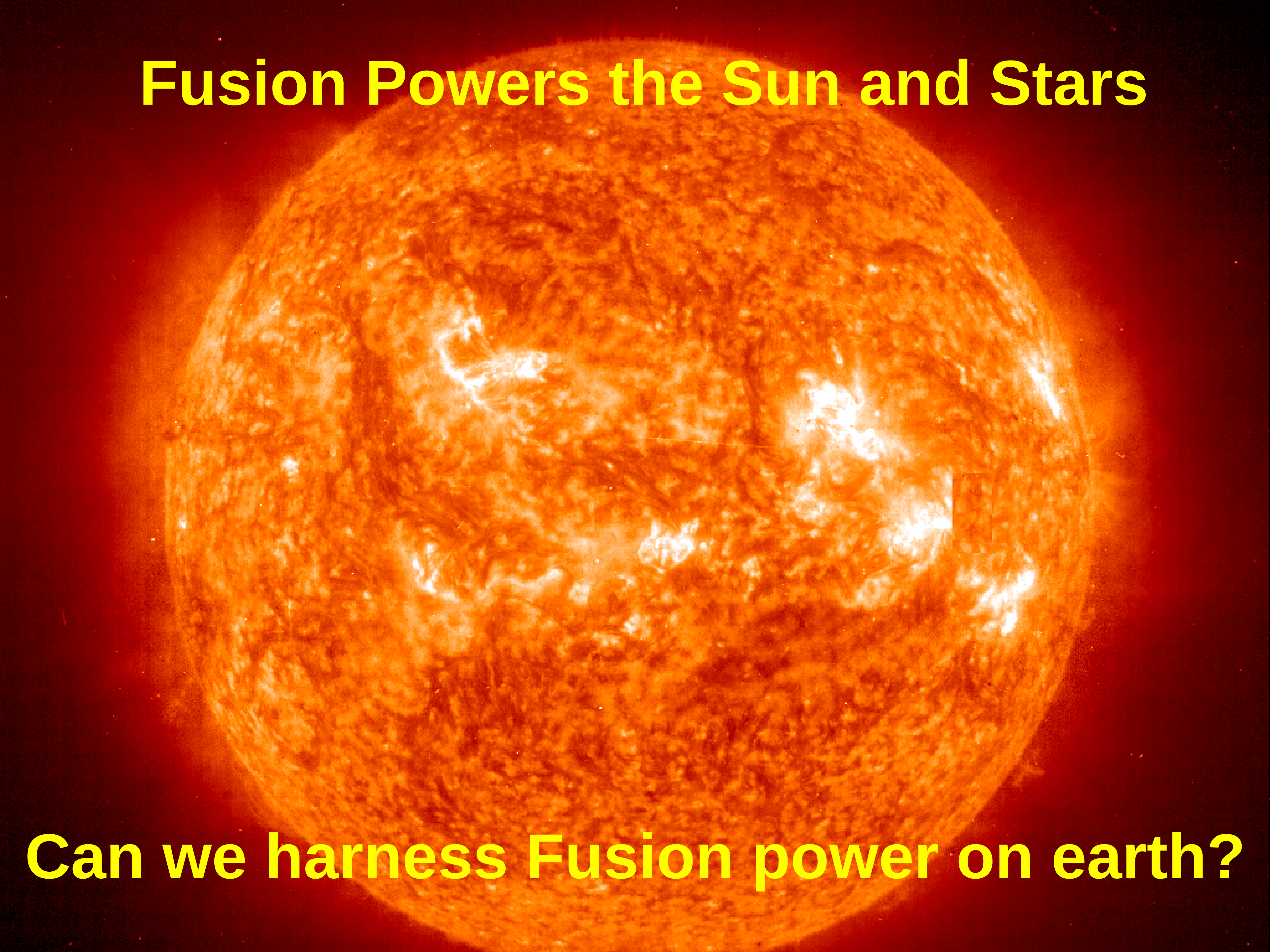
CEMRACS ‘10

Marseille, France

July 19, 2010

Fusion Powers the Sun and Stars

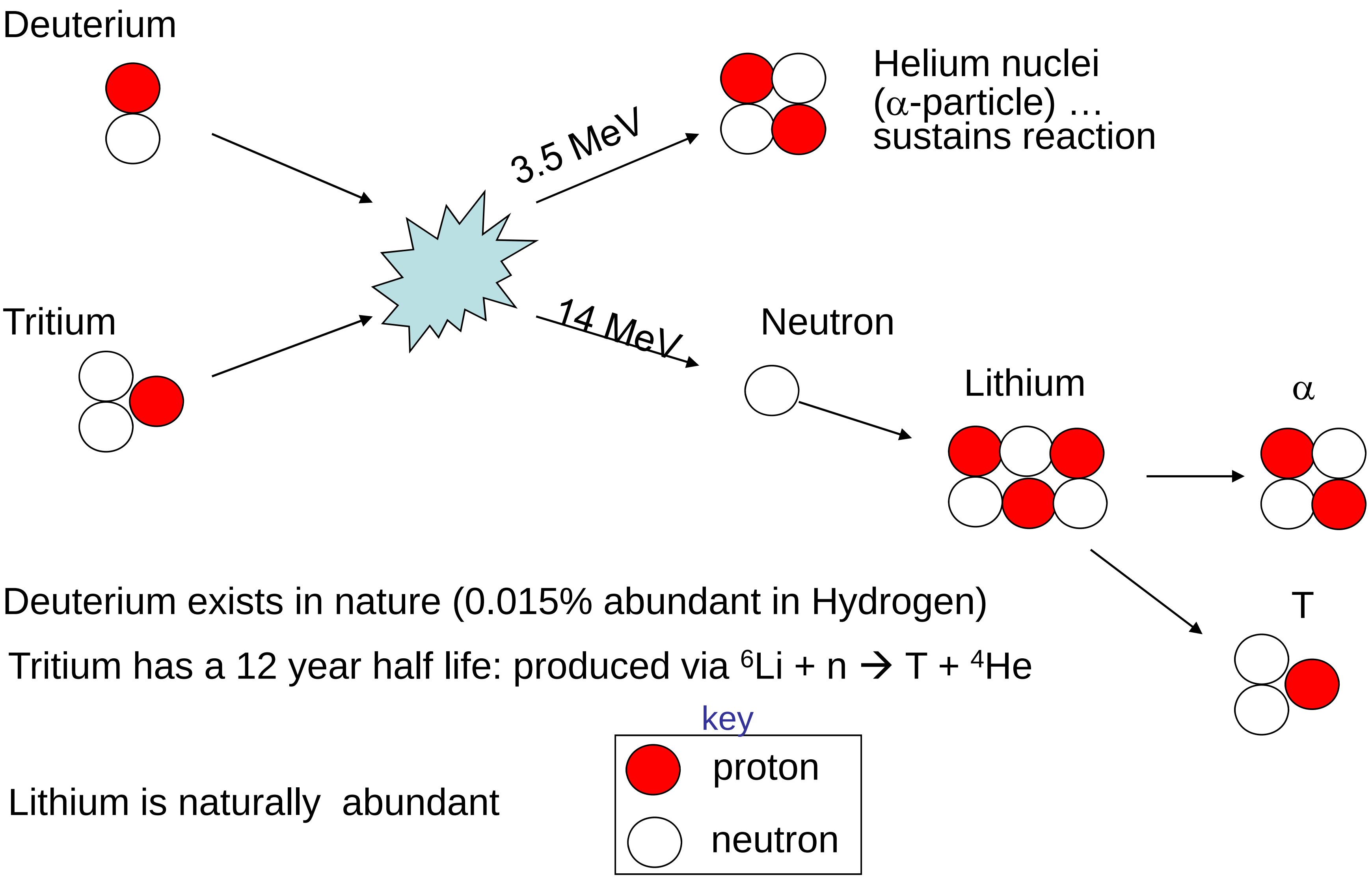


Can we harness Fusion power on earth?

The Case for Fusion Energy

- Worldwide demand for energy continues to increase
  - Due to population increases and economic development
  - Most population growth and energy demand is in urban areas
    - Implies need for large, centralized power generation
- Worldwide oil and gas production is near or past peak
  - Need for alternative source: coal, fission, fusion
- Increasing evidence that release of greenhouse gases is causing global climate change . . . “Global warming”
  - Historical data and 100+ year detailed climate projections
  - This makes nuclear (fission or fusion) preferable to fossil (coal)
- Fusion has some advantages over fission that could become critical:
  - Inherent safety (no China syndrome)
  - No weapons proliferation considerations (security)
  - Greatly reduced waste disposal problems (no Yucca Mt.)

Controlled Fusion uses isotopes of Hydrogen in a High Temperature Ionized Gas (Plasma)



Deuterium

Tritium

3.5 MeV

14 MeV

Helium nuclei
(α-particle) …
sustains reaction

Neutron

Lithium

α

T

key

● proton

○ neutron

Deuterium exists in nature (0.015% abundant in Hydrogen)

Tritium has a 12 year half life: produced via $^{6}\mathrm{Li} + \mathrm{n} \rightarrow \mathrm{T} + {}^{4}\mathrm{He}$

Lithium is naturally abundant

Controlled Fusion Basics

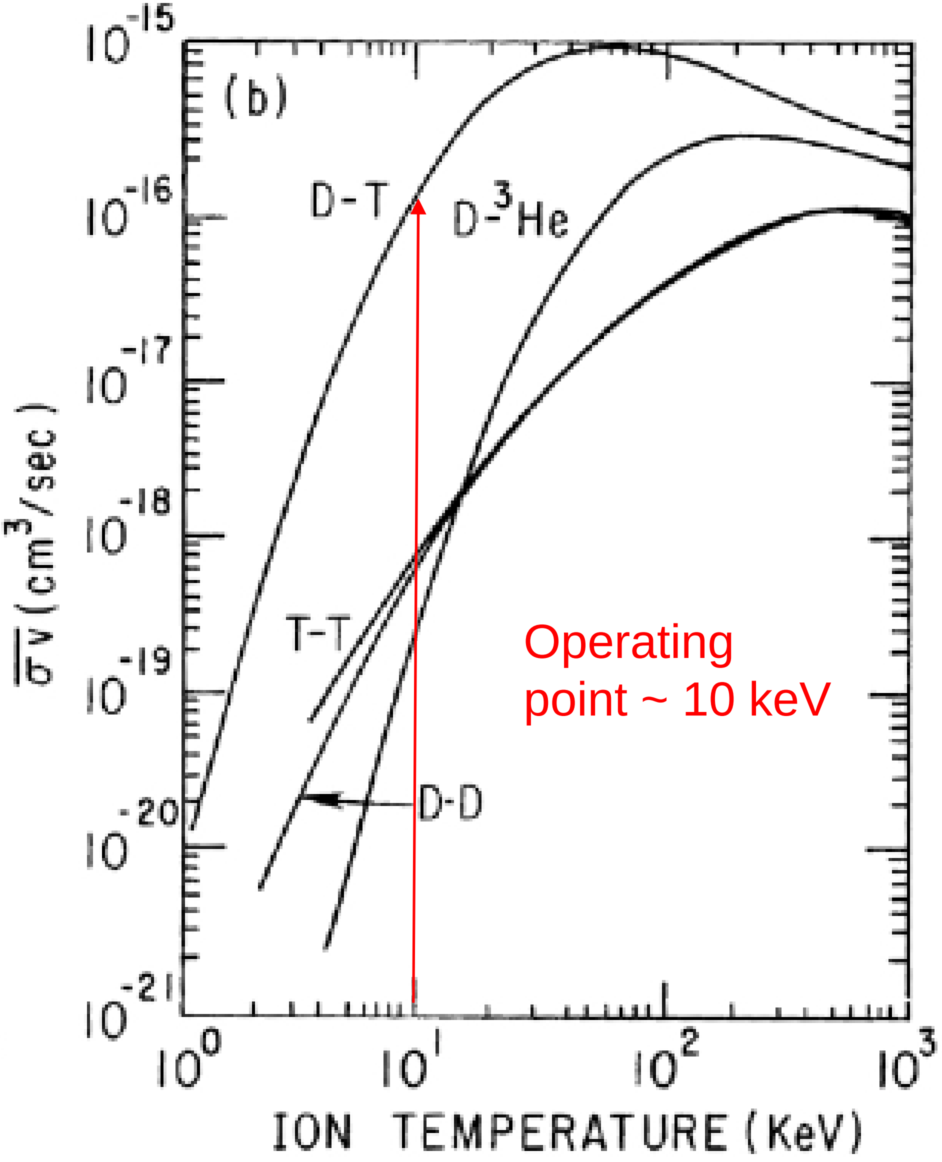
Create a mixture of D and T (plasma), heat it to high temperature, and the D and T will fuse to produce energy.

$P_{DT} = n_D n_T <\sigma v>(U_\alpha + U_n)$

at 10 keV, $<\sigma v> \sim T^2$

$P_{DT} \sim$ (plasma pressure)^2

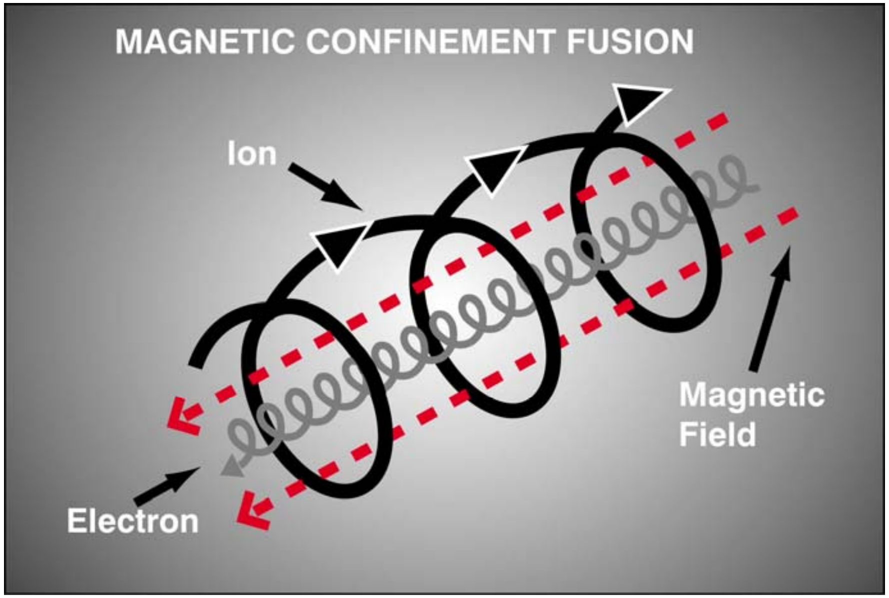
Need ~ 5 atmosphere @ 10 keV

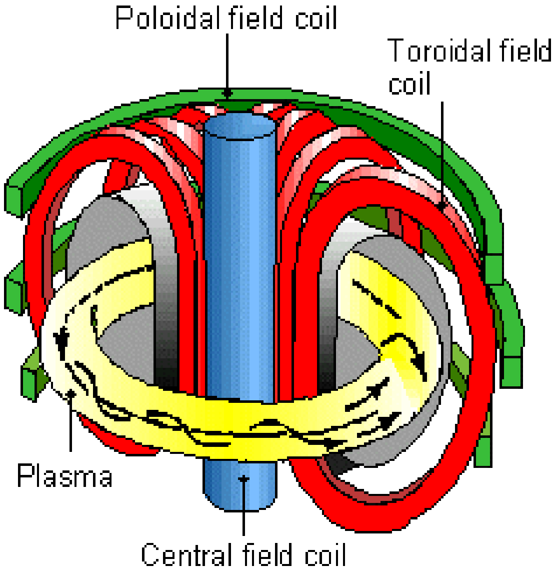


Operating point ~ 10 keV

Note: 1 keV = 10,000,000 deg(K)

Toroidal Magnetic Confinement



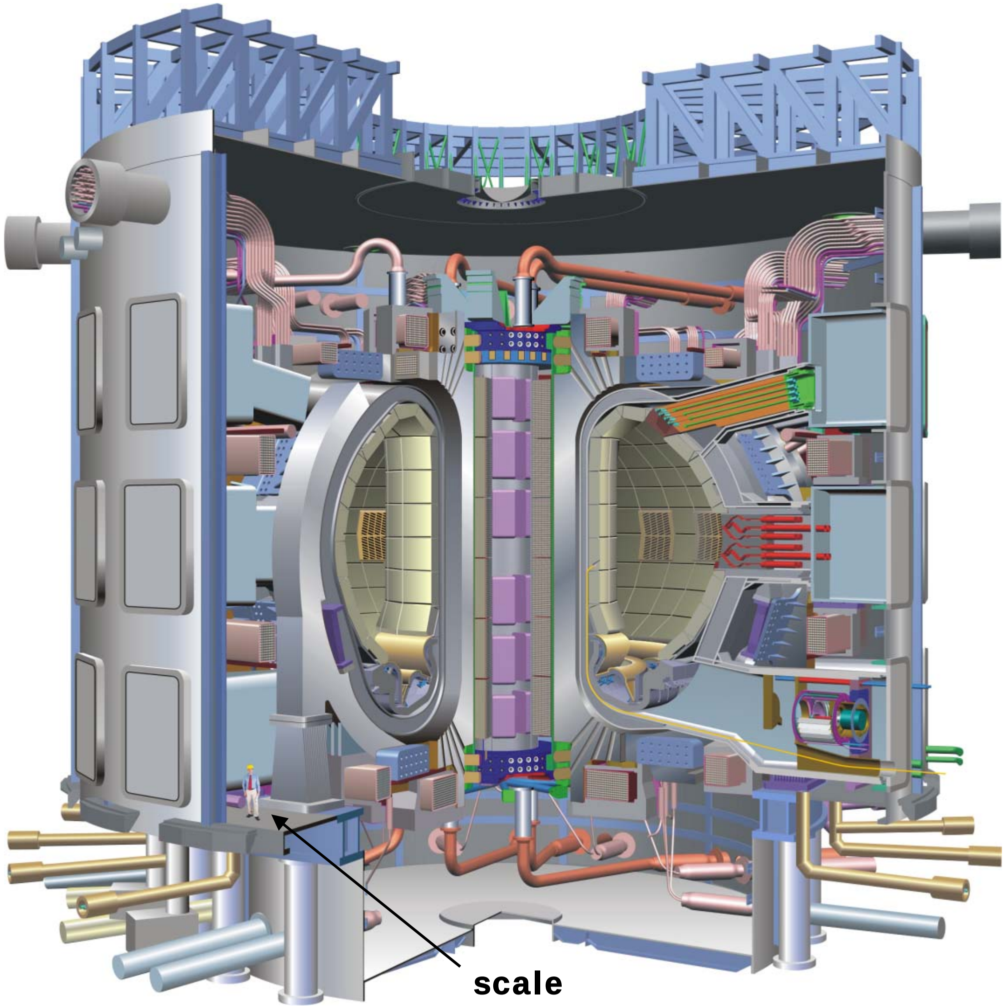


Charged particles have helical orbits in a magnetic field; they describe circular orbits perpendicular to the field and free-stream in the direction of the field.

TOKAMAK creates toroidal magnetic fields to confine particles in the 3rd dimension. Includes an induced toroidal plasma current to heat and confine the plasma

“TOKAMAK”: Russian abbreviation for “toroidal chamber”

ITER is now under construction



International
Thermonuclear
Experimental **R**eactor:

- European Union
- Japan
- United States
- Russia
- Korea
- China
- India

- World's largest tokamak
- all super-conducting coils

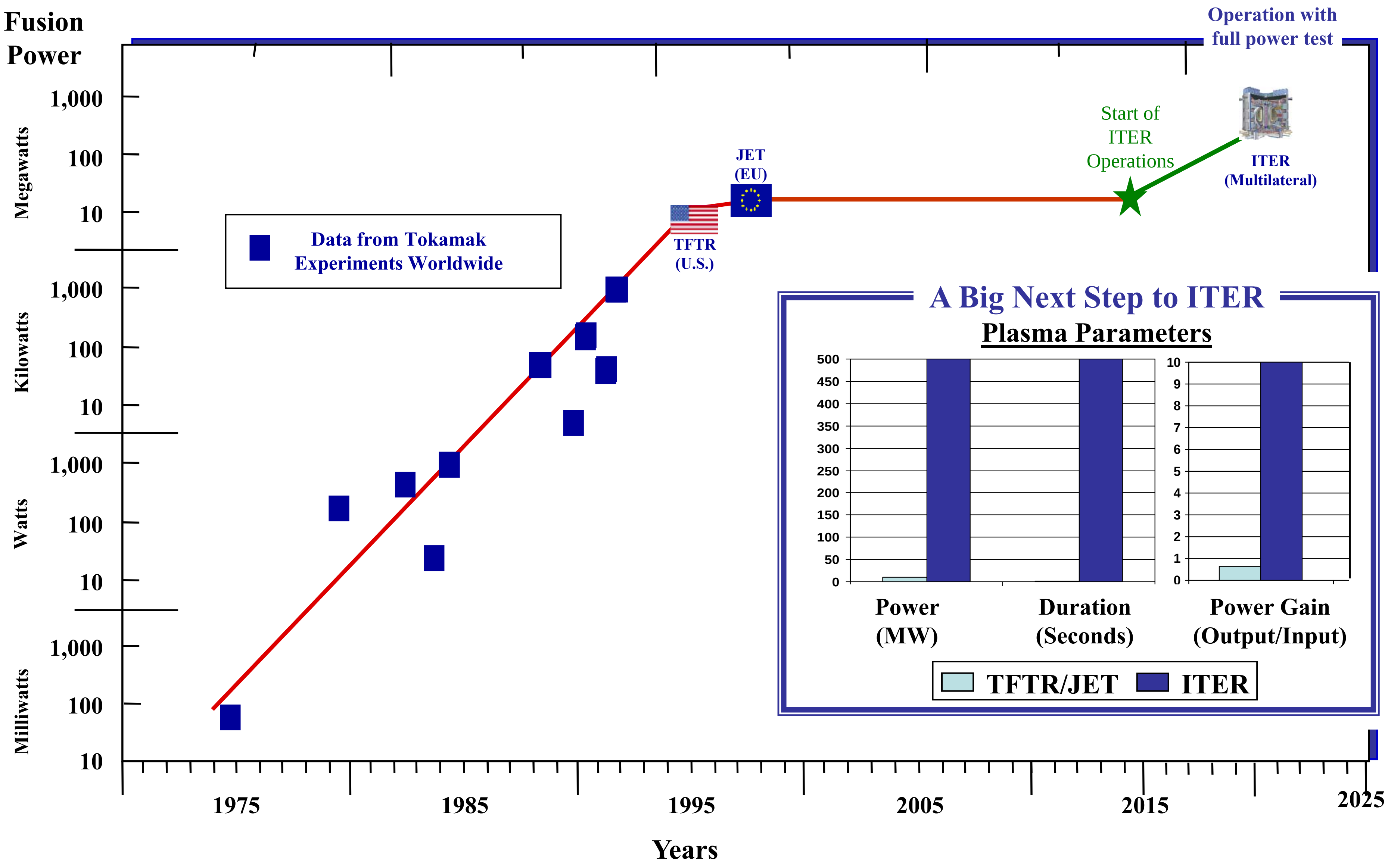
- 500 MW fusion output
- Cost: $ 5-10 B
- Originally to begin operation in 2015 (now 2028 full power)

ITER has a site…
Cadarache, France

June 28, 2005

Ministerial Level Meeting
Moscow, Russia

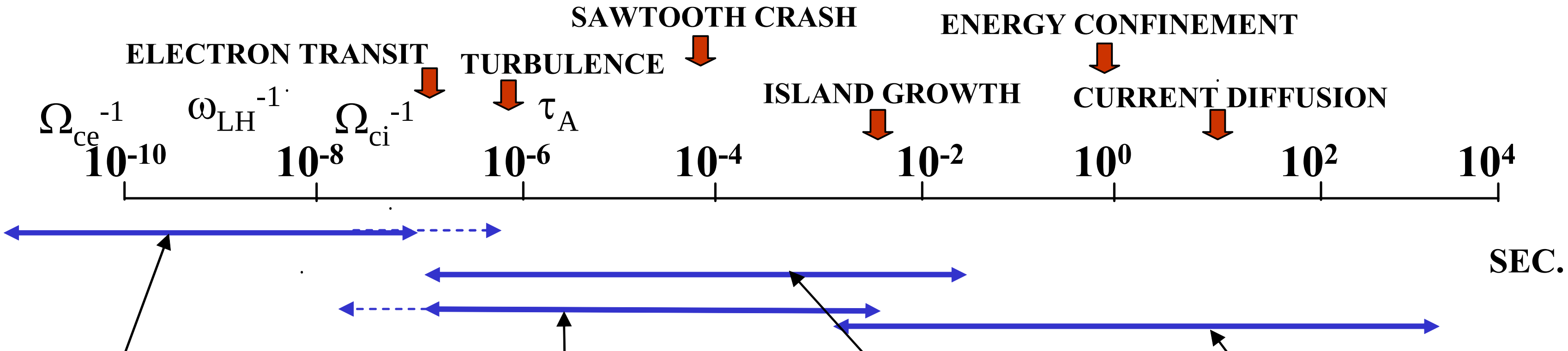
Progress in Magnetic Fusion Research and Next Step to ITER

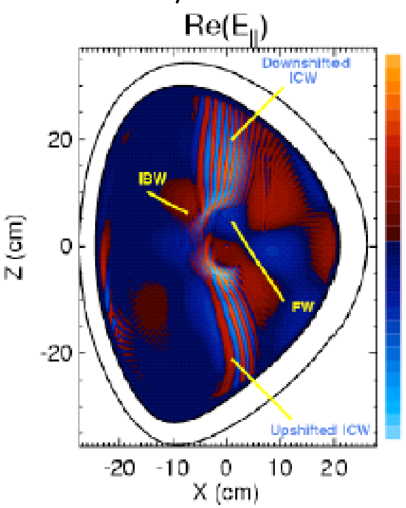


Simulations are needed in 4 areas

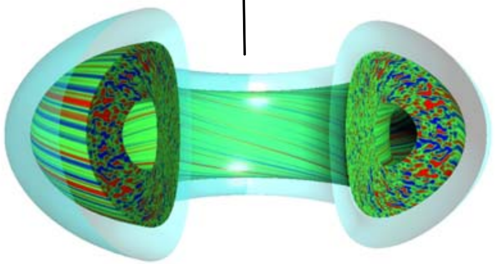
- How to heat the plasma to thermonuclear temperatures (~ 100,000,000^{o} C)
- How to reduce the background turbulence
- How to eliminate device-scale instabilities
- How to optimize the operation of the whole device

These 4 areas address different timescales and are normally studied using different codes

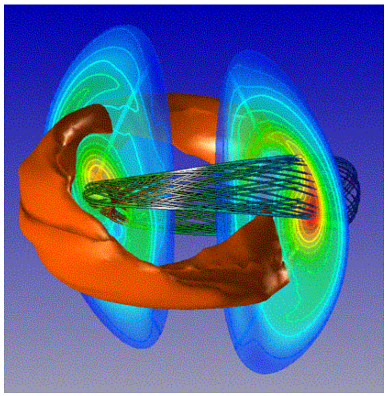




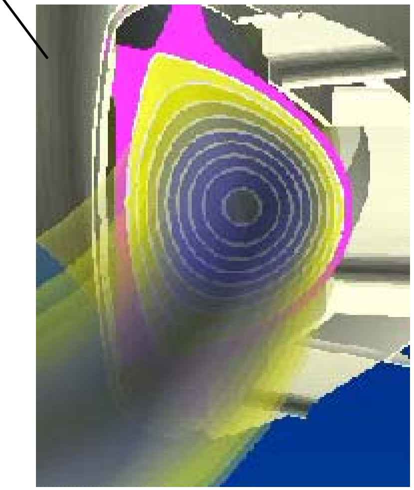
(a) RF codes



(b) Micro-turbulence codes

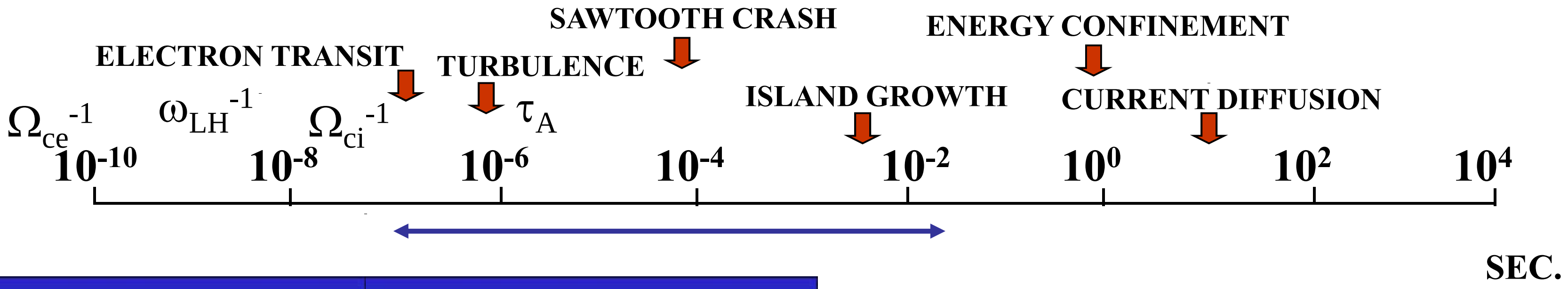


(c) Extended-MHD codes



(d) Transport Codes

Extended MHD Codes solve 3D fluid equations for device-scale stability



SAWTOOTH CRASH

ENERGY CONFINEMENT

ELECTRON TRANSIT

TURBULENCE

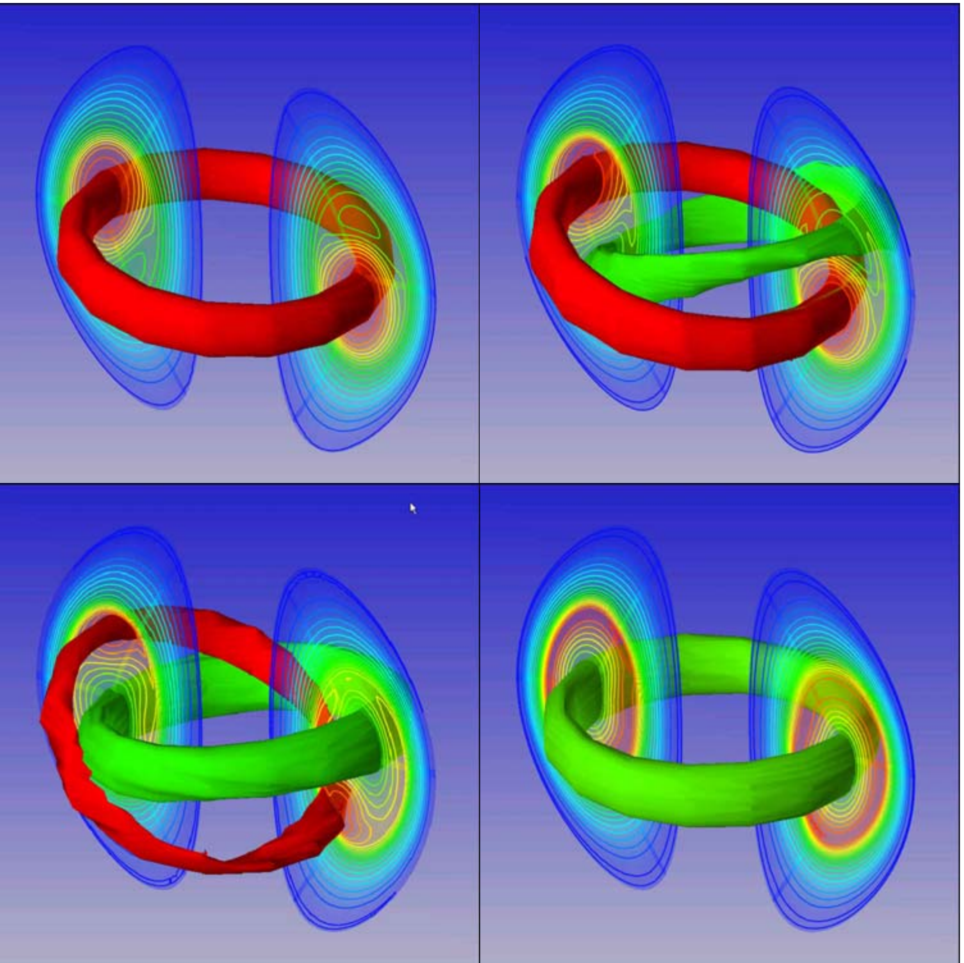
ISLAND GROWTH

CURRENT DIFFUSION

Ω_{ce}^{-1} ω_{LH}^{-1} Ω_{ci}^{-1} τ_A

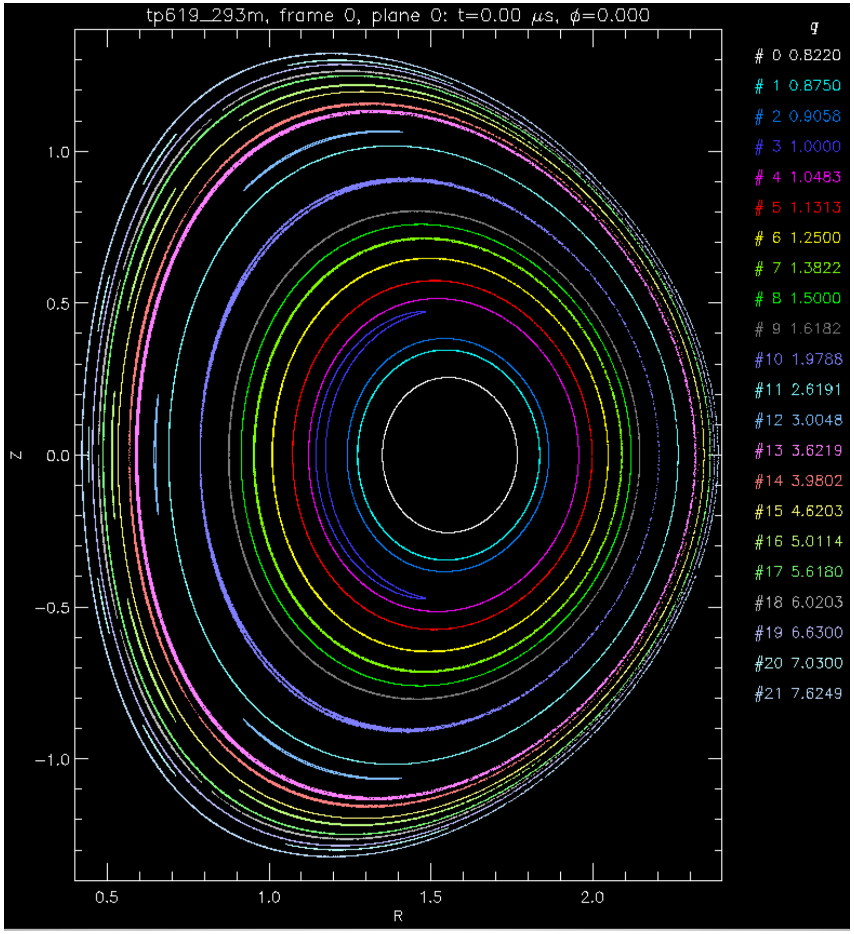
10^{-10} 10^{-8} 10^{-6} 10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4}

SEC.



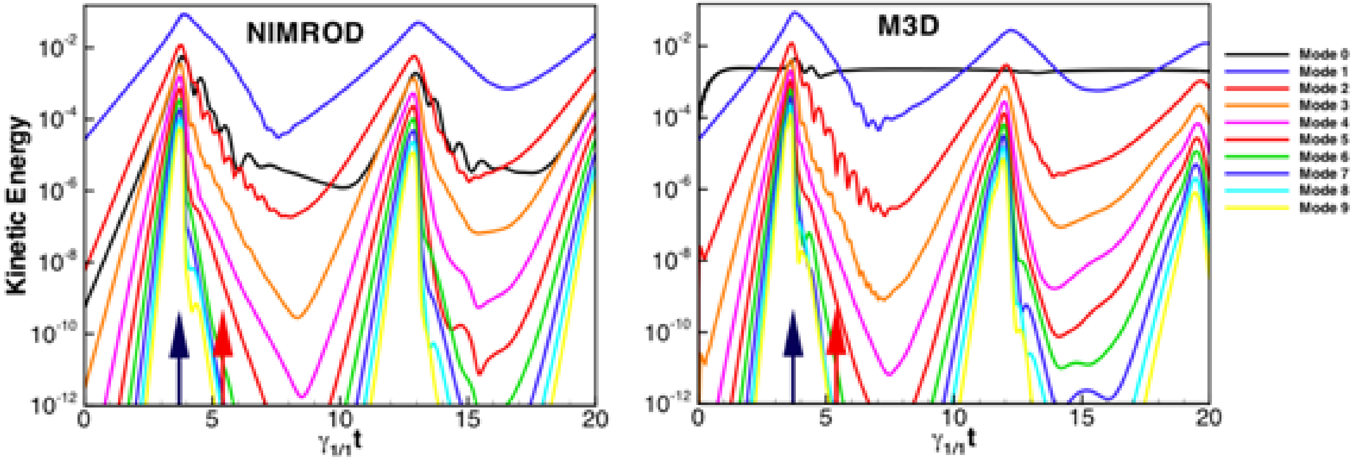
- Sawtooth cycle is one example of global phenomena that need to be understood
- Can cause degradation of confinement, or plasma termination if it couples with other modes
- There are several codes in the US and elsewhere that are being used to study this and related phenomena:
  - NIMROD
  - M3D

Quicktime Movie shows Poincare plot of magnetic field at one toroidal location

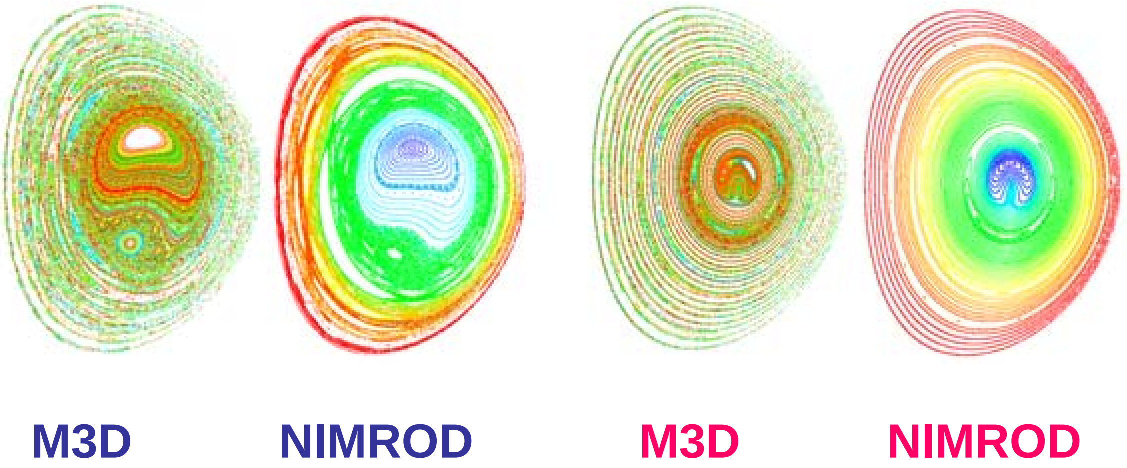


- Example of a recent 3D calculation using M3D code
- "Internal Kink" mode in a small tokamak (Sawtooth Oscillations)
- Good agreement between M3D, NIMROD, and experimental results
- 500 wallclock hours and over 200,000 CPU-hours

Excellent Agreement between NIMROD and M3D



Kinetic energy vs time in lowest toroidal harmonics



Flux
Surfaces
during crash
at 2 times

2-Fluid MHD Equations:

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

continuity

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla\bullet\mathbf{B} = 0 \qquad \mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

Maxwell

$$nM_i\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B} - \nabla\bullet\mathbf{\Pi}_{GV} + \mu \nabla^2 \mathbf{V}$$

momentum

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne}\left(\mathbf{J} \times \mathbf{B} - \nabla p_e\right)$$

Ohm's law

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla\bullet\left(\frac{3}{2} p_e \mathbf{V}\right) = -p_e \nabla\bullet\mathbf{V} + \eta J^2 - \nabla\bullet\mathbf{q}_e + Q_\Delta$$

electron energy

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla\bullet\left(\frac{3}{2} p_i \mathbf{V}\right) = -p_i \nabla\bullet\mathbf{V} + \mu \left|\nabla V\right|^2 - \nabla\bullet\mathbf{q}_i - Q_\Delta$$

ion energy

Ideal MHD

Resistive MHD

2-fluid MHD

n	number density	$\mathbf{V}$	fluid velocity	μ	viscosity
$\mathbf{B}$	magnetic field	p_e	electron pressure	η	resistivity
$\mathbf{J}$	current density	p_i	ion pressure	$\mathbf{q_i}, \mathbf{q_e}$	heat fluxes
$\mathbf{E}$	electric field	$p \equiv p_e + p_i$		Q_Δ	equipartition
$nM_i \equiv \rho$	mass density	e	electron charge	μ_0	permeability

Ideal MHD Equations:

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\mathbf{V}) = 0$$

continuity

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla \bullet \mathbf{B} = 0 \qquad \mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

Maxwell

$$nM_i \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B}$$

momentum

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$$

Ohm's law

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\frac{3}{2} p\mathbf{V} \right) = -p \nabla \bullet \mathbf{V}$$

energy

Ideal MHD

n	number density	$\mathbf{V}$	fluid velocity
$\mathbf{B}$	magnetic field	p_e	electron pressure
$\mathbf{J}$	current density	p_i	ion pressure
$\mathbf{E}$	electric field	$p \equiv p_e + p_i$	
$nM_i \equiv \rho$	mass density		

μ_0 permeability

Ideal MHD Equations:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0 \qquad \text{continuity}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \nabla \bullet \mathbf{B} = 0 \qquad \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \qquad \text{Maxwell}$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}\right) + \nabla p = \mathbf{J} \times \mathbf{B} \qquad \text{momentum}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \qquad \text{Ohm's law}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\frac{3}{2} p \mathbf{V}\right) = -p \nabla \bullet \mathbf{V} \qquad \text{energy}$$

$$s \equiv p \rho^{-5/3} \quad \Rightarrow \quad \frac{\partial s}{\partial t} + \mathbf{V} \bullet \nabla s = 0 \qquad \text{entropy}$$

Ideal MHD	n	number density	$\mathbf{V}$	fluid velocity	$\mathbf{E},\mathbf{J}$	can be eliminated
	$\mathbf{B}$	magnetic field	p_e	electron pressure	$\partial \rho / \partial t$	is redundant
	$\mathbf{J}$	current density	p_i	ion pressure	$\nabla \bullet \mathbf{B}$	is redundant
	$\mathbf{E}$	electric field	$p \equiv p_e + p_i$		μ_0	permeability
	$nM_i \equiv \rho$	mass density				

Ideal MHD Equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V} \times \mathbf{B} \right)$$

$$\rho(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \bullet \nabla \mathbf{V}) + \nabla p = \frac{1}{\mu_0} \left(\nabla \times \mathbf{B} \right) \times \mathbf{B}$$

$$\frac{3}{2} \frac{\partial p}{\partial t} + \nabla \bullet \left(\frac{3}{2} p \mathbf{V} \right) = - p \nabla \bullet \mathbf{V}$$

$$\frac{\partial s}{\partial t} + \mathbf{V} \bullet \nabla s = 0$$

$$\rho = \left(p / s \right)^{3/5}$$

ρ	mass density
$\mathbf{B}$	magnetic field
$\mathbf{V}$	fluid velocity
s	entropy density
p	fluid pressure

$\nabla \bullet \mathbf{B}$ is redundant

μ_0 permeability

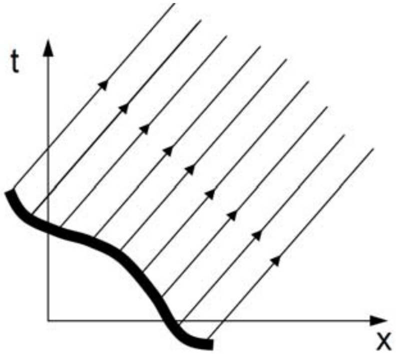
Quasi-linear
Symmetric → real characteristics
Hyperbolic

Ideal MHD characteristics:

The characteristic curves are the surfaces along which the solution is propagated. In 1D, the characteristic curves would be lines in (x,t)

Boundary data (normally IC and BC) can be given on any curve that each characteristic curve intersects only once:

$$\frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}=0$$



➔ Cannot be tangent to characteristic curve

To calculate characteristics in 3D, we suppose that the boundary conditions are given on a 3D surface $\phi(\mathbf{r},t)=\phi_0$ and ask under what conditions this is *insufficient* to determine the solution away from this surface. If so, ϕ is a characteristic surface.

Perform a coordinate transformation: $(\mathbf{r},t)\rightarrow(\phi,\chi,\sigma,\tau)$ and look for power series solution away from the boundary surface $\phi=\phi_0$

$$\mathbf{v}(\phi,\chi,\sigma,\tau)=\mathbf{v}_0(\chi,\sigma,\tau)+(\phi-\phi_0)\frac{\partial\mathbf{v}}{\partial\phi}\bigg|_{\phi_0}+\underbrace{(\chi-\chi_0)\frac{\partial\mathbf{v}}{\partial\chi}\bigg|_{\phi_0}+(\sigma-\sigma_0)\frac{\partial\mathbf{v}}{\partial\sigma}\bigg|_{\phi_0}+(\tau-\tau_0)\frac{\partial\mathbf{v}}{\partial\tau}\bigg|_{\phi_0}}$$

If this cannot be constructed, then ϕ is a characteristic surface

These can all be calculated since they are surface derivatives within $\phi=\phi_0$

Ideal MHD characteristics-2:

Introduce a characteristic surface $\phi(\mathbf{r},t) \;=\; \phi_0$

$\mathbf{B}=\left(0,0,B\right)$ **B** is in $\hat{z}$ direction

spatial normal $\hat{\mathbf{n}} = \nabla\phi \,/\, \left|\nabla\phi\right|$

$\hat{\mathbf{n}}=\left(n_x,0,n_z\right)$ propagation in (x,z)

characteristic speed: $u \equiv \left(\phi_t + \mathbf{V}\bullet\nabla\phi\right)/\left|\nabla\phi\right|$

$$V_A \equiv B/\sqrt{\mu_0\rho}$$

$$c_S \equiv \sqrt{\tfrac{5}{3}\,p/\rho}$$

$$(\;)' = \frac{\partial}{\partial\phi}(\;)$$

Ideal MHD → $\mathbf{A}\bullet\mathbf{X} = \cdots$

All terms containing derivatives involving ϕ

All known quantities

$$\mathbf{A}=\begin{bmatrix} -u & 0 & 0 & -n_zV_A & 0 & n_xV_A & n_xc_S & 0 \\ 0 & -u & 0 & 0 & -n_zV_A & 0 & 0 & 0 \\ 0 & 0 & -u & 0 & 0 & 0 & n_zc_S & 0 \\ -n_zV_A & 0 & 0 & -u & 0 & 0 & 0 & 0 \\ 0 & -n_zV_A & 0 & 0 & -u & 0 & 0 & 0 \\ n_xV_A & 0 & 0 & 0 & 0 & -u & 0 & 0 \\ n_xc_S & 0 & n_zc_S & 0 & 0 & 0 & -u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u \end{bmatrix} \qquad \mathbf{X}=\begin{bmatrix} \rho V_x' \\ \rho V_y' \\ \rho V_z' \\ \sqrt{\rho/\mu_0}\,B_x' \\ \sqrt{\rho/\mu_0}\,B_y' \\ \sqrt{\rho/\mu_0}\,B_z' \\ c_S^{-1}p' \\ s' \end{bmatrix}$$

if det $\mathbf{A}$ = 0 $\;\rightarrow \phi$ is characteristic surface

Ideal MHD wave speeds:

$$\det \mathbf{A} = 0$$

$$D = u^2\left(u^2 - V_{An}^2\right)\left[u^4 - \left(V_A^2 + c_S^2\right)u^2 + V_{An}^2 c_S^2\right] = 0$$

$$\mathbf{B} = \left(0, 0, B\right)$$

$$\hat{\mathbf{n}} = \left(n_x, 0, n_z\right)$$

$$V_A \equiv B / \sqrt{\mu_0 \rho}$$

$$c_S \equiv \sqrt{\tfrac{5}{3} p / \rho}$$

$$V_{An}^2 \equiv n_Z^2 V_A^2$$

$u^2 = u_0^2 = 0$ entropy disturbance

$u^2 = u_A^2 = V_{An}$ Alfven wave

$u^2 = u_s^2 = \tfrac{1}{2}\left(V_A^2 + c_S^2\right) - \tfrac{1}{2}\left[\left(V_A^2 + c_S^2\right)^2 - 4V_{An}^2 c_S^2\right]^{1/2}$ slow wave

$u^2 = u_f^2 = \tfrac{1}{2}\left(V_A^2 + c_S^2\right) + \tfrac{1}{2}\left[\left(V_A^2 + c_S^2\right)^2 - 4V_{An}^2 c_S^2\right]^{1/2}$ fast wave

In normal magnetically confined plasmas, we take the low-β limit $c_S^2 \ll V_A^2$

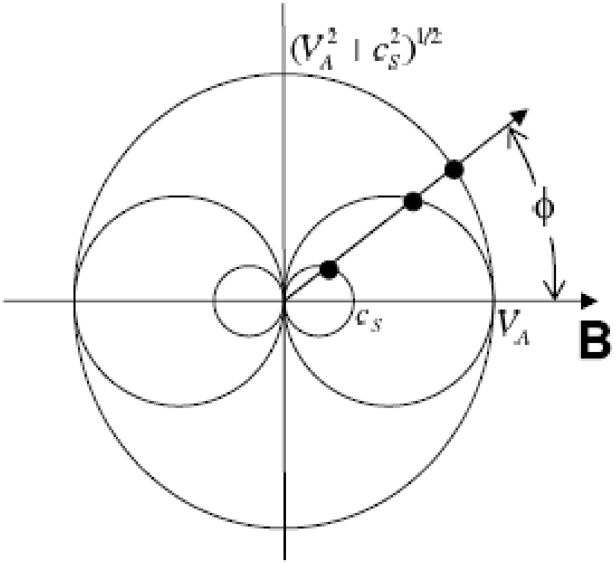
⇒

$u^2 = u_A^2 = V_{An}$ Alfven wave

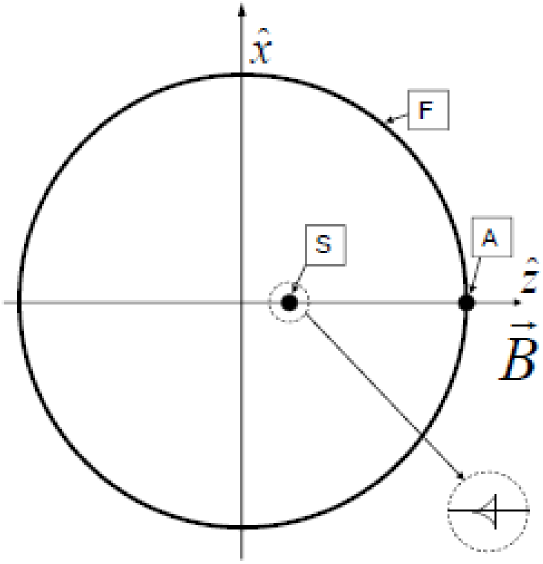
$u^2 = u_s^2 \simeq n_z^2 c_S^2$ slow wave

$u^2 = u_f^2 \simeq V_A^2 + n_x^2 c_S^2$ fast wave

Ideal MHD surface diagrams



Reciprocal normal surface diagram



Ray surface diagram

$$u^2 = u_A^2 = V_{An} \quad \text{Alfven wave}$$

$$u^2 = u_s^2 \simeq n_z^2 c_S^2 \quad \text{slow wave}$$

$$u^2 = u_f^2 \simeq V_A^2 + n_x^2 c_S^2 \quad \text{fast wave}$$

$$\mathbf{B} = (0, 0, B)$$

$$\hat{\mathbf{n}} = (n_x, 0, n_z)$$

$$V_A \equiv B / \sqrt{\mu_0 \rho}$$

$$c_S \equiv \sqrt{\tfrac{5}{3} p / \rho}$$

$$V_{An}^2 \equiv n_Z^2 V_A^2$$

Ideal MHD eigenvectors

$$\mathbf{B} = (0,0,B)$$

$$\hat{\mathbf{n}} = (n_x, 0, n_z)$$

$$V_A \equiv B / \sqrt{\mu_0 \rho}$$

$$c_S \equiv \sqrt{\tfrac{5}{3} p / \rho}$$

$$V_{An}^2 \equiv n_Z^2 V_A^2$$

$$\mathbf{X} = \begin{bmatrix} \rho V_x' \\ \rho V_y' \\ \rho V_z' \\ \sqrt{\rho/\mu_0}\, B_x' \\ \sqrt{\rho/\mu_0}\, B_y' \\ \sqrt{\rho/\mu_0}\, B_z' \\ c_S^{-1} p' \\ s' \end{bmatrix} = \begin{array}{ccccc} \text{entropy} & \text{Alfven} & \begin{array}{c}\text{fast}\\ n_x = 1\\ n_z = 0\end{array} & \begin{array}{c}\text{fast}\\ n_x = 0\\ n_z = 1\end{array} & \begin{array}{c}\text{slow}\\ n_x = 0\\ n_z = 1\end{array} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \pm 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ c_S / V_A \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ \pm 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \pm 1 \\ 0 \end{bmatrix} \end{array}$$

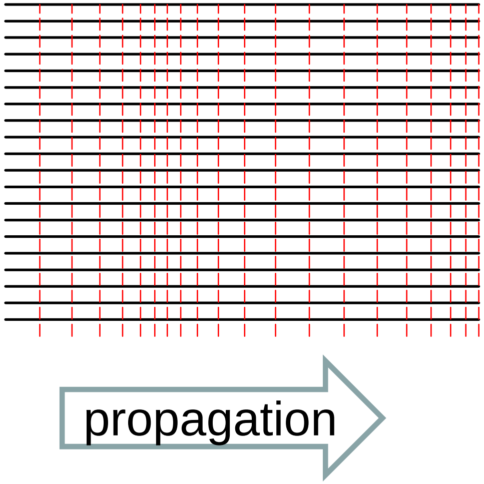
The Alfven wave only propagates parallel to the magnetic field, and does so by bending the field. It is purely transverse (incompressible)

Only the fast wave can propagate perpendicular to the background field, and does so by compressing and expanding the field

The slow wave does not perturb the magnetic field, only the pressure

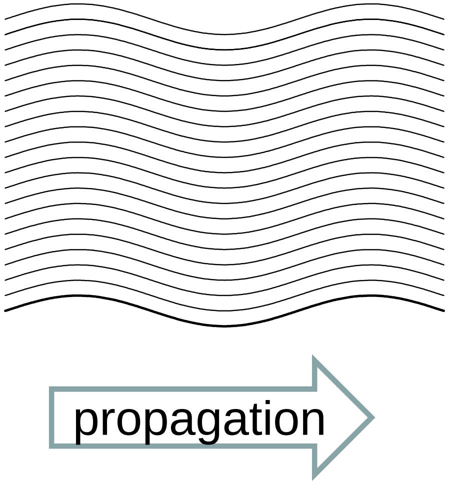
Background magnetic field direction

Slow Wave



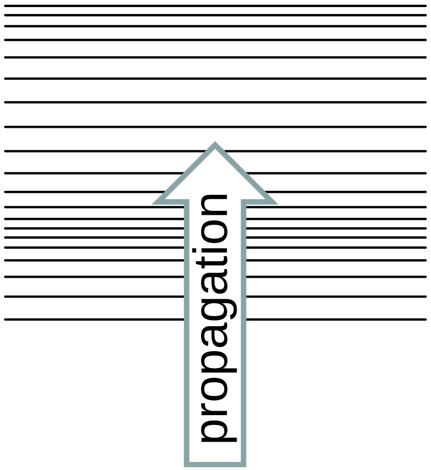
- only propagates parallel to **B**
- only compresses fluid in parallel direction
- does not perturb magnetic field

Alfven Wave



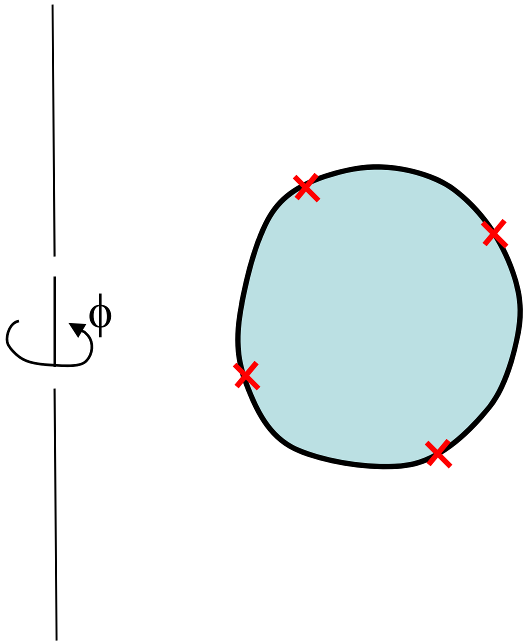
- only propagates parallel to **B**
- incompressible
- only bends the field, does not compress it

Fast Wave



- can propagate perpendicular to **B**
- only compresses fluid in ⊥ direction
- compresses the magnetic field
- **This is the troublesome wave!**

Tokamaks have Magnetic Surfaces, or Flux Surfaces

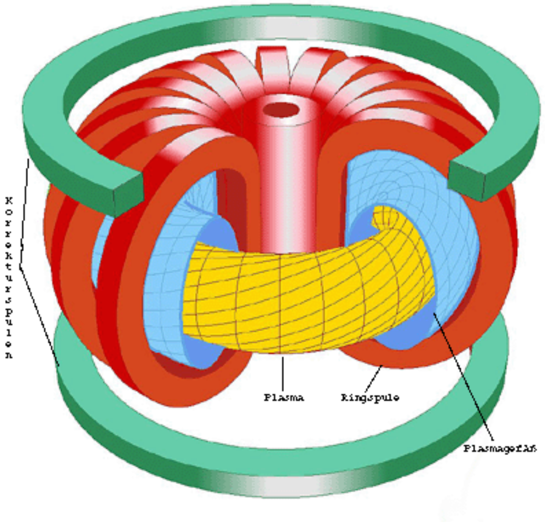


Magnetic field is primarily into the screen, however it has a twist to it. After many transits, it forms 2D surfaces in 3D space.

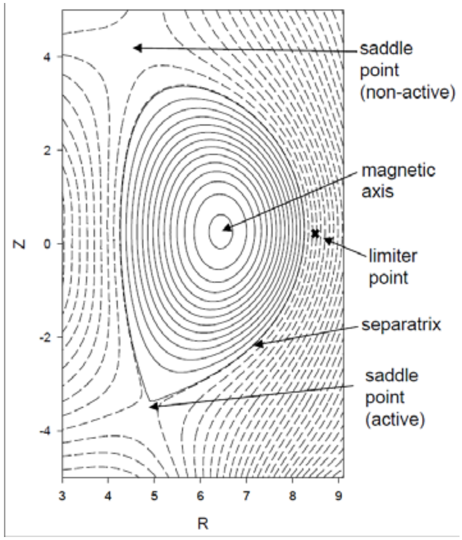
Because the particles are free to stream along the field, the temperatures and densities are nearly uniform on these surfaces.

Only the Fast Wave can propagate across these surfaces, but it will have a very small amplitude compared to the other waves.

Must deal with Fast Wave



Tokamak schematic



Tokamak cross section

- The field lines in a tokamak are dominantly in the toroidal direction.
- The magnetic field forms “flux surfaces”.
- Only the fast wave can propagate across these surfaces.
- Since the gradients across surfaces are large (requiring high resolution), the time-scales associated with the fast wave are very short
- However, the amplitude will always be small because it compresses the field.

➔ The presence of the fast wave makes explicit time integration not practical

Summary

- Nuclear fusion is a promising energy source that will be demonstrated in the coming decades by way of the tokamak (ITER)
- Global dynamics of the plasma in the tokamak are described by a set of fluid like equations called the MHD equations
- A subset of the full-MHD equations with the dissipative terms removed are called the ideal-MHD equations
- These have wave solutions that illustrate that there are 3 fundamentally different types of waves.
- Unstable plasma motions are always associated with the slow wave and Alfven wave.
- The fast wave is a major source of trouble computationally because it is the fastest and the only one that propagates across the surfaces
- Largely because of the fast wave, implicit methods are essential

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Suite 300, Boca Raton, FL 33487
270 Madison Avenue
New York, NY 10016
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

K10512
ISBN: 978-1-4398-1021-7

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Computational Methods in Plasma Physics

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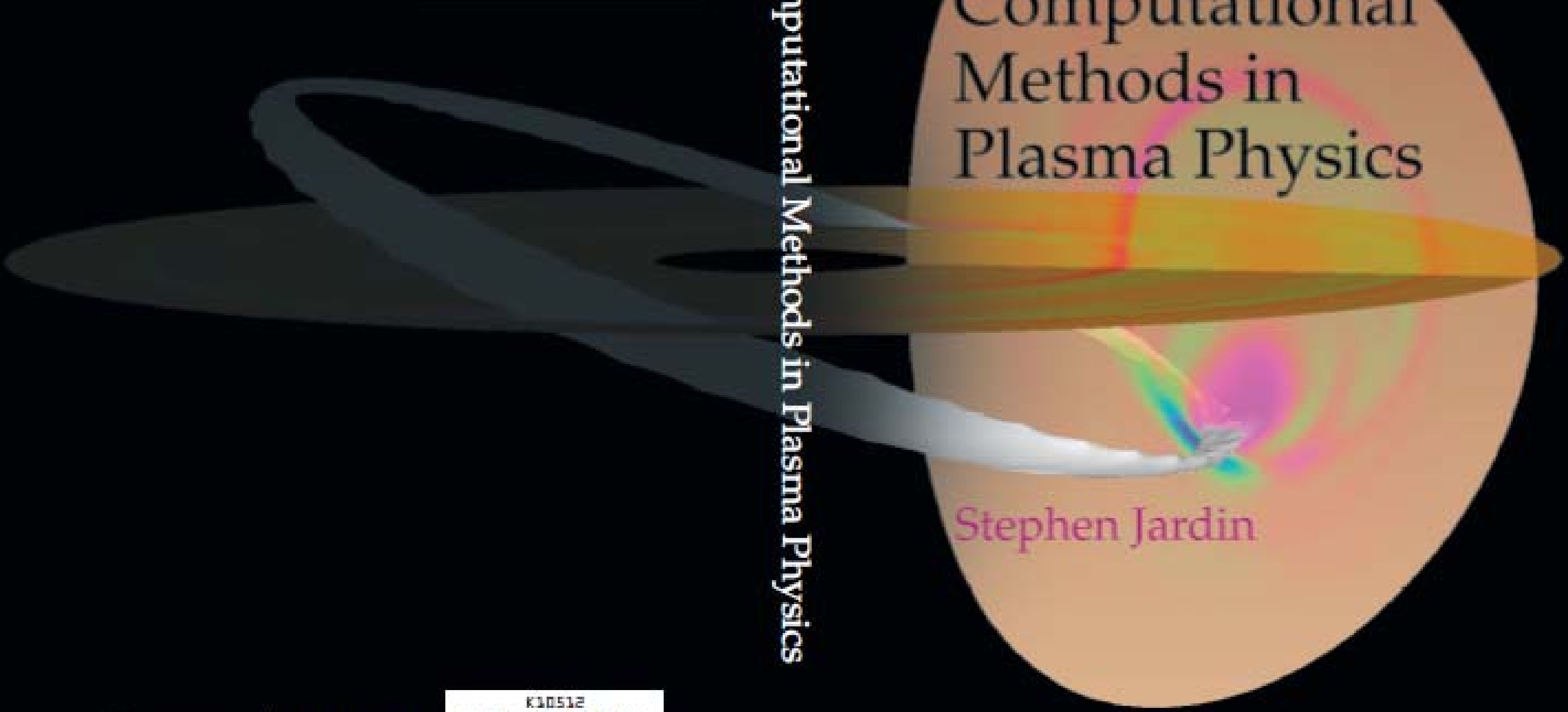

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